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Instability of isolated three-dimensional magnetic null points

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Although most magnetic neutral points occurring in nature seem to form part of a continuum, recent studies of reconnection have centred on static equilibria in the neighbourhood of an isolated three-dimensional null point. The linear stability of this configuration is studied here. It is found that one may choose a flux surface so that transverse oscillations localized around the surface and polarized within it must grow exponentially in time. This means that any static equilibrium containing an isolated three-dimensional null point is linearly unstable.

1. Introduction

Magnetic null points are important, among other things, because reconnection takes place there. They must exist wherever the local topology of the magnetic field is not trivial, such as in the classical magnetospheric models of Dungey (1961) and Axford and Hines (1961) for the insertion of the geomagnetic field within the global solar wind. Reconnection is still a widely debated subject (Heikkila 1997), but the pivotal role of neutral points is not in question. Those models, as well as several others, feature a whole line of null points, but recently some studies have been undertaken on isolated null points (Priest and Titov 1996). In this paper we intend to show that in the simplest situation (an inviscid, perfectly conducting fluid) three-dimensional isolated neutral points are linearly unstable: field-line oscillations may accumulate there, rapidly destroying any static equilibrium around those points. This is in contrast to the two-dimensional situation, or to the lines of null points present in the models mentioned above. Hence this kind of equilibria are unlikely to occur in real phenomena, and studies of reconnection at those points should take this fact into account. A recent study (Parnell *et al.* 1997) also finds that isolated magnetic null points are linearly unstable; however, its approach is essentially different, and both it and the present paper shed light on complementary aspects of the evolution of these neutral points.

2. The model

We consider an isolated null point Q of the magnetic field $\mathbf{B}$. The three eigenvalues of the Jacobian matrix $\nabla\mathbf{B}$ at the point Q must sum to zero because $\mathbf{B}$ is solenoidal, and one of them must be real because $\mathbf{B}$ is real. If the remaining two are complex conjugates, the field roughly describes a helix around the real eigendirection that flattens at the critical point. The occurrence of this kind of point at natural configurations is rather unusual, so we consider only the so-called hyperbolic case, when

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the three eigenvalues are real. Then we have again two possibilities. First, if one eigenvalue is zero, the other two have opposite signs to each other, and the field has x-type topology in the plane generated by the associated eigenvectors. This happens when there is a line of null points, as described in Sec. 1. Secondly, at a true isolated hyperbolic point, one of the eigenvalues has a sign opposite to that of the remaining two: the first eigenvector forms the *spine* and the remaining two the *fan plane* (this terminology is as in Priest and Titov 1996). To visualize the local topology, consider the field $\mathbf{B} = (x, cy, -(c+1)z)$ with $c > 0$; the spine is the z axis and the fan the plane $z = 0$. The field flows towards Q along the spine, and radiates from Q in the fan plane. Obviously the direction of the field may be reversed by considering $-\mathbf{B}$. We let α equal the eigenvalue of greatest absolute value, and consider the plane generated by the corresponding eigenvector and another eigenvector associated with an eigenvalue β with $|\beta| < |\alpha|$. Certainly these exist because the three eigenvalues are different from zero, in contrast to what happens at a planar x-point. We assume now that the field is in a static equilibrium and consider small variations of it. The consequent linearized magnetohydrodynamic equations are well known: we shall refer to Hameiri (1985) as one of the most complete and rigorous treatments of this topic. Although only the axisymmetric case is studied there in detail, its mathematical approach for perturbations localized around a flux surface and a field line apply in much more general situations. Assuming a polytropic equation of state $p = S\rho^{\gamma}$ and taking the usual constants equal to 1, we get, after taking a Fourier–Laplace transform (i.e. considering a time behaviour of the form $e^{\lambda t}$),

$$\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{b} + \mathbf{b} \cdot \boldsymbol{\nabla}\mathbf{B} - \boldsymbol{\nabla} p_* - \lambda\rho\mathbf{u} = \mathbf{0}, \tag{1}$$

$$\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{B} + \frac{1}{\gamma p}(\mathbf{u} \cdot \boldsymbol{\nabla} p)\mathbf{B} - \lambda\left(\mathbf{b} + \frac{1}{\gamma p}(\mathbf{B} \cdot \mathbf{b})\mathbf{B} - \frac{1}{\gamma p} p_* \mathbf{B}\right) = \mathbf{0}, \tag{2}$$

$$\mathbf{B} \cdot (\mathbf{B} \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{B}) - \mathbf{u} \cdot \boldsymbol{\nabla} p - (\gamma p + B^2)\boldsymbol{\nabla} \cdot \mathbf{u} - \lambda p_* = 0, \tag{3}$$

where $\mathbf{b}$ and $\mathbf{u}$ are respectively the perturbed magnetic field and velocity, p_* is the perturbed total pressure (kinetic plus magnetic), and Q and ρ are the equilibrium pressure and density. The incompressible limit is achieved by taking $\gamma \to \infty$, and $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. All the λs for which there is a solution (in a very wide sense) of the above equations form the spectrum of the MHD equations. Any such λ with positive real part guarantees instability of the equilibrium. Particular choices of perturbed fields yield parts of the spectrum that are easier to handle: if we take any flux surface and consider small values of the magnitudes differentiated across it (i.e. the normal component of $\mathbf{u}$ and p_*), we are left with a reduced system

$$L\,(\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{b} + \mathbf{b} \cdot \boldsymbol{\nabla}\mathbf{B}) = \lambda\rho\mathbf{u}, \tag{4}$$

$$L\left(\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{B} + \frac{1}{\gamma p}(\mathbf{u} \cdot \boldsymbol{\nabla} p)\mathbf{B}\right) = \lambda\left(\mathbf{b} + \frac{1}{\gamma p}(\mathbf{B} \cdot \mathbf{b})\mathbf{B}\right), \tag{5}$$

where L is the projection that eliminates the component perpendicular to the flux surface. This surface may cut itself in the long run, but the existence of a bounded solution of the above system always guarantees that λ is part of the essential spectrum (related to the so-called ballooning modes; special flux surfaces produce Alfvén modes). The system is essentially an ordinary differential one where we differentiate along the static field $\mathbf{B}$. In our case we choose the field line Γ ending at Q whose tangent vector is the α eigenvector, and any flux surface Σ containing Γ such that

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the β eigenvector is tangent to Σ at the point Q. Thus the tangent plane to Σ at Q is generated by the α and β eigenvectors. This flux surface may be the fan (if α and β have the same sign) or may link the spine with one line of the fan otherwise. In fact, only a small portion of the flux surface around the field line Γ is relevant. Γ may go to another null point, or may behave in a more complicated manner; this is important for the setting of the boundary values of the equation. Recall that $\mathbf{b}$ and $\mathbf{u}$ should be small compared with the equilibrium quantities, which means in particular that $\mathbf{b}$ must have a zero at Q at least of the same order as that of $\mathbf{B}$, which we assume to be one. The same should happen at the other end if Γ finishes at another critical point; or it must remain bounded if Γ has no end. If we denote by τ the natural parameter along Γ (so that $\mathbf{B}\cdot\boldsymbol{\nabla} = d/d\tau$) then $\mathbf{B}$ behaves near Q as $e^{\pm\alpha\tau}$ (depending on the sign of α), and Q corresponds to the value $\pm\infty$. However, it is more useful to use the arc length as a parameter. If we set Q at $s = 0$, B behaves as $|\alpha|s$ along Γ near Q, and as $|\beta|s$ at the other field line.

3. Behaviour of solutions near the null point

Assume that we have a solution of (4) and (5), with the boundary conditions described above for one eigenvalue λ. In fact, it is enough that λ belongs to the spectrum of the system. This spectrum does not need to be formed solely by eigenvalues, because the interval of the parameter τ is $(-\infty, \infty)$. This kind of boundary-value problem has a long history, especially in the simplest case of a second-order scalar differential equation (see e.g. Coddington and Levinson 1955; Dunford and Schwarz 1963), and many results are known about the existence of essential and point spectra. We do not need to go deeper into them; we merely assume that a solution can be found for some $\lambda \neq 0$. For points of the essential spectrum we should need to deal with approximate solutions, which we omit for simplicity. A further simplification is performed: we shall take the flux surface as precisely a plane in a neighbourhood of Q. In this way we may guarantee that $\mathbf{b}$ and $\mathbf{u}$ lie within this plane and omit the projections in (4) and (5). The results are equally true for the general case, but several geometric terms must be added to the final system to account for the curvature of the surface, which only serve to obscure the main facts. Finally, to avoid compressibility effects, we assume the equilibrium density ρ to be constant near Q. For the equilibrium to be linearly stable we need $\mathrm{Re}\,\lambda \leqslant 0$. Let us now reduce the problem to a second-order differential equation in $\mathbf{b}$. By taking the value of $\mathbf{u}$ given by (4) and substituting it into (5), we are left with

$$B^2\mathbf{b}'' + BB'\mathbf{b}' + \frac{1}{\gamma p}B\mathbf{B}(\mathbf{b}'\cdot\boldsymbol{\nabla}p) + \mathbf{b}\cdot B\boldsymbol{\nabla}\mathbf{B}' - \mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} + \frac{1}{\gamma p}\mathbf{B}[(\mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{B})\cdot\boldsymbol{\nabla}p] - \lambda^2\rho\mathbf{b} - \frac{\lambda^2\rho}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{B} = \mathbf{0}, \tag{6}$$

where $'$ denotes d/ds. Although this is a rather complicated equation, notice that several terms are multiplied by factors of the order of B, which are small near $s = 0$. Assume now that all the coefficients are analytic functions of s in this neighbourhood of Q. Since, as stated before, B behaves like $|\alpha|s$ near Q (where $s = 0$), the equation may be written as

$$\alpha^2 s^2\mathbf{b}'' + s\mathbf{A}_1(s)\mathbf{b}' + \mathbf{A}_2(s)\mathbf{b} = \mathbf{0}, \tag{7}$$

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where $\mathbf{A}_1$ and $\mathbf{A}_2$ are analytic matrices in s. This type of equation is well known: 0 is a singular regular point, and locally the solutions may be found by Frobenius' method: they have the form $s^\mu \mathbf{\Phi}(s)$, where $\mathbf{\Phi}$ is analytic, $\mathbf{\Phi}(0) \neq \mathbf{0}$, and μ and $\mathbf{\Phi}(0)$ are solutions of the so-called indicial equation

$$[\alpha^2\mu(\mu-1)\mathbf{I} + \mathbf{A}_1(0)\mu + \mathbf{A}_2(0)]\mathbf{\Phi}(0) = \mathbf{0}, \tag{8}$$

where $\mathbf{I}$ is the unit matrix. There exist some caveats concerning the difference of the solutions μ of the indicial equation, which do not need to bother us now. Since in our case, the only term in $\mathbf{A}_1$ of zero order in s is $\alpha^2\mathbf{I}$, and in $\mathbf{A}_2$ is $-(\boldsymbol{\nabla}\mathbf{B})^2 - \lambda^2\rho\mathbf{I}$, the indicial equation becomes

$$\left\{\left[\alpha^2\mu(\mu-1) + \alpha^2\mu - \lambda^2\rho\right]\mathbf{I} - (\boldsymbol{\nabla}\mathbf{B})^2\right\}\mathbf{\Phi}(0) = \mathbf{0}. \tag{9}$$

Finally we assume that $\mathbf{b}$ is transverse to the equilibrium field $\mathbf{B}$, i.e. not collinear with the α eigenvector near Q. This excludes magnetosonic oscillations and certainly occurs when the fluid is incompressible. For instance, Alfvénic waves are transverse to the original field and propagate along its lines. This means that $(\boldsymbol{\nabla}\mathbf{B})^2\mathbf{\Phi}(0)$ cannot be $\alpha^2\mathbf{\Phi}(0)$, and since the other eigenvalue of $\boldsymbol{\nabla}\mathbf{B}(0)$ is β, necessarily $(\boldsymbol{\nabla}\mathbf{B})^2\mathbf{\Phi}(0) = \beta^2\mathbf{\Phi}(0)$. We are left with $\alpha^2\mu^2 - \lambda^2\rho - \beta^2 = 0$, i.e.

$$\mu = \pm\frac{(\lambda^2\rho + \beta^2)^{1/2}}{\alpha}. \tag{10}$$

Since, as stated before, $\mathbf{b}$ must have at least a simple zero at $s = 0$ for the linearization to be valid, the only admissible solution corresponds to positive μ. Also, $\mu \geqslant 1$, and therefore $\lambda^2\rho \geqslant \alpha^2 - \beta^2$, which, since $\beta^2 < \alpha^2$, requires that $\lambda^2 > 0$. If λ satisfies (4) and (5) for some $(\mathbf{b}, \mathbf{u})$ then so does $-\lambda$ for $(-\mathbf{b}, \mathbf{u})$. Hence we may always choose a positive λ, which means that there is instability, and we may even guarantee a minimum lower growth exponent: $[(\alpha^2 - \beta^2)/\rho]^{1/2}$.

Let us now discuss the meaning of this instability. Obviously the problem lies in Q and not in the rest of Γ. When α and β are both negative (respectively both positive), the field lines in the flux surface approach (respectively leave) Q by becoming tangent to the β eigenvector, with the exception of Γ itself: for a simple example of this situation, consider the field $(-x, -cy)$ with $0 < c < 1$; the x axis is Γ. If one of α or β is negative and the other positive, Q does not attract field lines, but they stay closer to the β eigenvector than to Γ: for example, take $(-x, cy)$ with $0 < c < 1$. The common feature to both cases is that transverse oscillations along field lines accumulate at Q coming from opposite directions, and at two different speeds $\alpha s/\rho^{1/2}$ and $\beta s/\rho^{1/2}$. In contrast, in the case $|\beta| \geqslant |\alpha|$ oscillations come mostly from the direction of Γ and at the same speed. Apparently in the first case, even if the amplitude of the oscillation along each field line tends to zero when approaching Q, they reinforce in such a manner that the null point cannot remain static. Thus imaginary eigenvalues $\lambda = i\omega$ are excluded. Only continuous growth is permitted, and this in turn causes the perturbed field to increase exponentially and destroys the equilibrium. The case $|\alpha| = |\beta|$ is the only one allowing oscillatory perturbations, whereas for $|\alpha| < |\beta|$ these would be damped.

One might ask why this result does not apply to the limiting case $\beta = 0$. The answer is that it does – and every point of a neutral line is unstable: but transverse oscillations merely move the neutral line along itself, so the line as a whole is stable. In this case the requirement for the perturbed field to be of the same order as the original one is excessive: the line as a whole may remain in equilibrium for transverse

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oscillations, so that an external observer would note nothing. This certainly does not happen for a single point.

4. Conclusions

Prompted by recent work on reconnection near isolated critical points of the magnetic field, we have investigated the stability of static equilibra around hyperbolic magnetic null points. We have found that if the fluid is locally incompressible in the vicinity of the null point, one may choose a flux surface so that transverse oscillations polarized along this surface grow exponentially and therefore destroy the equilibrium rapidly. In the compressible case magnetosonic waves should also be taken into account. This instability seems to arise because field line oscillations accumulate at the null point in such a way that it cannot remain static. In contrast, classical planar x-points are marginally stable, and lines formed by neutral points are stable as a whole. Hence hyperbolic null points are unlikely to occur in a realistic static equilibrium.

References

Axford, W. I. and Hines, C. O. 1961 A unifying theory of high-latitude geophysical phenomena and geomagnetic storms. *Can. J. Phys.* **39**, 1433.

Coddington, E. A. and Levinson, N. 1955 *Theory of Ordinary Differential Equations*. McGraw-Hill, New York.

Dunford, N. and Schwarz, J. T. 1963 *Linear Operators II*. Interscience, New York.

Dungey, J. W. 1961 Interplanetary magnetic field and the auroral zones. *Phys. Rev. Lett.* **6**, 47.

Hameiri, H. 1985 On the essential spectrum of ideal magnetohydrodynamics. *Commun. Pure Appl. Maths* **38**, 43.

Heikkila, W. J. 1997 The Reconnection Myth. *Eos* **78**, 15.

Parnell, C. E., Neukirch, T., Smith, J. M., and Priest, E. R. 1997 Structure and collapse of three-dimensional magnetic neutral points. *Geophys. Astrophys. Fluid Dyn.* **84**, 245.

Priest, E. R. and Titov, V. S. 1996 Magnetic reconnection at three-dimensional null points. *Phil. Trans. R. Soc. Lond.* **A354**, 2591.

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