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Noether formalism with gauge-invariant variations

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Abstract. The energy–momentum tensor is usually obtained by symmetrizing the canonical tensor, which is sometimes difficult. The fact that the energy–momentum tensor must always be gauge invariant, and insight gained by investigating the Maxwell kinetic equations, lead one to conclude that this tensor should be readily obtainable for a certain class of Lagrange densities. The appropriate method to achieve this consists in performing gauge-invariant variations within the framework of Noether's theory. These variations are equivalent to shifts in space and time, and are considered here instead of the usual shift variations, which are not gauge invariant. These features indicate the existence of a certain underlying structure which is made evident by the proposed method. The Lagrange densities of the class in question are characterized by having a gauge-invariant contribution additional to the Maxwell part. For vanishing potentials, such Lagrange densities lead directly to a symmetric canonical tensor; this implies that there are only scalar fields. Maxwell's equations, being the basic part of such theories, are treated first. Then, the coupled system of electromagnetic fields and the usual scalar fields describing charged matter is considered. Finally, a different kind of scalar field which occurs in a structural non-standard form in collisionless phase-space theories (Vlasov or collisionless Boltzmann-like equations) is treated.

1. Introduction

This paper was motivated by the observation made by Pfirsch [1] and Pfirsch and Morrison [2] in the context of Maxwell kinetic theories that the gauge-invariant part of the canonical tensor obtained with shift variations by Noether's theorem is just the symmetric energy–momentum tensor. Further motivation arose from problems encountered by the present authors when investigating general kinetic and gyrokinetic theories. The Lagrange densities for the theories of interest belong to a class which has a gauge-invariant contribution additional to the Maxwell part. For vanishing potentials, such Lagrange densities lead directly to a symmetric canonical tensor. The procedure presented here is applicable to this whole class, i.e. to all Lagrangians for the usual scalar fields. The important point is that these Lagrangians do not contain, except for the Maxwell fields, any vectors, four-vectors or higher-order tensors. The Dirac equation is, therefore, also excluded because of the occurrence of the four-vector consisting of the Dirac matrices γ_ν.

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Since the energy–momentum tensor must always be gauge invariant, it is concluded that, for the Lagrangians in question, it should be possible to obtain this tensor directly, instead of having to symmetrize a non-gauge-invariant canonical tensor. The crucial step is to carry out Noether's procedure with gauge-invariant variations. These variations are equivalent to shifts in space and time, and are considered instead of the usual shift variations, which are not gauge invariant. Here, the variations consist of a combination of the usual infinitesimal shift variations and certain infinitesimal gauge transformations. With the three kinds of variations denoted by

$$\delta_{\mathrm{comb}},\ \delta_{\mathrm{shift}},\ \delta_{\mathrm{gauge}}, \tag{1}$$

application of these variations to the Lagrange densities $\mathscr{L}$ considered yields, also without inserting the Euler–Lagrange equations, because of $\delta_{\mathrm{gauge}}\mathscr{L} = 0$,

$$\delta_{\mathrm{comb}}\mathscr{L} = \delta_{\mathrm{shift}}\mathscr{L} + \delta_{\mathrm{gauge}}\mathscr{L} = \delta_{\mathrm{shift}}\mathscr{L}. \tag{2}$$

This shows the equivalence of δ_{comb} and δ_{shift}. Furthermore, we also use the notation

$$\delta^{\mathrm{EL}}_{\mathrm{comb}}\mathscr{L} \tag{3}$$

to denote the result of the gauge-invariant variations *after* the Euler–Lagrange equations have been inserted. Conservation laws then follow from the equality

$$\delta^{\mathrm{EL}}_{\mathrm{comb}}\mathscr{L} = \delta_{\mathrm{shift}}\mathscr{L} \quad \text{or} \quad \delta^{\mathrm{EL}}_{\mathrm{comb}}\mathscr{L} = \delta^{(\mathrm{EL})}_{\mathrm{shift}}\mathscr{L}, \tag{4}$$

where $\delta^{(\mathrm{EL})}_{\mathrm{shift}}\mathscr{L}$ indicates the use of some (but not all) Euler–Lagrange equations, as done in the last example concerning the Maxwell–Vlasov theory.

A further, interesting way to derive a symmetric energy–momentum tensor from the Lagrangian density is Hilbert's method [3, 4]. His procedure can be applied to relativistic theories that have the same formal structure within the framework of both special and general relativity, e.g. in the electromagnetic case and for the Klein–Gordon equation. A similarly uncomplicated application of Hilbert's method to the relativistic Maxwell–Vlasov theory is, however, not possible. For this theory, even the derivation of the non-relativistic symmetric energy–momentum tensor as a limiting case of the relativistic one is not straightforward (a discussion of both cases is given in Sec. 5 below). Furthermore, Hilbert's method is generally not applicable to non-relativistic theories, for which only the symmetry of the spatial components of the energy–momentum tensor is required.

The basic part of the Lagrangians under discussion is the contribution from the Maxwell fields, which will therefore first be separately treated. Then, the coupled system of electromagnetic fields and the usual scalar fields describing charged matter is considered. Finally, a different kind of scalar field which occurs in a structural non-standard form in collisionless phase-space theories (Vlasov or collisionless Boltzmann-like equations) is treated. The fully symmetric tensor is obtained for relativistic theories, whereas in the non-relativistic case the symmetry concerns only the spatial components. We begin with the definition of the infinitesimal gauge transformation.

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2. The infinitesimal gauge transformation

A gauge transformation of the potentials $\mathbf{A}(\mathbf{x},t)$ and $\Phi(\mathbf{x},t)$ can be obtained by means of a generating function $W(\mathbf{x},t)$:

$$\mathbf{A}'(\mathbf{x},t)=\mathbf{A}(\mathbf{x},t)+\frac{\partial W(\mathbf{x},t)}{\partial \mathbf{x}},\quad \Phi'(\mathbf{x},t)=\Phi(\mathbf{x},t)-\frac{1}{c}\frac{\partial W(\mathbf{x},t)}{\partial t}. \tag{5}$$

There will be additional relations for certain quantities entering special theories with matter included. For the examples considered below these relations are given in the corresponding sections.

For a four-vector representation of the potentials

$$A_0(x^0,x^1,x^2,x^3)=-\Phi(\mathbf{x},t),$$

$$(A_1(x^0,x^1,x^2,x^3),\ A_2(x^0,x^1,x^2,x^3),\ A_3(x^0,x^1,x^2,x^3))=\mathbf{A}(\mathbf{x},t),$$

$$x^0=ct,\quad (x^1,x^2,x^3)=\mathbf{x},$$

this gauge transformation is

$$A'_\mu=A_\mu+\frac{\partial W}{\partial x^\mu}. \tag{6}$$

Since the shift variations are linear in the potentials, the gauge transformations must also be linear in them in order to allow a combination of both that can lead to a gauge-invariant expression. In addition, W must be a scalar. This is guaranteed by the choice

$$W(\mathbf{x},t)=\boldsymbol{\epsilon}\cdot\mathbf{A}-\epsilon^0\Phi=\epsilon^\mu A_\mu, \tag{7}$$

with

$$\epsilon^0=c\,\delta t,\quad (\epsilon^1,\epsilon^2,\epsilon^3)=\boldsymbol{\epsilon}=\delta\mathbf{x}.$$

Here, δt and $\delta\mathbf{x}$ are, respectively, the time shift and the spatial shift. The corresponding variations of $\mathbf{A}$ and Φ are

$$\delta_{\text{gauge}}\mathbf{A}=\frac{\partial}{\partial\mathbf{x}}(\boldsymbol{\epsilon}\cdot\mathbf{A}-\epsilon^0\Phi),\quad \delta_{\text{gauge}}\Phi=-\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\epsilon}\cdot\mathbf{A}-\epsilon^0\Phi), \tag{8}$$

or else

$$\delta_{\text{gauge}}A_\mu=\frac{\partial}{\partial x^\mu}(\epsilon^\lambda A_\lambda). \tag{9}$$

The shift variations are

$$\delta_{\text{shift}}\mathbf{A}=-\boldsymbol{\epsilon}\cdot\frac{\partial\mathbf{A}}{\partial\mathbf{x}}-\epsilon^0\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \delta_{\text{shift}}\Phi=-\boldsymbol{\epsilon}\cdot\frac{\partial\Phi}{\partial\mathbf{x}}-\epsilon^0\frac{1}{c}\frac{\partial\Phi}{\partial t}, \tag{10}$$

or else

$$\delta_{\text{shift}}A_\mu=-\epsilon^\lambda\frac{\partial}{\partial x^\lambda}A_\mu. \tag{11}$$

Although the standard variations, which consist only of the shifts, would be perfectly adequate, they have, however, the disadvantage of usually leading to a non-symmetric, gauge-dependent canonical tensor. Our combined variations consisting of the usual, gauge-dependent Euler shift variations (see e.g. [2,(22)]), plus an additional infinitesimal gauge transformation, have the very convenient property of being gauge invariant, and thus produce gauge-invariant expressions for the variation of the Lagrange density which *directly* lead to the symmetric, gauge-independent energy–momentum tensor. This is especially advantageous in cases in

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which the derivation of the symmetric energy–momentum tensor from the canonical tensor is far from trivial, e.g. in drift-kinetic and gyrokinetic theories.

The expressions for $\delta_{\text{shift}}\mathbf{A}$ and $\delta_{\text{gauge}}\mathbf{A}$, (8)–(11), combine to give the gauge-invariant expressions

$$\delta_{\text{comb}}\mathbf{A} = \delta_{\text{shift}}\mathbf{A} + \delta_{\text{gauge}}\mathbf{A} = \boldsymbol{\epsilon} \times \mathbf{B} + \epsilon^0 \mathbf{E}, \tag{12}$$

$$\delta_{\text{comb}}\Phi = \delta_{\text{shift}}\Phi + \delta_{\text{gauge}}\Phi = \boldsymbol{\epsilon} \cdot \mathbf{E}, \tag{13}$$

or else

$$\delta_{\text{comb}}A_\mu = \delta_{\text{shift}}A_\mu + \delta_{\text{gauge}}A_\mu = -\epsilon^\lambda \left[\frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\mu}\right] = -\epsilon^\lambda F_{\lambda\mu}. \tag{14}$$

Equation (12), multiplied by e/c, corresponds to a momentum change owing to the Lorentz force, experienced by a charged particle or mass element in the time $\delta t = \epsilon^0/c$ when it moves with the velocity $\mathbf{v} = \boldsymbol{\epsilon}/\delta t$. Equation (13), multiplied by e, corresponds to the energy change experienced by the same entity in the time δt. The form of these two equations is the same as the covariant form (see e.g. [5]) of the four-force density of electrodynamics,

$$\mathbf{k} = \frac{1}{c}\mathbf{j} \times \mathbf{B} + \rho\mathbf{E}, \quad k^0 = \mathbf{j} \cdot \mathbf{E}.$$

The change in the fields obtained from $\delta_{\text{comb}}\mathbf{A}$ and $\delta_{\text{comb}}\Phi$ is, as it should be, not affected by the gauge transformation:

$$\delta_{\text{comb}}\mathbf{B} = \boldsymbol{\nabla} \times \delta_{\text{comb}}\mathbf{A} = -\epsilon^0 \frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\epsilon} \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{x}} = \delta_{\text{shift}}\mathbf{B}, \tag{15}$$

$$\delta_{\text{comb}}\mathbf{E} = -\boldsymbol{\nabla}\delta_{\text{comb}}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\delta_{\text{comb}}\mathbf{A} = -\epsilon^0 \frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\epsilon} \cdot \frac{\partial \mathbf{E}}{\partial \mathbf{x}} = \delta_{\text{shift}}\mathbf{E}. \tag{16}$$

3. The Maxwell energy–momentum tensor

For investigating symmetry properties it is appropriate to use the component representation of the potentials and fields, thereby distinguishing between upper indices (contravariant components) and lower indices (covariant components),

$$\begin{aligned} B^i &= \frac{1}{2}\varepsilon^{ikl}\left(\frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l}\right) = \frac{1}{2}\varepsilon^{ikl}F_{kl}, \\ E_i &= -\left(\frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i}\right) = -F_{0i} = F_{i0}, \end{aligned} \tag{17}$$

where i, k, l take any of the values $1, 2, 3$ and ε^{ikl} is the fully antisymmetric ε-tensor. It should be recalled that the metric tensor $g^{\mu\nu}$ for the system of coordinates (ct, x^1, x^2, x^3) is diagonal, with elements $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, which means that the 0th components change their sign when the index 0 is shifted up or down.

The Maxwell Lagrange density is then

$$\mathscr{L}_{\text{M}} = \frac{1}{8\pi}[\mathbf{E}^2 - \mathbf{B}^2] = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}, \tag{18}$$

where the summation over μ, ν is from 0 to 3.

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With (14) taken into account, the variation $\delta\mathscr{L}_{\mathrm{M}}$ of $\mathscr{L}_{\mathrm{M}}$ yields

$$\begin{aligned}\delta\mathscr{L}_{\mathrm{M}} &= -\frac{1}{16\pi}[F_{\mu\nu}\delta F^{\mu\nu} + F^{\mu\nu}\delta F_{\mu\nu}] = -\frac{1}{8\pi}F^{\mu\nu}\delta F_{\mu\nu} \\ &= -\frac{1}{8\pi}F^{\mu\nu}\left[\frac{\partial(\delta A_\nu)}{\partial x^\mu} - \frac{\partial(\delta A_\mu)}{\partial x^\nu}\right] = -\frac{1}{4\pi}F^{\mu\nu}\frac{\partial(\delta A_\nu)}{\partial x^\mu} \\ &= -\frac{1}{4\pi}\left[\frac{\partial}{\partial x^\mu}[F^{\mu\nu}\delta A_\nu] - \delta A_\nu\frac{\partial F^{\mu\nu}}{\partial x^\mu}\right]. \end{aligned} \tag{19}$$

The Euler–Lagrange equations obtained with general variations δA_ν from the action principle $\delta\int\mathscr{L}_{\mathrm{M}}\,d^3x\,dt = 0$ are then

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = 0. \tag{20}$$

Inserting the Euler–Lagrange equations yields

$$\delta\mathscr{L}_{\mathrm{M}}^{\mathrm{EL}} = -\frac{1}{4\pi}\frac{\partial}{\partial x^\mu}[F^{\mu\nu}\delta A_\nu]. \tag{21}$$

The usual procedure is to insert here non-gauge-invariant shift variations $\delta A_\nu = -\epsilon^\lambda\partial A_\nu/\partial x^\lambda$, which lead to the non-symmetric canonical tensor. The tensor $F^{\mu\nu}\partial A_\nu/\partial x^\lambda$ is then made symmetric (and gauge invariant) by combining it with a tensor $\sim F^{\mu\nu}\partial A_\lambda/\partial x^\nu = \partial[F^{\mu\nu}A_\lambda]/\partial x^\nu$, which contains both symmetric and non-symmetric contributions, and whose four-divergence $\partial^2[F^{\mu\nu}A_\lambda]/\partial x^\mu\partial x^\nu$ vanishes. Instead of doing that, we take the δA_ν's to be the gauge-invariant result of a shift plus an infinitesimal gauge transformation as given by (14). This yields

$$\delta\mathscr{L}_{\mathrm{M}}^{\mathrm{EL}} = \delta_{\mathrm{comb}}\mathscr{L}_{\mathrm{M}}^{\mathrm{EL}} = \epsilon^\lambda\frac{1}{4\pi}\frac{\partial}{\partial x^\mu}[F^{\mu\nu}F_{\lambda\nu}]. \tag{22}$$

On the other hand, one can derive $\delta\mathscr{L}_{\mathrm{M}}$ directly:

$$\delta_{\mathrm{shift}}\mathscr{L}_{\mathrm{M}} = -\epsilon^\lambda\frac{\partial\mathscr{L}_{\mathrm{M}}}{\partial x^\lambda} = -\epsilon^\lambda\frac{\partial\mathscr{L}_{\mathrm{M}}}{\partial x^\mu}\delta^\mu_\lambda. \tag{23}$$

By setting the expressions in (22) and (23) equal to each other, according to (4), one obtains

$$\frac{\partial}{\partial x^\mu}\left[\mathscr{L}_{\mathrm{M}}\delta^\mu_\lambda + \frac{1}{4\pi}F^{\mu\nu}F_{\lambda\nu}\right] = 0. \tag{24}$$

These conservation laws can be written as

$$\frac{\partial}{\partial x^\mu}T^\mu_\lambda = 0, \tag{25}$$

with

$$T^\mu_\lambda := \mathscr{L}_{\mathrm{M}}\,\delta^\mu_\lambda + \frac{1}{4\pi}F^{\mu\nu}F_{\lambda\nu} = -\frac{1}{16\pi}F_{\sigma\rho}F^{\sigma\rho}\delta^\mu_\lambda + \frac{1}{4\pi}F^{\mu\nu}F_{\lambda\nu}. \tag{26}$$

The fully contravariant tensor $T^{\mu\lambda}$ is found to be

$$T^{\mu\lambda} = g^{\lambda\tau}T^\mu_\tau = -\frac{1}{16\pi}F_{\sigma\rho}F^{\sigma\rho}g^{\lambda\mu} + \frac{1}{4\pi}F^{\mu\nu}F^\lambda_\nu. \tag{27}$$

With the relation

$$F^{\mu\nu}F^\lambda_\nu = F^\mu_\nu F^{\lambda\nu} \tag{28}$$

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taken into account, the tensor $T^{\mu\lambda}$ is then symmetric:

$$T^{\mu\lambda} = T^{\lambda\mu}. \tag{29}$$

The relation (27) is the well-known expression for the Maxwell energy–momentum tensor.

Our method of considering gauge-invariant variations of the potentials thus directly yields the symmetric energy–momentum tensor. It is not necessary to do the usual procedure of first obtaining the canonical tensor, which is then symmetrized [6].

4. Coupling to scalar matter fields

Although we could in principle consider more general cases, we consider here only the usual relativistic scalar matter fields. The gauge-invariant Lagrange densities $\mathscr{L}_\psi$ for these fields ψ_n, with first-order derivatives, have the structure

$$\mathscr{L}_\psi = \mathscr{L}(\psi_n, \psi_n^*, K_n), \tag{30}$$

where the index n can take values between 1 and N, N being the number of different fields, and with

$$K_n = K_n^* = \left(\frac{\partial \psi_n}{\partial x_\nu} - i\alpha_n A^\nu \psi_n\right)\left(\frac{\partial \psi_n^*}{\partial x^\nu} + i\alpha_n A_\nu \psi_n^*\right). \tag{31}$$

Here, $\alpha_n = e_n/\hbar c$ is a constant characterizing each field. The summation convention is again applied with ν taking values from 0 to 3. The quantity K_n is a relativistic scalar. As in the previous section, one has $x^0 = ct$ and $A_0 = -\Phi$. Therefore, one gets $x_0 = -ct$ and $A^0 = \Phi$ because of $g_{00} = -1$. The combination of the arguments must ensure that $\mathscr{L}_\psi$ is real and gauge invariant.

A simple example is the Lagrangian for the Klein–Gordon equation [7]. We consider it here for several particle species n coupled only through the electromagnetic field:

$$\mathscr{L}_\psi = -\sum_{n=1}^{N} \frac{\hbar^2}{2m_n}\left[\left(\frac{\partial \psi_n^*}{\partial x^\nu} + i\frac{e_n}{\hbar c}A_\nu \psi_n^*\right)\left(\frac{\partial \psi_n}{\partial x_\nu} - i\frac{e_n}{\hbar c}A^\nu \psi_n\right) + \frac{1}{2}\frac{m_n^2 c^2}{\hbar^2}\psi_n^*\psi_n\right]. \tag{32}$$

This equation is invariant under the gauge transformation $A_\nu \to A_\nu + \partial W/\partial x^\nu$, $\psi_n \to \psi_n \exp(i(e_n/\hbar c)W)$.

The Lagrangian of the total system, including the electromagnetic fields, is

$$\mathscr{L} = \mathscr{L}_{\mathrm{M}} + \mathscr{L}_\psi, \tag{33}$$

with $\mathscr{L}_{\mathrm{M}}$ given by (18).

For the derivation of the energy–momentum tensor, the contribution of each field ψ_n can be considered separately. The summation over n is therefore dropped in the following and we write for the general variation of $\mathscr{L}_\psi$

$$\delta\mathscr{L}_\psi = [-\tfrac{1}{4}m_n c^2 \psi_n \delta\psi_n^* + \text{c.c.}] - \beta_n(\delta K_n)_{\delta\psi_n, \delta\psi_n^*, \delta A_\nu, \delta A^\nu}, \tag{34}$$

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c.c. being the complex conjugate, $\beta_n = \hbar^2/2m_n$,

$$(\delta K_n)_{\delta\psi_n^*} = \frac{\partial}{\partial x^\nu}\left[\delta\psi_n^*\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right)\right] - \delta\psi_n^*\left(\frac{\partial}{\partial x^\nu} - i\alpha_n A_\nu\right)\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right), \tag{35}$$

$$(\delta K_n)_{\delta\psi_n} = \left[(\delta K_n)_{\delta\psi_n^*}\right]^* \tag{36}$$

and

$$(\delta K_n)_{\delta A_\nu,\delta A^\nu} = \delta A_\nu i\alpha_n\left[-\psi_n\left(\frac{\partial\psi_n^*}{\partial x_\nu} + i\alpha_n A^\nu\psi_n^*\right) + \psi_n^*\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right)\right], \tag{37}$$

where use has been made of $\delta A^\nu\,\partial/\partial x^\nu = \delta A_\nu\partial/\partial x_\nu$ and $\delta A_\nu A^\nu = \delta A^\nu A_\nu$, valid for each ν.

Variation of the total action integral with respect to A_ν yields the inhomogeneous Maxwell equations,

$$\frac{1}{4\pi}\frac{\partial F^{\mu\nu}}{\partial x^\mu} = i\beta_n\alpha_n\left[\psi_n\left(\frac{\partial\psi_n^*}{\partial x_\nu} + i\alpha_n A^\nu\psi_n^*\right) - \psi_n^*\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right)\right]. \tag{38}$$

Variation of the action integral with respect to ψ_n^* yields the Euler–Lagrange equations for the scalar fields ψ_n,

$$-\frac{1}{4}m_n c^2\psi_n + \beta_n\left(\frac{\partial}{\partial x^\nu} - i\alpha_n A_\nu\right)\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right) = 0. \tag{39}$$

With (38) and (39) taken into account the general variation of the Lagrangian $\mathscr{L}$ reduces to

$$\delta^{\mathrm{EL}}\mathscr{L} = -\frac{1}{4\pi}\frac{\partial}{\partial x^\nu}[F^{\nu\mu}\delta A_\mu] - \beta_n\frac{\partial}{\partial x^\nu}\left[\delta\psi_n\left(\frac{\partial\psi_n^*}{\partial x_\nu} + i\alpha_n A^\nu\psi_n^*\right) + \delta\psi_n^*\left(\frac{\partial\psi_n}{\partial x_\nu} - i\alpha_n A^\nu\psi_n\right)\right]. \tag{40}$$

From this relation the energy–momentum tensor is obtained by choosing for the variations $\delta\mathscr{L}_{\mathrm{M}}$, $\delta\mathscr{L}_\psi$, δA_ν, $\delta\psi_n$ the combination of shift and gauge transformations as defined by the generating function W of (7). The first term in (40) and $\delta\mathscr{L}_{\mathrm{M}}$ again yield the contribution (26) to the total energy–momentum tensor. The contribution from $\delta\mathscr{L}_\psi$ is, analogous to that of (23),

$$\delta_{\mathrm{comb}}\mathscr{L}_\psi = \delta_{\mathrm{shift}}\mathscr{L}_\psi = -\epsilon^\lambda\frac{\partial\mathscr{L}_\psi}{\partial x^\lambda} = -\epsilon^\lambda\frac{\partial\mathscr{L}_\psi}{\partial x^\nu}\delta^\nu_\lambda = -\frac{\partial}{\partial x^\nu}\left(\epsilon^\lambda\ \mathscr{L}_\psi\ \delta^\nu_\lambda\right), \tag{41}$$

which yields as in (26) together with (28) a symmetric contribution to the energy–momentum tensor.

For the terms in (40) containing $\delta\psi_n^*$ and $\delta\psi_n$ it is essential to use the combination of shift and gauge transformations. The gauge transformations are

$$\psi_n' = e^{i\alpha_n W}\psi_n, \quad \psi_n'^* = e^{-i\alpha_n W}\psi_n'^*, \tag{42}$$

and therefore, with W given by (7),

$$\delta_{\mathrm{gauge}}\psi_n = \epsilon^\lambda A_\lambda i\alpha_n\psi_n, \quad \delta_{\mathrm{gauge}}\psi_n^* = -\epsilon^\lambda A_\lambda i\alpha_n\psi_n^*. \tag{43}$$

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The shift transformations are

$$\delta_{\text{shift}}\psi_n = -\epsilon^\lambda \frac{\partial \psi_n}{\partial x^\lambda}, \quad \delta_{\text{shift}}\psi_n^* = -\epsilon^\lambda \frac{\partial \psi_n^*}{\partial x^\lambda}, \tag{44}$$

and therefore, in agreement with the definition in (2),

$$\begin{aligned} \delta_{\text{comb}}\psi_n &= \delta_{\text{shift}}\psi_n + \delta_{\text{gauge}}\psi_n = -\epsilon^\lambda \left(\frac{\partial \psi_n}{\partial x^\lambda} - i\alpha_n A_\lambda \psi_n \right), \\ \delta_{\text{comb}}\psi_n^* &= \delta_{\text{shift}}\psi_n^* + \delta_{\text{gauge}}\psi_n^* = -\epsilon^\lambda \left(\frac{\partial \psi_n^*}{\partial x^\lambda} + i\alpha_n A_\lambda \psi_n^* \right). \end{aligned} \tag{45}$$

The terms in question in (40) can be written as

$$\frac{\partial}{\partial x^\nu} \epsilon^\lambda \big(T_\lambda^{(1)\nu}\big), \tag{46}$$

with

$$\begin{aligned} T_\lambda^{(1)\nu} = \beta_n \Bigg[&\left(\frac{\partial \psi_n}{\partial x^\lambda} - i\alpha_n A_\lambda \psi_n \right) \left(\frac{\partial \psi_n^*}{\partial x_\nu} + i\alpha_n A^\nu \psi_n^* \right) \\ &+ \left(\frac{\partial \psi_n^*}{\partial x^\lambda} + i\alpha_n A_\lambda \psi_n^* \right) \left(\frac{\partial \psi_n}{\partial x_\nu} - i\alpha_n A^\nu \psi_n \right) \Bigg]. \end{aligned} \tag{47}$$

Lowering the index ν in this expression immediately shows that

$$T_{\nu\lambda}^{(1)} = T_{\lambda\nu}^{(1)}. \tag{48}$$

The total energy–momentum tensor is again obtained from the equality $\delta_{\text{comb}}^{\text{EL}} (\mathscr{L}_{\text{M}} + \mathscr{L}_\psi) = \delta_{\text{shift}} (\mathscr{L}_{\text{M}} + \mathscr{L}_\psi)$. The combination of the expressions (26) and (47) and the contribution according to (41) yield, with $\beta_n = \hbar^2/2m_n$ and $\alpha_n = e_n/\hbar c$ inserted,

$$\begin{aligned} T_\lambda^\nu = \frac{1}{4\pi} F^{\nu\rho} F_{\lambda\rho} - \frac{1}{16\pi} F_{\sigma\rho} F^{\sigma\rho} \delta_\lambda^\nu + \frac{\hbar^2}{2m_n} \Bigg[&\left(\frac{\partial \psi_n}{\partial x^\lambda} - i\frac{e_n}{\hbar c} A_\lambda \psi_n \right) \left(\frac{\partial \psi_n^*}{\partial x_\nu} + i\frac{e_n}{\hbar c} A^\nu \psi_n^* \right) \\ &+ \left(\frac{\partial \psi_n^*}{\partial x^\lambda} + i\frac{e_n}{\hbar c} A_\lambda \psi_n^* \right) \left(\frac{\partial \psi_n}{\partial x_\nu} - i\frac{e_n}{\hbar c} A^\nu \psi_n \right) \Bigg] + \mathscr{L}_\psi \delta_\lambda^\nu. \end{aligned} \tag{49}$$

It is again noted here that for obtaining this result it was necessary to use the combined variations of all contributions, those of the Maxwell fields and those of the scalar fields.

The coupling of electromagnetic fields to matter fields in two cases which are useful for plasma applications has been investigated in [8] and [9] in a different context, relating to laser–plasma and neutrino–plasma interactions, respectively. In [8, (19c)], and in [9, (4.22)], expressions which explicitly exhibit the gauge-dependent parts are given for the Euler shift variations of the electromagnetic potentials. It is precisely this kind of gauge-dependent contributions in the case of electromagnetic fields that is eliminated by our method of combined Euler shift variations and infinitesimal gauge transformations. Apart from this, the gauge-dependent contributions stemming from the shift variation of the scalar fields are also eliminated with the same gauge transformations.

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5. Coupling to a system of Vlasov equations

For very large mean free paths of the particles a good description of plasmas is given in terms of the Vlasov equations, which are collisionless Boltzmann equations for each particle species n. These equations describe the behaviour of the distribution functions $f_n(\mathbf{x}, \mathbf{p}, t)$ in phase space, or $f_n(\mathbf{x}, \mathbf{v}, t)$ in position–velocity space. In the latter case, the non-relativistic Vlasov equations are

$$\frac{\partial f_n}{\partial t} + \mathbf{v} \cdot \frac{\partial f_n}{\partial \mathbf{x}} + \frac{e_n}{m_n} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_n}{\partial \mathbf{v}} = 0, \tag{50}$$

where the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are solutions of the inhomogeneous Maxwell equations with the charge and current densities given by

$$\rho(\mathbf{x}, t) = \sum_n e_n \int f_n(\mathbf{x}, \mathbf{v}, t)\, d^3v, \quad \mathbf{j}(\mathbf{x}, t) = \sum_n e_n \int \mathbf{v} f_n(\mathbf{x}, \mathbf{v}, t)\, d^3v. \tag{51}$$

It was shown by Pfirsch [1], Pfirsch and Morrison [2] and Correa-Restrepo et al. [10] that such a theory and similar more complicated theories can be described by a gauge-invariant Lagrangian. As already mentioned in the Introduction, some observations made in [1] and [2] and problems encountered by the authors of this article in connection with ongoing investigations of general kinetic and gyrokinetic theories motivated the present paper.

Concerning the particles, one has to work with scalar fields. However, the equations for these fields have a completely different structure from that given by the Lagrange density $\mathscr{L}_\psi$ in (30) in the preceding section, and the details given there do not apply here. On the other hand, one still has to work with scalar fields, which means that, except for the Maxwell fields, the theory does not contain any kind of vectors. The method of combining shift and gauge transformations to obtain the symmetric energy–momentum tensor should therefore again be successful. We treat here both the relativistic and the non-relativistic Vlasov–Maxwell theory.

5.1. Lagrange density of the combined Vlasov–Maxwell theory

The Lagrange density for the combined Vlasov–Maxwell theory is given by

$$\mathscr{L}(\mathbf{x}, t) = \mathscr{L}_{\mathrm{M}}(\mathbf{x}, t) + \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t), \tag{52}$$

where $\mathscr{L}_{\mathrm{M}}(\mathbf{x}, t)$ is again the Maxwell Lagrange density (18). The Vlasov Lagrange density is given by the following expression:

$$\mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) = -\sum_n \int d^3\alpha\, f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) \left[\frac{\partial S_n(\mathbf{x}, \boldsymbol{\alpha}, t)}{\partial t} + H_n \left(\mathbf{x}, \frac{\partial S_n}{\partial \mathbf{x}}, t \right) \right], \tag{53}$$

where $H_n(\mathbf{x}, \mathbf{p}, t) = e_n \Phi(\mathbf{x}, t) + \hat{H}_n(\mathbf{p} - (e_n/c)\mathbf{A}(\mathbf{x}, t))$ is the Hamiltonian for particles of species n. In the non-relativistic case $\hat{H}_n$ is given by

$$\hat{H}_n = \frac{1}{2m_n} \left[\mathbf{p} - \frac{e_n}{c} \mathbf{A}(\mathbf{x}, t) \right]^2, \tag{54}$$

while in the relativistic case one has

$$\hat{H}_n = c \sqrt{\left(\mathbf{p} - \frac{e_n}{c} \mathbf{A}(\mathbf{x}, t) \right)^2 + m_n^2 c^2}. \tag{55}$$

As will be seen below, the functions $S_n(\mathbf{x}, \mathbf{a}, t)$ satisfy the Hamilton–Jacobi equations, $\partial S_n/\partial t + H_n(\mathbf{x}, \partial S_n/\partial \mathbf{x}, t) = 0$, as a result of the variational principle. These

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functions are only used as a tool. They will not appear in the final expressions and, therefore, one does not have to solve the Hamilton–Jacobi equations; only certain general properties of the functions $S_n(\mathbf{x}, \mathbf{a}, t)$ will play a role. The $\boldsymbol{\alpha}$'s are constants of integration for the Hamilton–Jacobi equations. $S_n(\mathbf{x}, \mathbf{a}, t)$ is a mixed-variable generating function for a canonical transformation to new Hamiltonians equal to zero. The new momenta $\boldsymbol{\alpha}$ and coordinates $\boldsymbol{\beta} = \partial S_n(\mathbf{x}, \boldsymbol{\alpha}, t)/\partial \boldsymbol{\alpha}$ are therefore constants of motion. The functions $f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t)$ are primary forms of the distribution functions with the argument $\boldsymbol{\alpha}$ instead of $\mathbf{p}$ or $\mathbf{v}$. The quantities to be varied in Hamilton's principle are, besides Φ and $\mathbf{A}$, the functions $S_n(\mathbf{x}, \boldsymbol{\alpha}, t)$ and $f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t)$. The Lagrangian is gauge invariant with $\mathbf{A}'(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) + \nabla W(\mathbf{x}, t)$, $\Phi'(\mathbf{x}, t) = \Phi(\mathbf{x}, t) - (1/c)\partial W(\mathbf{x}, t)/\partial t$ and the gauge transformation of $S_n(\mathbf{x}, \boldsymbol{\alpha}, t)$,

$$S_n'(\mathbf{x}, \boldsymbol{\alpha}, t) = S_n(\mathbf{x}, \boldsymbol{\alpha}, t) + \frac{e_n}{c} W(\mathbf{x}, t). \tag{56}$$

5.2. The variation of the Vlasov Lagrange density

The variation of the Vlasov Lagrange density can be written as

$$\delta \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) = \delta_{\delta\Phi} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) + \delta_{\delta\mathbf{A}} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) + \delta_{\delta f_{\mathrm{pn}}} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) + \delta_{\delta S_n} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t), \tag{57}$$

with

$$\delta_{\delta\Phi} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) = -\sum_n \int d^3\alpha \, f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) e_n \delta\Phi(\mathbf{x}, t), \tag{58}$$

$$\delta_{\delta\mathbf{A}} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) = -\sum_n \int d^3\alpha \, f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) \left[-\frac{e_n}{c} \delta\mathbf{A}(\mathbf{x}, t) \right] \cdot \frac{\partial H_n}{\partial \mathbf{p}}, \tag{59}$$

$$\delta_{\delta f_{\mathrm{pn}}} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) = -\sum_n \int d^3\alpha \, \delta f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) \left[\frac{\partial S_n(\mathbf{x}, \boldsymbol{\alpha}, t)}{\partial t} + H_n \right] \tag{60}$$

and

$$\begin{aligned} \delta_{\delta S_n} \mathscr{L}_{\mathrm{V}}(\mathbf{x}, t) &= -\sum_n \int d^3\alpha \, f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) \left[\frac{\partial \, \delta S_n \, (\mathbf{x}, \boldsymbol{\alpha}, t)}{\partial t} + \frac{\partial \, \delta S_n}{\partial \mathbf{x}} \cdot \frac{\partial H_n}{\partial \mathbf{p}} \right] \\ &= -\sum_n \int d^3\alpha \left[-\delta S_n \, (\mathbf{x}, \boldsymbol{\alpha}, t) \left[\frac{\partial f_{\mathrm{pn}}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(f_{\mathrm{pn}} \frac{\partial H_n}{\partial \mathbf{p}} \right) \right] \right. \\ &\quad \left. + \frac{\partial}{\partial t}(f_{\mathrm{pn}} \delta S_n) + \frac{\partial}{\partial \mathbf{x}} \cdot \left(f_{\mathrm{pn}} \delta S_n \frac{\partial H_n}{\partial \mathbf{p}} \right) \right], \end{aligned} \tag{61}$$

where

$$\frac{\partial H_n}{\partial \mathbf{p}} = \left. \frac{\partial H_n(\mathbf{x}, \mathbf{p}, t)}{\partial \mathbf{p}} \right|_{\mathbf{p} = \partial S_n(\mathbf{x}, \boldsymbol{\alpha}, t)/\partial \mathbf{x}}. \tag{62}$$

The use of these expressions in Hamilton's principle together with the variation of the Maxwell Lagrange density, (18), yields the electric charge density from the variation $\delta\Phi$, the current density from $\delta\mathbf{A}$, the Hamilton–Jacobi equations from δf_{pn} and the Vlasov equations in the variables $\mathbf{x}, \boldsymbol{\alpha}, t$ from δS_n. One obtains

$$\frac{1}{4\pi} \nabla \cdot \mathbf{E} = \sum_n e_n \int d^3\alpha \, f_{\mathrm{pn}}(\mathbf{x}, \boldsymbol{\alpha}, t) = \rho, \tag{63}$$

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$$\frac{c}{4\pi}\left[\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right]=\sum_n e_n\int d^3\alpha\, f_{\mathrm{p}n}(\mathbf{x},\boldsymbol{\alpha},t)\frac{\partial H_n}{\partial\mathbf{p}}=\mathbf{j}, \tag{64}$$

$$\frac{\partial S_n(\mathbf{x},\boldsymbol{\alpha},t)}{\partial t}+H_n=0 \tag{65}$$

and

$$\frac{\partial f_{\mathrm{p}n}}{\partial t}+\frac{\partial}{\partial\mathbf{x}}\cdot\left(f_{\mathrm{p}n}\frac{\partial H_n}{\partial\mathbf{p}}\right)=0. \tag{66}$$

In (64), $\partial H_n/\partial\mathbf{p}=\mathbf{v}$ is valid in both the non-relativistic and relativistic cases. In the latter case, this yields for the relation between the canonical and the kinematic momenta $\mathbf{p}-(e_n/c)\mathbf{A}=m_n\mathbf{v}/\sqrt{1-\mathbf{v}^2/c^2}$.

After the contribution of $\delta\mathscr{L}_{\mathrm{V}}$ to the Euler–Lagrange equations (63)–(66) is taken into account, the variation $\delta\mathscr{L}_{\mathrm{V}}(\mathbf{x},t)$ reduces to

$$\delta\mathscr{L}_{\mathrm{V}}^{\mathrm{EL}}(\mathbf{x},t)=-\sum_n\int d^3\alpha\left[\frac{\partial}{\partial t}(f_{\mathrm{p}n}\delta S_n)+\frac{\partial}{\partial\mathbf{x}}\cdot\left(f_{\mathrm{p}n}\delta S_n\frac{\partial H_n}{\partial\mathbf{p}}\right)\right]. \tag{67}$$

The contribution from the shift transformation to δS_n is

$$\delta_{\mathrm{shift}}S_n=-\epsilon^\lambda\frac{\partial S_n}{\partial x^\lambda}=-\boldsymbol{\epsilon}\cdot\frac{\partial S_n}{\partial\mathbf{x}}-\epsilon^0\frac{1}{c}\frac{\partial S_n}{\partial t}. \tag{68}$$

The contribution from the gauge transformation to δS_n is obtained from (7) and (56) as

$$\delta_{\mathrm{gauge}}S_n=\frac{e_n}{c}\epsilon^\mu A_\mu=\frac{e_n}{c}[\boldsymbol{\epsilon}\cdot\mathbf{A}-\epsilon^0\Phi], \tag{69}$$

and, therefore,

$$\begin{aligned}\delta_{\mathrm{comb}}S_n&=\delta_{\mathrm{shift}}S_n+\delta_{\mathrm{gauge}}S_n=-\epsilon^\mu\left(\frac{\partial S_n}{\partial x^\mu}-\frac{e_n}{c}A_\mu\right)\\&=-\boldsymbol{\epsilon}\cdot\left(\frac{\partial S_n}{\partial\mathbf{x}}-\frac{e_n}{c}\mathbf{A}\right)-\epsilon^0\left(\frac{\partial S_n}{\partial ct}+\frac{e_n}{c}\Phi\right).\end{aligned} \tag{70}$$

The expression $\partial S_n/\partial\mathbf{x}$ becomes the canonical momentum $\mathbf{p}$ after a transformation is made from the variable $\boldsymbol{\alpha}$ to $\mathbf{p}$,

$$\frac{\partial S_n}{\partial\mathbf{x}}=\mathbf{p}. \tag{71}$$

This change of variables will now be done for all the expressions obtained hitherto, first transforming the Euler–Lagrange equation for the primary distribution function $f_{\mathrm{p}n}$, (66),

$$\frac{\partial f_{\mathrm{p}n}}{\partial t}+\frac{\partial}{\partial\mathbf{x}}\cdot\left(f_{\mathrm{p}n}\frac{\partial H_n}{\partial\mathbf{p}}\right)=0. \tag{72}$$

This equation is solved, in particular, by the Van Vleck determinant

$$w_n(\mathbf{x},\boldsymbol{\alpha},t)=\left\|\frac{\partial^2 S_n}{\partial x_i\partial\alpha_m}\right\|. \tag{73}$$

The proof is simple and can be found in [11] or [12]. This quantity allows a constant

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of motion $\hat{f}_n$ to be obtained from the density $f_{\mathrm{p}n}$ by setting

$$f_{\mathrm{p}n} = \hat{f}_n w_n. \tag{74}$$

The equation for the new function $\hat{f}_n$ is

$$\frac{\partial \hat{f}_n}{\partial t} + \frac{\partial H_n}{\partial \mathbf{p}} \cdot \frac{\partial \hat{f}_n}{\partial \mathbf{x}} = 0. \tag{75}$$

Since $\boldsymbol{\alpha}$ is a constant of motion, this means that $\hat{f}_n(\mathbf{x}, \boldsymbol{\alpha}, t)$ is also a constant of motion. The transformation (71) allows $\boldsymbol{\alpha}$ to be eliminated by introducing $\mathbf{p}$ instead; this yields a new function $f_n(\mathbf{x}, \mathbf{p}, t)$ replacing $\hat{f}_n(\mathbf{x}, \boldsymbol{\alpha}, t)$ by

$$\hat{f}_n(\mathbf{x}, \boldsymbol{\alpha}, t) = f_n(\mathbf{x}, \mathbf{p}, t), \tag{76}$$

where, of course, $f_n(\mathbf{x}, \mathbf{p}, t)$ is now also a constant of motion. The equation for this quantity is therefore

$$\frac{\partial f_n}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_n}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial f_n}{\partial \mathbf{p}} = 0$$

or

$$\frac{\partial f_n}{\partial t} + \frac{\partial H_n(\mathbf{x}, \mathbf{p}, t)}{\partial \mathbf{p}} \cdot \frac{\partial f_n}{\partial \mathbf{x}} - \frac{\partial H_n(\mathbf{x}, \mathbf{p}, t)}{\partial \mathbf{x}} \cdot \frac{\partial f_n}{\partial \mathbf{p}} = 0, \tag{77}$$

which is the Vlasov equation for $f_n(\mathbf{x}, \mathbf{p}, t)$.

Another effect of the transformation (71) is the transformation of the integrals,

$$\int d^3\alpha \, f_{\mathrm{p}n} \cdots \rightarrow \int d^3p \, \frac{1}{w} f_{\mathrm{p}n} \cdots = \int d^3p \, f_n \cdots. \tag{78}$$

This yields, with (63) and (64), the usual expressions for ρ and $\mathbf{j}$, and (67), because of the property of w solving (72), becomes

$$\delta \mathscr{L}_{\mathrm{V}}^{\mathrm{EL}}(\mathbf{x}, t) = -\sum_n \int d^3p \left[\frac{\partial}{\partial t}(f_n \delta S_n) + \frac{\partial}{\partial \mathbf{x}} \cdot \left(f_n \delta S_n \frac{\partial H_n}{\partial \mathbf{p}} \right) \right]. \tag{79}$$

With $x^0 = ct$, $v^0 = c$ (therefore $v_0 = -c$ because of $g_{00} = -1$) and $\partial H_n / \partial \mathbf{p} = \mathbf{v}$, and μ in the range $0, \ldots, 3$, this can be written as

$$\delta \mathscr{L}_{\mathrm{V}}^{\mathrm{EL}}(\mathbf{x}, t) = -\sum_n \int d^3p \, \frac{\partial}{\partial x^\mu} [f_n \delta S_n v^\mu]. \tag{80}$$

Here, δS_n is given by the expression for the combined variation $\delta_{\mathrm{comb}} S_n$ from (70). In the relativistic case, the following substitutions can be made:

$$\frac{\partial S_n}{\partial t} = -H_n \quad \text{and} \quad \frac{\partial S_n}{\partial \mathbf{x}} - \frac{e_n}{c}\mathbf{A} = \mathbf{p} - \frac{e_n}{c}\mathbf{A} = \frac{\hat{H}_n}{c^2}\frac{\partial H_n}{\partial \mathbf{p}} = \frac{\hat{H}_n}{c^2}\mathbf{v} = \frac{m_n \mathbf{v}}{\sqrt{1 - (\mathbf{v}^2/c^2)}}, \tag{81}$$

while in the non-relativistic case one also has $\partial S_n / \partial t = -H_n$, but

$$\frac{\partial S_n}{\partial \mathbf{x}} - \frac{e_n}{c}\mathbf{A} = \mathbf{p} - \frac{e_n}{c}\mathbf{A} = m_n \mathbf{v}. \tag{82}$$

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This yields

$$(\delta_{\rm comb}S_n)_{\rm rel.} = -\boldsymbol{\epsilon}\cdot\frac{\hat{H}_n}{c^2}\mathbf{v} + \epsilon^0\frac{\hat{H}_n}{c}$$
$$= -\frac{\hat{H}_n}{c^2}[\epsilon^i v_i + \epsilon^0 v_0]$$
$$= -\frac{\hat{H}_n}{c^2}\epsilon^\lambda v_\lambda = -m_n\frac{\epsilon^\lambda v_\lambda}{\sqrt{1-\mathbf{v}^2/c^2}} \tag{83}$$

and

$$(\delta_{\rm comb}S_n)_{\rm non\text{-}rel.} = -\boldsymbol{\epsilon}\cdot m_n\mathbf{v} + \epsilon^0\frac{\hat{H}_n}{c}$$
$$= -m_n\epsilon^i v_i - \epsilon^0 v_0\frac{\hat{H}_n}{c^2}$$
$$= -\epsilon^\lambda\left[m_n\delta^i_\lambda v_i + \delta^0_\lambda v_0\frac{\hat{H}_n}{c^2}\right]$$
$$= -\epsilon^\lambda m_n\left[\delta^i_\lambda v_i - \delta^0_\lambda\frac{\mathbf{v}^2}{2c}\right]. \tag{84}$$

The variation $\delta\mathscr{L}^{\rm EL}_{\rm V}$ in the relativistic case is then given by

$$\left(\delta\mathscr{L}^{\rm EL}_{\rm V}(\mathbf{x},t)\right)_{\rm rel.} = \epsilon^\lambda\frac{\partial}{\partial x^\mu}\left(T_{\rm V}{}^\mu_\lambda\right)_{\rm rel.}, \tag{85}$$

with

$$\left(T_{\rm V}{}^\mu_\lambda\right)_{\rm rel.} = \sum_n\int d^3p\, f_n\frac{\hat{H}_n}{c^2}v^\mu v_\lambda = \sum_n\int d^3p\, f_n m_n\frac{v^\mu v_\lambda}{\sqrt{1-\mathbf{v}^2/c^2}}. \tag{86}$$

In the non-relativistic case, one obtains

$$\left(\delta\mathscr{L}^{\rm EL}_{\rm V}(\mathbf{x},t)\right)_{\rm non\text{-}rel.} = \epsilon^\lambda\frac{\partial}{\partial x^\mu}\left(T_{\rm V}{}^\mu_\lambda\right)_{\rm non\text{-}rel.}, \tag{87}$$

with

$$\left(T_{\rm V}{}^\mu_\lambda\right)_{\rm non\text{-}rel.} = \sum_n\int d^3p\left[m_n f_n v^\mu\left(\delta^i_\lambda v_i - \delta^0_\lambda\frac{\mathbf{v}^2}{2c}\right)\right], \tag{88}$$

with $i = 1,2,3$ and μ in the range 0 to 3. The fully contravariant tensor $T_{\rm V}{}^{\mu\lambda}$ is obtained as

$$T_{\rm V}{}^{\mu\lambda} = g^{\lambda\alpha}T_{\rm V}{}^\mu_\alpha. \tag{89}$$

The tensor $(T_{\rm V}{}^{\mu\lambda})_{\rm rel.}$ for the relativistic case is symmetric in μ and λ for μ and λ equal to $0,1,2,3$.

In the non-relativistic case, one has $(T_{\rm V}{}^{\mu\lambda})_{\rm non\text{-}rel.} = \sum_n\int d^3p\, m_n f_n\tau^{\mu\lambda}$, with $\tau^{\mu\lambda} := v^\mu(g^{\lambda i}v_i - g^{\lambda 0}v^2/2c)$, which yields $\tau^{00} = \mathbf{v}^2/2$, $\tau^{0i} = cv_i$, $\tau^{i0} = v^i\mathbf{v}^2/2c$ and $\tau^{ij} = v^i v_j$. $(T_{\rm V}{}^{\mu\lambda})_{\rm non\text{-}rel.}$ is symmetric in μ and λ for μ and λ equal to $1,2,3$. This is the only symmetry required for non-relativistic theories. The lack of symmetry with respect to the index 0, $(T_{\rm V}{}^{0i})_{\rm non\text{-}rel.} \neq (T_{\rm V}{}^{i0})_{\rm non\text{-}rel.}$, is a consequence of not taking into account the rest energy m_nc^2 in the non-relativistic Hamiltonian, and of having $\partial S_n/\partial\mathbf{x} - (e_n/c)\mathbf{A} = m_n\mathbf{v}$ instead of $m_n\mathbf{v}/\sqrt{1-\mathbf{v}^2/c^2} \approx m_n\mathbf{v}(1+v^2/2c^2)$.

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Because of the Hamilton–Jacobi equation, valid for the actual dynamic development of the system, $\mathscr{L}_{\mathrm{V}}(\mathbf{x},t)=0$ for all $\mathbf{x}$ and t. Hence, using only this Euler–Lagrange equation, one has

$$\delta^{(\mathrm{EL})}_{\mathrm{comb}}\mathscr{L}_{\mathrm{V}}(\mathbf{x},t)=\delta^{(\mathrm{EL})}_{\mathrm{shift}}\mathscr{L}_{\mathrm{V}}(\mathbf{x},t)=0. \tag{90}$$

According to (4) in the Introduction, the Vlasov contribution to the energy–momentum tensor is therefore $T_{\mathrm{V}}{}^{\mu\lambda}$. This expression added to that in (27) then yields the total Maxwell–Vlasov energy–momentum tensor.

A simple modification leading to a completely covariant description of the Vlasov–Maxwell variational method, but leaving the energy–momentum tensor unchanged, can be found in the Summary of [2].

The non-relativistic Maxwell–Vlasov energy–momentum tensor has also been derived by Brizard [13] using a 'constrained' variational principle, but with a non-gauge-invariant Lagrangian which does not lead in a straightforward way to a symmetric energy–momentum tensor. However, the symmetrization can be done in the usual way by adding to the canonical tensor a term of the form $\partial(F^{\mu\alpha}A^{\nu}/4\pi)/\partial x^{\alpha}$, whose four-divergence vanishes, $\partial^2(F^{\mu\alpha}A^{\nu}/4\pi)/\partial x^{\mu}\partial x^{\alpha}=0$, and by making use of Maxwell's equations.

6. Summary

Standard methods do not yield in a straightforward manner the symmetric energy–momentum tensor. Here, a new method for directly obtaining in a very straightforward way the symmetric energy–momentum tensor for an important class of Lagrangian densities is given. These Lagrange densities are characterized by having a gauge-invariant contribution additional to the Maxwell part. For vanishing potentials, such Lagrange densities must lead directly to a symmetric canonical tensor; this implies that there are only scalar fields. The basic feature of the method consists in using only *gauge-invariant* variations by combining shifts in space–time with certain infinitesimal gauge transformations when considering the changes produced in the system by shifts in space–time. Three examples are treated in detail: Maxwell's equations, being the basic part of such theories, are considered first. Then, the coupled system of electromagnetic fields and the usual relativistic scalar matter fields describing charged matter is considered. Finally, a different kind of scalar field which occurs in a structural non-standard form in collisionless phase-space theories (relativistic and non-relativistic Vlasov or collisionless Boltzmann-like equations) is treated.

References

[1] Pfirsch, D. 1984 New variational formulation of Maxwell–Vlasov and guiding center theories. Local charge and energy conservation laws. *Z. Naturforsch.* **39a**, 1.

[2] Pfirsch, D. and Morrison, P. J. 1985 Local conservation laws for the Maxwell–Vlasov and collisionless kinetic guiding-center theories. *Phys. Rev.* A **32**, 1714.

[3] Hilbert, D. 1924 Die Grundlagen der Physik. *Math. Ann.* **92**, 1.

[4] Iwanenko, D. and Sokolow, A. 1953 *Klassische Feldtheorie*. Berlin: Akademie, p. 131.

[5] Madelung, E. 1964 *Die Mathematischen Hilfsmittel des Physikers*. Berlin, Goettingen, Heidelberg: Springer, p. 419.

https://doi.org/10.1017/S0022377803002617 Published online by Cambridge University Press

[6] Landau, L. D. and Lifschitz, E. M. 1961 *The Classical Theory of Fields*. London: Pergamon Press, pp. 84–86.

[7] Morse, P. M. and Feshbach, H. 1953 *Methods of Theoretical Physics,* Part I. New York, Toronto, London: McGraw-Hill, p. 316.

[8] Brizard, A. J. 1998 A new Lagrangian formulation for laser–plasma interactions. *Phys. Plasmas* **5**, 1110.

[9] Brizard, A. J., Murayama, H. and Wurtele, J. S. 2000 Magnetic field generation from self-consistent collective neutrino–plasma interactions. *Phys. Rev.* E **61**, 4410.

[10] Correa-Restrepo, D., Pfirsch, D. and Wimmel, H. K. 1986 Combined Maxwell and kinetic guiding center theory with polarization drift: regularized variational formulation with local charge and energy conservation. *Physica* A **136**, 453.

[11] Pfirsch, D. 1968 De Theoremate conservandi quodam differentiali Theoriae ab Hamilton atque Jacobi doctae. *Arch. Rational Mech. Anal.* **29**, 323.

[12] Van Vleck, J. H. 1928 The correspondence principle in the statistical interpretation of quantum mechanics. *Proc. Natl Acad. Sci.* **14**, 178.

[13] Brizard, A. J. 2000 New variational principle for the Vlasov–Maxwell equations. *Phys. Rev. Lett.* **84**, 5768.

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