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Rost nilpotence for twisted Milnor hypersurfaces*

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Abstract. **We show that the strong Rost nilpotence holds for motives of some hyperplane sections of twisted Milnor hypersurfaces. Hence, we provide a new family of examples of smooth projective algebraic varieties which satisfy the strong Rost nilpotence principle. As an application, we compute the p-canonical dimension for such varieties.**

These notes are essentially based on the results from four papers: [VZ08, XZ24, Ma25, Gi25] to which we also refer for most of the notations and definitions.

Let V be a vector space of dimension $n+1 \geq 3$ over a field k. Consider the partial flag variety

$$E = \{W_1 \subsetneq W_n \subsetneq V\colon \dim W_i = i\}.$$

Let $\varphi \in \mathrm{End}(V)$ be an endomorphism with $n+1$ distinct eigenvalues over the algebraic closure of k such that the subalgebra $L := k[\varphi] \subseteq \mathrm{End}(V)$ is Galois of dimension $n+1$ over k. Consider the hyperplane section of E

$$Y = \{W_1 \subsetneq W_n \subsetneq V\colon \dim W_i = i,\ \varphi(W_1) \subsetneq W_n\}.$$

By the main result of [Ma25] (Thm.1.1 loc.cit.), the Chow motive of Y (with integer coefficients) decomposes as

$$\mathrm{M}(Y) = \bigoplus_{i=0}^{n-2} \mathrm{M}(\mathbb{P}^n_k)(i) \oplus \mathrm{M}(\mathrm{Spec}\, L)(n-1). \tag{0.1}$$

Assume now that φ is of the form $\varphi' \oplus diag(a_{m+1}, \dots, a_n)$, $n > m \geq 2$, where $a_i \in k^\times$, and $L' = k[\varphi']$ is a proper Galois algebra of degree $m+1$ in the sense of [KMRT, Ch.V.18B] that is a finite étale algebra with suitable group action. So L is the product of L' with $n-m$ copies of k. Then the computation of $\mathrm{M}(Y)$ of the respective hyperplane section can be reduced to that of its Galois part $\mathrm{M}(Y')$ using the following analogue of the Rost filtration for quadrics:

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In projective coordinates $([x_0 : \ldots : x_n], [y_0 : \ldots : y_n]) \in \mathbb{P}^n \times \mathbb{P}^n$ the hyperplane section $Y = Y_{m,n}$ is given by the system of equations

$$\begin{cases} \sum_{i=0}^{n} x_i y_i = 0 \quad \text{(this defines } E\text{)} \\ (x_0, \ldots, x_m)\varphi'(y_0, \ldots, y_m)^t + \sum_{i=m+1}^{n} a_i x_i y_i = 0. \end{cases}$$

Define Z to be the closed subset of $Y_{m,n}$ given by $x_n = 0$. Consider the open complement $Y_{m,n} \setminus Z$ given by $x_n \neq 0$. It projects onto $\mathbb{P}^{n-1} = \{[y_0 : \ldots : y_{n-1}]\}$ with the fibre $\mathbb{A}^{n-1}$. Now let $Z' = \mathbb{P}^{n-1} \times pt$ be the closed subset in Z given by $y_n \neq 0$ and $y_i = 0, i < n$. Then the map forgetting the last coordinates x_n and y_n defines the projection $Z \setminus Z' \to Y_{m,n-1}$ with the fibre $\mathbb{A}^1$. The relative cellular decomposition of [EKM, Cor.66.4] then implies the motivic decomposition:

$$\mathrm{M}(Y_{m,n}) = \mathrm{M}(\mathbb{P}^{n-1})(n-1) \oplus \mathrm{M}(Y_{m,n-1})(1) \oplus \mathrm{M}(\mathbb{P}^{n-1}). \tag{0.2}$$

Following [Gi25, Def.2.1.(b)] we say that the strong Rost Nilpotence principle holds for a Chow motive M over k if for all field extensions l/k, the kernel of the restriction morphism

$$res_{l/k} \colon \mathrm{End}_k(M) \longrightarrow \mathrm{End}_l(M_l)$$

is a nilpotent ideal. We say that a motive M is Artin-Tate if

$$M = M_1(i_1) \oplus \cdots \oplus M_r(i_r), \quad i_1 < \ldots < i_r,$$

where the motives M_i are direct summands of motives of 0-dimensional varieties.

***Lemma* 1** *The strong Rost Nilpotence holds for Artin-Tate motives.*

Proof Since $\mathrm{Hom}(M_s(i_s), M_t(i_t)) = 0$ if $s \neq t$, for any $\psi \in \mathrm{End}(M)$, we have $\psi = \psi_1 \oplus \ldots \oplus \psi_r$, where each $\psi_i \in \mathrm{End}(M_i)$. Hence, we may assume M is the motive of some 0-dimensional variety X. But the restriction morphism $res_{l/k}$ is known to be injective in this case (the Chow group of X is free). ∎

Combining (0.1), Lemma 1 and applying [Gi25, Thm.2.4] we obtain:

***Proposition* 2** *The strong Rost Nilpotence holds for the motive* $\mathrm{M}(Y)$.

We now twist the variety Y by a 1-cocycle corresponding to the central simple algebra A of degree $n+1$ so that L is its maximal commutative subalgebra. In other words, the variety E_A is the partial flag of right ideals

$$E_A = \{I_1 \subsetneq I_n \subsetneq A : \mathrm{rdim}\, I_i = i\},$$

and Y_A is given by $\varphi I_1 \subset I_n$ (if A is split, $I_i = \mathrm{Hom}(V, W_i)$). Observe that Y_A is a generalization of the major example of [XZ24].

***Lemma* 3** *Let X be an irreducible smooth projective variety over k. Assume that X satisfies the strong Rost nilpotence generically (that is the $k(X)$-variety $X_{k(X)}$ satisfies the strong Rost nilpotence). Then X satisfies the strong Rost nilpotence.*

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Proof Following the proof of [VZ08, Prop.3.1] the restriction morphism factors as the composite

$$\mathrm{End}_k(\mathrm{M}(X)) \xrightarrow{res_{X/k}} \mathrm{End}_X(\mathrm{M}(X_X)) \xrightarrow{res_{k(X)/X}} \mathrm{End}_{k(X)}(\mathrm{M}(X_{k(X)}))$$

where the middle term denotes endomorphisms in the category of Chow motives over the relative base X, the first map is injective, and the second map is induced by passing to the generic point of X. Observe that the kernel of $res_{k(X)/X}$ is the nilpotent ideal by the proof of [VZ08, Lem.3.2] extended in [Gi25, §3.5].

Now since X satisfies the strong Rost nilpotence generically, the kernel of the restriction $res_{l/k}\colon \mathrm{End}_k(\mathrm{M}(X)) \to \mathrm{End}_l(\mathrm{M}(X_l))$ is contained in the kernel of $res_{F/k}$, where F contains both l and $k(X)$, so is nilpotent. ∎

***Theorem* 4** *The strong Rost nilpotence holds for the motive* $\mathrm{M}(Y_A)$.

Proof From [XZ24, §2] it follows that the motive $\mathrm{M}(Y_A)$ splits as

$$\mathrm{M}(Y_A) = N \oplus \bigoplus_{i=0}^{n-3} \mathrm{M}(\mathrm{SB}(A))(i),$$

where the motive N is supported only in the middle codimension (over the algebraic closure of k, N splits as a direct sum of Tate motives twisted by $n-2$).

Observe that there is a map $Y_A \to \mathrm{SB}(A)$ induced by sending a partial flag $I_1 \subsetneq I_n$ to I_1. Therefore, $\mathrm{SB}(A)$ has a $k(Y_A)$-valued point, and the algebra A splits over $k(Y_A)$. Hence the variety Y_A turns over $k(Y_A)$ into the variety Y, and the strong Rost Nilpotence holds for Y_A generically. To finish we apply Lemma 3. ∎

Since the variety Y_A is geometrically split, and it satisfies the Rost Nilpotence principle, by [Ka13, Cor.2.6] the Krull-Schmidt principle holds for its motive with finite coefficients $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime. The decomposition of the motive of $\mathrm{SB}(A)$ appearing in [Ka96] can then be refined as

$$\mathrm{M}(\mathrm{SB}(A);\mathbb{F}_p) = \bigoplus_{r=0}^{m} \mathrm{M}(\mathrm{SB}(D_p);\mathbb{F}_p)(rd) \quad \text{where } d = p^{\nu_p(\mathrm{ind}(A))},\ m = \tfrac{\deg(A)}{d} - 1,$$

and D_p is the p-primary component of a division algebra Brauer equivalent to A.

***Corollary* 5** *The canonical p-dimension of the variety Y_A is equal to $d-1$.*

Proof Let U be the upper motive of $\mathrm{M}(Y_A;\mathbb{F}_p)$. It is isomorphic to the indecomposable motive $\mathrm{M}(\mathrm{SB}(D_p);\mathbb{F}_p)$. Observe that $\dim(\mathrm{SB}(D_p)) = d-1$. Being an indecomposable lower summand, the dual motive U^* of U is isomorphic to $U(\dim(Y_A)-d-1)$ by the Krull-Schmidt property (see [Ka10, Prop.5.2]). By [Ka10, Thm.5.1], the canonical p-dimension of Y_A is then equal to the dimension of U that is $d-1$. ∎

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