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CONNECTIVITY AND ECONOMIC GROWTH

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We present a simple theoretical model that illustrates the benefits of regional connectivity and specialization for growth. Starting with one community, we show how welfare measured by utility per head increases as the number of connected communities increases. We assume a common connectivity infrastructure through which a central planner is able to add new communities to the pool of already connected communities, a costly but rewarding activity that is funded by levying a tax on those already connected. We find that increasing production costs lead to faster transitions toward the steady state, whereas increasing transportation and communication costs tend to lengthen the transition. The results point to reductions in transportation and communication costs, in particular, as a suitable vehicle for speeding up growth and underline the existence of a positive scale effect that induces integration and reinforces both transitional and steady state growth.

Keywords: Growth, Connectivity, Specialization, Optimum Control

1. INTRODUCTION

Since the days of Adam Smith and David Ricardo, it has been widely known that the specialization of productive activities and the subsequent trading of the fruits of such specialization is able to generate higher benefits from resource use than would be possible in the case of pure self-sufficiency. Smith provides the famous example of the pin factory, where the set of all workers specialized in different subtasks of pin-making taken together is more productive than the same set of workers if each individual worker had to cover all subtasks by himself. With David Ricardo, the benefits from international specialization arise from the so-called comparative advantages of countries in particular parts of the tradable goods spectrum. The sources of welfare increase are indeed concentration of productive activity on "comparative advantaged" goods and subsequent trading/exchanging of the products/services produced among countries. This requires a high degree of connectedness between trading partners that is assumed a priori but does not have to exist in actual fact.

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Take rural areas in African developing countries, for example. Communities in such areas are often relatively disconnected from other communities, because both means of transportation such as cars, trains, and complementary infrastructure (roads, bridges, railroads) and means of telecommunication are often lacking. This implies that such communities are forced to be self-reliant to a large extent, and if such communities are relatively small, which they usually are, then it may be difficult for them to attain the level of welfare that is attainable for larger communities through the specialization of production activities.

In this paper, we want to look into this matter more closely, by formulating a very simple or even simplistic model of growth through specialization that relies on the provision of communication and transportation infrastructure for welfare growth to take off. The growth in welfare is due to communities becoming connected through communication channels and through transportation infrastructure, providing the possibility of trade in goods and services. Hence, in our model, the provision of information and transportation infrastructure is a *conditio sine qua non* for growth to occur.

To find out how large the impact of the provision of such infrastructure is on economic growth, we formulate a model that is largely based on a stripped-down version of Krugman (1979), except that we use it as a template for the description of how a community works when left on its own. Then we add an additional top layer to the Krugman template in which we allow different communities to communicate and trade with each other. Moreover, by allowing the local community to go global in this way, all communities that are connected to each other can concentrate on their comparative strengths and trade. By explicitly introducing costs of trading and costs of being connected, it is not self-evident that being connected pays off. Nonetheless, we try to find out how a benevolent central planner would have to play his role as the Great Connector (further called GC) to optimize the development over time of utility per head for all people touched by the GC. We will show that under particular parameter constraints, the GC will want to connect more and more people, because doing so is in the interest of the people already connected. We look into the steady state rate of expansion of such connections, but also at the corresponding transitional dynamics.

We find that increasing production costs leads to faster transitions toward the steady state whereas increasing transportation and communication costs tends to lengthen the transition. The results point to reduction in transportation and communication costs, in particular, as a suitable vehicle for speeding up growth. The results also show a strong effect of reductions in the cost of making new connections. This has a significant impact both on the steady state growth rate and on transitional growth, while significantly reducing the transition period. We also find a strong impact on transitional and steady state growth of the population size of individual communities. Although such a scale effect is deemed unrealistic in ordinary endogenous growth settings, including the Romer (1990) model, which is also driven by love of variety, like the present model, in a setting with fixed set-up costs as ours, such a scale effect makes ultimate sense, and provides a direct

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incentive for connecting an increasing number of communities thus allowing ongoing product specialization and a concurrent rise in welfare for the communities involved.

The organization of the paper is as follows. In Section 2 we describe the model. Section 3 provides the outcomes of a number of sensitivity analyses. Section 4 provides some policy implications and concluding remarks.

2. THE MODEL

2.1. Preliminaries

In this section we provide the elements that we will be using from the Krugman (1979) North–South model, further called the KNS model. We will leave out the technology features of the KNS model by assuming that the set of varieties of goods/services that could be imitated is so large and that people are so good at imitating that the rate of imitation is unbounded in principle, except for the fixed costs of producing a particular variety. The fixed cost element is an important departure from the original Krugman setting, because it provides the incentive for each of the individual communities that are connected to specialize in a unique subspectrum of goods. We also leave out the North–South asymmetry present in the KNS model, and so end up with a South–South type of model instead, where all countries connected to each other would in principle be able to cover the same spectrum of goods as they would have to if they were self-reliant.[1] To keep the model as simple as possible, we have just one factor of production, labor, and because we want to expand the setup with potentially infinitely many communities, we assume that all communities are identical, except for the fact that they may be connected or not and except for the subspectra of goods that they are producing.

As regards the demand for goods and services, we assume that if there are N different goods that could be produced by a community, then the utility that an individual belonging to that community could gain from spending a budget B on the consumption of these N varieties at levels x_i is given by

$$U = \left\{ \sum_{i=1}^{N} (x_i)^{\rho} \right\}^{1/\rho}. \tag{1}$$

The utility function is a standard CES function with symmetric contribution of each individual variety to utility. $\sigma = 1/(1-\rho)$ is the elasticity of substitution between varieties. The inverse demand function for a particular variety can be obtained by maximizing (1) subject to the budget constraint $B = \sum_{i=1}^{N} p_i \cdot x_i$, where B is the per capita budget, p_i is the price of variety i, and x_i is the corresponding volume of consumption of variety i:

$$\frac{\partial U}{\partial x_i} = \lambda \cdot p_i \Rightarrow p_i = \frac{\partial U}{\partial x_i} \Big/ \lambda \forall i = 1, \ldots, N. \tag{2}$$

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In (2), λ is the Lagrange multiplier associated with the budget constraint. Multiplying (2) by x_i and then summing over i and using Euler combined with the fact that U is linear homogeneous in x_i gives the familiar result that $\lambda = U/B$, i.e., the Lagrange multiplier equals utility per dollar spent on consumption in the optimum.

If prices are identical for all varieties consumed, i.e., $p_i = p \forall i$, then the level of consumption of each variety would be identical as well, i.e., $x_i = x \forall i$, and so the budget B would have to be distributed evenly over all varieties, implying that

$$x = \frac{B}{N \cdot p}. \tag{3}$$

When substituted into (1), (3) implies that

$$U = \frac{B}{p} \cdot N^{\frac{1}{\sigma - 1}}. \tag{4}$$

For $\sigma > 1$, (4) shows that increases in N have a positive impact on utility: the utility function exhibits love of variety. Let L now be the number of persons in the community. Then total utility for the entire community, UC, would be given by

$$UC = L \cdot U = L \cdot \frac{B}{p} \cdot N^{\frac{1}{\sigma - 1}}. \tag{5}$$

It follows that total utility shows the love-of-variety feature for $\sigma > 1$, because $\frac{\partial UC}{\partial N} > 0$ in that case. Henceforth, we assume that $\sigma > 1$.

2.2. Connecting with Another Community

A community connected to other communities has an incentive to produce a different set of varieties than the other communities to which it is connected, if there are fixed costs involved in producing each variety (as we assume). To obtain the varieties of goods produced elsewhere, that community will need to export part[2] of the domestic production of each variety belonging to its own production range, and will have to import the other varieties. Let x now represent the common consumption level of domestically produced (and therefore exported) goods, and let m be the common level of imported goods. Furthermore, let q be the corresponding price of m. In that case, the level of per capita utility would be given by

$$U = (N_x \cdot x^{\rho} + N_m \cdot m^{\rho})^{1/\rho}, \tag{6}$$

where N_x is the number of varieties that are domestically produced and partly exported, and N_m is the number of imported varieties. The corresponding per

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capita budget constraint B for this revised utility maximization problem is then given by

$$B = N_x \cdot p \cdot x + N_m \cdot m \cdot q. \tag{7}$$

Maximizing (6) subject to (7) by choosing the optimum level of each individual domestic variety $x_i \forall i | 1 \leq i \leq N_x$ and of each individual imported variety $m_j \forall j | 1 \leq j \leq N_m$, while taking account of the symmetry between products, results in

$$x = (p \cdot \lambda)^{-\sigma} \cdot U, \tag{8.A}$$

$$m = (q \cdot \lambda)^{-\sigma} \cdot U, \tag{8.B}$$

where λ is again the Lagrange multiplier associated with the budget constraint and x and m represent the common consumption levels of domestically produced goods x_i and imported goods m_j, whereas p and q represent their corresponding (common) prices. As before, $\lambda = \frac{U}{B} \Rightarrow \frac{1}{\lambda} = \frac{B}{U}$, i.e.,$\frac{1}{\lambda}$ can be interpreted as the unit cost of a unit of utility, i.e., a *util*.

Substituting (8.A) and (8.B) into (6), we find that

$$\frac{1}{\lambda} = \left(N_x \cdot p^{1-\sigma} + N_m \cdot q^{1-\sigma}\right)^{1/(1-\sigma)}, \tag{9}$$

which shows that the cost of a util is linear homogeneous in the prices of its components. The relationship $\lambda \cdot B = U$ can be used to substitute for U in (8.A) and (8.B), leaving the levels of consumption of domestic and imported goods as a function of the available budget and the corresponding consumer prices and the Lagrange multiplier only:

$$x = p^{-\sigma} \cdot \lambda^{1-\sigma} \cdot B = \frac{p^{-\sigma}}{N_x \cdot p^{1-\sigma} + N_m \cdot q^{1-\sigma}} \cdot B, \tag{10.A}$$

$$m = q^{-\sigma} \cdot \lambda^{1-\sigma} \cdot B = \frac{q^{-\sigma}}{N_x \cdot p^{1-\sigma} + N_m \cdot q^{1-\sigma}} \cdot B, \tag{10.B}$$

where the right-hand sides of (10.A) and (10.B) result from substituting (9) into (8.A) and (8.B).

2.3. Goods and Services Supply with Multiple Connected Communities

The supply of each individual good can be modeled using the standard assumption that each individual supplier is of measure zero; i.e., his own actions do not noticeably affect the average cost of a util, hence $\frac{1}{\lambda}$, hence λ. So, from the perspective of an individual supplier, both the budget and the Lagrange multiplier in the middle parts of (10.A) and (10.B) are given.

As regards the production technology, we assume that the production of a variety requires the input of labor at a wage rate w to perform three different functions. The production of each variety requires $\bar{l}$ units of labor as fixed real setup costs. Moreover, the real variable cost of producing a variety amounts to

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α units of labor per unit of output. Finally, if a unit is shipped to an external community, that requires β additional units of labor per unit of output to cover per-unit communication and transportation resource requirements. Note that for reasons of simplicity we assume that α and β are independent of the variety and the community.

For a profit-maximizing supplier of domestic goods, the resulting profit function will therefore be given by[3]

$$\pi = p \cdot x + q \cdot z - w \cdot (\alpha \cdot x + (\alpha + \beta) \cdot z + \bar{l}). \quad \textbf{(11)}$$

In (11), π represents the profit flow for the local producer of each variety. This producer sells his produce on the domestic market at price p, but also on the external market at price q.[4] The corresponding volumes sold are x and z. These price/volume combinations have to be consistent with the respective home demand equation, (10.A). For the export volume z, the corresponding demand for imported goods by the external communities would function as the relevant demand constraint, i.e., (10.B) would be relevant, but then with the foreign budget and foreign domestic prices and foreign import prices taking the place of those same variables from a home perspective. Given the symmetries between communities, and given that there are W communities of equal size and with equal preferences and technologies, it follows that x and z for the home community (and hence due to our symmetry assumptions for any other community as well) are given by

$$x = p^{-\sigma} \cdot \lambda^{1-\sigma} \cdot B \cdot L, \quad \textbf{(12.A)}$$

$$z = q^{-\sigma} \cdot \lambda^{1-\sigma} \cdot (W - 1) \cdot B \cdot L. \quad \textbf{(12.B)}$$

The only difference between (12.A) and (10.A) is the size of the budget, because the domestic producer of a variety has to service the entire home population (with total budget BL) as well as the total foreign population [with total budget $(W\text{-}1)BL$, because there are W-1 foreign communities from the point of view of each individual community].

Maximization of (11) after plugging in (12.A) and (12.B) by choosing p and q then results in the following profit-maximizing prices:

$$p = \frac{\sigma}{\sigma - 1} \cdot \alpha \cdot w, \quad \textbf{(13.A)}$$

$$q = \frac{\sigma}{\sigma - 1} \cdot (\alpha + \beta).w, \quad \textbf{(13.B)}$$

where $\alpha \cdot w$ and $(\alpha + \beta).w$ are the marginal costs of servicing the domestic and the foreign markets using domestic varieties. Moreover, because $\sigma > \sigma - 1 > 0$, both p and q exceed their corresponding marginal costs.

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2.4. Creating Additional Connections

If we have W connected communities, then the outside world to which each individual community is connected consists of *W-1* communities. In addition to this, if the local community exports N_x varieties to the outside world, then, because of the symmetry assumptions made before, the outside world must be exporting *(W-1)*N_x varieties to the local community. We can now determine the value of N_x by using the assumption of free entry, which implies that profits per variety drop to zero. By substituting (13) as well as our observation that $N_m = (W-1) \cdot N_x$ into (9), while taking into account that the total foreign budget equals *(W-1)BL*, whereas the Lagrange multipliers in all communities must be the same because of the symmetry assumptions made earlier, we find that

$$\pi = \frac{B \cdot L \cdot (1-\rho) - \bar{l} \cdot N_x \cdot w}{N_x} = 0 \Rightarrow N_x = \frac{B \cdot L}{\sigma \cdot w \cdot \bar{l}}. \tag{14}$$

Note that in (14), $\frac{B \cdot L}{\bar{l} \cdot w}$ is the absolute maximum of the number of varieties that could be produced by an isolated community, because for this value of N_x, total consumer expenditures BL are just enough to cover the total setup cost, leaving no resources to actually produce a strictly positive level of each variety. Note moreover that if the elasticity of substitution between varieties σ were to increase, then the number of varieties supplied to the market would fall, because in that case the margin of price over marginal cost would go down, ceteris paribus, which would make it harder to recover the fixed setup cost per variety.

Using (14), the total number of varieties (V) produced and consumed by W connected communities would be given by

$$V = W \cdot \frac{L}{\bar{l}} \cdot \frac{B}{w} \cdot \frac{1}{\sigma}. \tag{15}$$

Because profits are zero by assumption, all income generated must be wage income. Hence, the budget of each consumer in each community is given by $B = w$, which, when substituted into (15), gives rise to

$$V = W \cdot \frac{L}{\bar{l}} \cdot \frac{1}{\sigma}. \tag{16}$$

In (16), $\frac{WL}{\bar{l}}$ is the absolute maximum of the number of varieties that all communities taken together could support. Because $\sigma > 1$, the actual number of varieties produced by all communities taken together is strictly smaller than the maximum number of varieties.

We can now obtain utility per capita in each of the W connected communities by substituting (13.A), (13.B), and (14) into (6):

$$U = \{L/\bar{l}\}^{1/(\sigma-1)} \cdot (\sigma-1) \cdot \sigma^{\sigma/(1-\sigma)} \cdot \left(\alpha^{1-\sigma} + (W-1) \cdot (\alpha+\beta)^{1-\sigma}\right)^{1/(\sigma-1)}. \tag{17}$$

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Because $\sigma > 1$, it follows directly that $\partial U/\partial W > 0$, $\partial U/\partial L > 0$, $\partial U/\partial \bar{l} < 0$, $\partial U/\partial \alpha < 0$, and $\partial U/\partial \beta < 0$, which implies that already connected communities would have an incentive to extend the number of connections with other communities even further as $\partial U/\partial W > 0$. In addition, per capita utility would rise with the size of each individual community L, because bigger populations allow larger ranges of goods produced by each community, which gives rise to love-of-variety-driven increases in per capita utility. The positive connection between growth and population size is present in the endogenous growth models of Romer (1990) and Aghion and Howitt (1992), and it has been severely criticized by Jones (1995) for lack of empirical backing, as far as the link between growth and population size in combination with a fixed share of R&D workers in that population is concerned. In a setting such as ours, however, growth arises not out of a productivity rise of the resources used in production, but rather out of a more precise servicing of demand niches through ongoing product specialization, which in turn is enabled by an ongoing integration of production and consumption activities. In this way, more varieties involving fixed costs can be produced at levels that fall proportionally less than the increase in the number of varieties, so that a net increase in the satisfaction of consumer demand may actually be the result. In such a setting, the scale effect, which is usually considered to be a weak point of endogenous growth models, makes perfect sense, and, one may argue, is consistent with the observation that growth and international trade are generally positively correlated.

2.5. Optimum Network Expansion Rates

Effects on per capita utility. $\partial U/\partial W > 0$ having been established, the optimum expansion rate of W would be infinitely high unless it were costly to make additional connections. For finite values of the expansion rate, it would be useful to know how the optimum expansion rate would depend on the structural parameters of the model, because those parameters may reflect behavior or some characteristic of technology that may be sensitive to some form of policy intervention. Therefore, we make the assumption that it takes some labor resources to connect thus far disconnected communities, for example, by building ground stations in the newly connected communities as well as transportation infrastructure.[5] As before, we make the simplest assumption possible, i.e., that the resources needed to make new connections are proportional to the number of newly connected communities. Thus we get

$$\dot{W} = \delta \cdot L_W. \tag{18}$$

In (18), L_W are the total labor resources used for expanding the number of connected communities. Let each connected community contribute a fraction v of its available labor force to this activity. Then we must have that $L_W = v \cdot L \cdot W$, and so it follows from (18) that

$$\frac{\dot{W}}{W} = \delta \cdot L \cdot v. \tag{19}$$

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We assume that the workers making new connections earn a competitive wage, and that total wage costs of these workers will be covered by a tax at rate τ on the wage income of all workers involved in goods production. The tax proceeds should be just enough to provide a net wage rate to the connection workers equal to the after-tax wage rate of the workers involved in goods production. Therefore we require that

$$\tau \cdot w_\tau \cdot (1 - v) \cdot L \cdot W = (1 - \tau) .w_\tau \cdot v \cdot L \cdot W \Rightarrow \tau = v, \qquad (20)$$

where the left-hand side of (20) represents the proceeds of taxing the workers in the goods industry, and the middle part of (20) represents the net wage income of the connection workers. In (20) we have added a subscript τ to the gross wage rate to indicate that gross wages might differ from the situation without any taxes. However, as before, wages will drop out of the expression for per capita utility [cf. (17)], but not the tax rate, because the per capita budget decreases by a proportion equal to the tax rate, whereas the prices of exports and imports would still be proportional to the gross wage rate w_τ [cf. (13.A) and (13.B)]. Hence, when we redo the analysis, but with $B_\tau = (1 - \tau)\, w_\tau$ replacing B and w_τ replacing w, we find that the new number of varieties produced by each community becomes a fraction $1 - \tau = 1 - v$ of the old number of varieties:

$$N_x = \frac{L \cdot (1 - \tau)}{\bar{l} \cdot \sigma}. \qquad (21)$$

Hence, introducing a tax (to finance an increase in W) reduces the number of varieties supplied by each community. Utility per capita would fall on that account. However, the number of communities connected (i.e., W) increases and therefore also, potentially at least, the total number of varieties available to all connected communities [cf. (17)]. Substituting (21) and (13) into (6), while taking into account that $N_m = (W - 1) \cdot N_x$, we find that the new expression for utility per capita, U_τ, becomes[6]

$$U_\tau = (1 - \tau).\{L \cdot (1 - \tau)/\bar{l}\}^{1/(\sigma-1)} \cdot (\sigma - 1) \cdot \sigma^{\sigma/(1-\sigma)} \cdot \left(\alpha^{1-\sigma} + (W - 1) \cdot (\alpha + \beta)^{1-\sigma}\right)^{1/(\sigma-1)}. \qquad (22)$$

It follows from (22) that $\partial U/\partial \tau < 0$ for two reasons. First, the budget/resources available for (producing) consumption (goods) fall by a fraction τ. Because $U = \lambda\, B$, it follows that when the per capita budget B drops by a fraction τ then per capita utility also has to fall by the same proportion, ceteris paribus. This effect is represented by the leftmost term $(1 - \tau)$ of the right-hand side of (22). Second, the number of varieties that can be produced falls by the same proportion too, because a fraction $v = \tau$ of the workforce is now diverted to making connections and hence is not available for the production of goods and services. The tax term inside the curly brackets in (22) represents the latter effect.

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The optimum control model. We can now construct an optimum control problem in which a central planner maximizes the present value of total utility in all connected communities, by using (22) scaled up by the total number of people belonging to the connected communities W, whereas the change over time in W is given by (19). The utility function employed is an ordinary CIES intertemporal utility function, where per capita utility at some moment in time is an imperfect substitute for per capita utility at other moments in time. Moreover, per capita utility is discounted. The corresponding present-value Hamiltonian reads

$$H = \frac{e^{-\mu \cdot t} \cdot U^{1-\theta} \cdot L \cdot W}{1-\theta} + \psi \cdot W \cdot \delta \cdot L \cdot v. \tag{23}$$

In (23), μ is the rate of discount, whereas $1/\theta$ is the intertemporal elasticity of substitution and ψ is the co-state associated with the state variable W. The single control variable of the system is v [as we assume that $\tau = v$ at all times; cf. (20)]. The corresponding FOCs of this problem are implicitly given by the requirements $\partial H/\partial v = 0$, $\partial H/\partial W = -\dot{\psi}$, and $\partial H/\partial \psi = \dot{W}$ and the transversality condition that requires $\lim_{t\to\infty} \psi_t \cdot W_t = 0$. The FOCs give rise to a set of nonlinear differential equations, which, under certain conditions, converges to a steady state for $t \to \infty$. The system of differential equations is given by

$$\dot{\psi} = \psi \cdot A \cdot \left(-\frac{W \cdot (1-v)}{W-1+B} - (1 + C \cdot (1-v)) \cdot \sigma \right), \tag{24.A}$$

$$\dot{\psi} = \psi \cdot \left(\frac{\dot{W}}{\sigma \cdot (1+C) \cdot (W-1+B)} + \frac{C \cdot \dot{v}}{(1+C) \cdot (1-v)} - \mu \right), \tag{24.B}$$

$$\dot{W} = W \cdot A \cdot \sigma \cdot v, \tag{24.C}$$

with $A = L \cdot \delta/\sigma$, $B = ((\alpha+\beta)/\alpha)^{\sigma-1}$, and $C = (\theta \cdot \sigma - 1)\big/((\theta-1)\cdot\sigma)$. (24.A) comes from the dynamic constraint $\partial H/\partial W = -\dot{\psi}$, whereas (24.B) can be obtained by taking the time derivative of $\partial H/\partial v = 0$and (24.C) results from$\partial H/\partial \psi = \dot{W}$. It should be noted that for W approaching infinity, the ratio $Z \triangleq \frac{W}{W-1+B}$ will approach 1 because B, as defined earlier, is effectively a constant. Consequently, we can reformulate the system given by (24.A)–(24.C) as an equivalent system in terms of just Z and v, by substituting out the equation of motion for $\dot{\psi}$ and using the definition of Z in terms of W:

$$\dot{Z} = L \cdot \delta \cdot v \cdot Z \cdot (1-Z), \tag{25.A}$$

$$\dot{v} = \frac{(1-v)\cdot((\theta-1)\cdot\mu\cdot(\sigma-1)\cdot\sigma + L\cdot\delta\cdot(1+Z\cdot(\theta-1)-\sigma)\cdot(1-\sigma+v\cdot(\theta\cdot\sigma-1)))}{v\cdot(\theta-1)\cdot\sigma\cdot(\theta\cdot\sigma-1)}. \tag{25.B}$$

Because both Z and v will be constant in the steady state, (25.A) and (25.B) can be used to solve for the steady state directly by setting $\dot{Z} = \dot{v} = 0$, giving

$$ZSS = 1, \tag{26.A}$$

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$$\upsilon SS = \frac{(\sigma - 1) \cdot (L \cdot \delta \cdot (\theta - \sigma) - (\theta - 1) \cdot \mu \cdot \sigma)}{L \cdot \delta \cdot (\theta - \sigma) \cdot (\theta \cdot \sigma - 1)}, \tag{26.B}$$

$$gWSS = \frac{(\sigma - 1) \cdot (L \cdot \delta \cdot (\theta - \sigma) - (\theta - 1) \cdot \mu \cdot \sigma)}{(\theta - \sigma) \cdot (\theta \cdot \sigma - 1)}, \tag{26.C}$$

where (26.C) comes from substituting (26.B) into (24.C) and where the suffix SS refers to the steady state and gX refers to the proportional growth rate of X.

Parameter constraints. If we required the model to react in a nonperverse manner to productivity rises or to changes in the discount rate, we would have to restrict the parameters so that $\frac{\partial gWSS}{\partial \delta} > 0 \Rightarrow \frac{\sigma - 1}{\theta\sigma - 1} > 0$ and $\frac{\partial gWSS}{\partial \mu} < 0 \Rightarrow \frac{(\theta-1)\cdot(\sigma-1)}{(\theta-\sigma)\cdot(\theta\cdot\sigma-1)} > 0$. Because $\sigma > 1$ by assumption, it must be the case that $\theta \cdot \sigma > 1$ too. But then the implicit parameter constraint $\frac{\partial gWSS}{\partial \mu} < 0$ implies that $\frac{\theta-1}{\theta-\sigma} > 0 \Rightarrow \theta < 1 < \sigma \vee \theta > \sigma > 1$. So these are two of the necessary conditions for the model to be able to generate sensible results. A further condition comes from the transversality condition, which requires that $g\psi SS + gWSS < 0$, which can be shown to imply that $\mu > \frac{L\cdot\delta\cdot(\theta-\sigma)}{\sigma-1}$. Hence the rate of discount must be high enough to make $\psi \cdot W$ vanish in the limit.

The steady state can be described as the point in the Z,v-plane where all movement in Z and v would vanish. This point is defined by the intersection of the relations between Z and v as given by (25.A) and (25.B) that can be obtained by setting $\dot{Z} = \dot{v} = 0$. Thus we find for the Z-locus and the v-locus that

$$Z = 1, \tag{27.A}$$

$$Z = (\sigma - 1) \cdot \left(\frac{1}{\theta - 1} - \frac{\mu \cdot \sigma}{L \cdot \delta \cdot (1 - \sigma + v \cdot (\theta \cdot \sigma - 1))} \right), \tag{27.B}$$

$$\frac{dZ}{dv} = \frac{\mu \cdot (\sigma - 1) \cdot \sigma \cdot (\theta \cdot \sigma - 1)}{L \cdot \delta \cdot (\sigma - 1 + v \cdot (1 - \theta \cdot \sigma))^2}, \tag{27.C}$$

where (27.C) provides the slope of the v-locus (cf. 27.B), which must be positive under the present parameter constraints. (27.B) shows that the graph of Z against v has a vertical asymptote at $\bar{v} = \frac{\sigma-1}{\theta\cdot\sigma-1}$. In addition to this, if we set $v = 0$ in (27.B), we find that for $\sigma > 1$ and $\theta > 1$, the corresponding value of Z must be positive, whereas Z could be negative for the case where $\sigma > 1 > \theta$ if θ were small enough. To rule out such an anomaly, we will drop the possibility that $\sigma > 1 > \theta$ and further investigate the case $\theta > \sigma > 1$.

A further requirement is that one would ideally want a value of Z on the v-locus for $v = 0$ that was non-negative and a value of $Z \geq 1$ for $v = 1$, so that a steady state existed that required a tax rate that was positive but less than 100%. These additional logical constraints follow directly from the phase diagram of Z and v presented in Figure 1.

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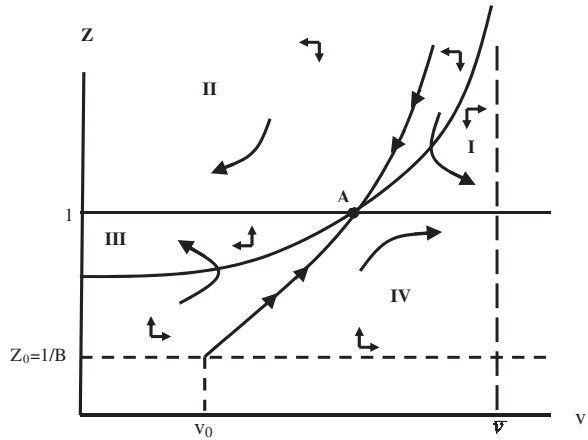


FIGURE 1. Phase diagram.

The transition path toward the steady state. The phase diagram in Figure 1 shows a striped vertical, which is associated with the vertical asymptote of the v-locus at $v = \bar{v}$. The constraints $\theta > \sigma > 1$ now imply that $0 < \bar{v} < 1$, which in turn implies that $vSS < \bar{v} < 1$. The horizontal dotted line represents the value of Z for $W = 1$, i.e., the initial situation where just one country starts making connections with other countries. The Z-locus is represented by the horizontal line at $Z = 1$ and the v-locus by the curve through point A on the Z-locus that gets closest to the vertical asymptote at $\bar{v}$. Both loci divide the plane into four regions, those above and below the Z-locus, and those to the left and to the right of the v-locus. Now consider a point above the Z-locus. For such a point, movement in the Z-dimension will be negative [cf. (25.A)]. For points below the Z-locus, movements in the Z-direction will be positive. Now consider a point on the v-locus for a given value of Z, while noting that (25.B) in combination with our adopted parameter restrictions implies that

$$\frac{\partial \dot{v}}{\partial Z} = \frac{L\,(1-v)\,\delta\,(v\,(\theta\sigma-1)-(\sigma-1))}{v\sigma\,(\theta\sigma-1)}$$

$$= \frac{L\,(1-v)\,\delta(v-\overline{v})\,(\theta\sigma-1)}{v\sigma\,(\theta\sigma-1)} < 0 \quad \text{for} \quad v < \bar{v} \quad \text{and} \quad 0 < v < 1. \quad \textbf{(28)}$$

Moving upward (increase Z) from some point on the v-locus would bring us into the part of the plane above *and* to the left of the v-locus, and the movement in v would become negative. For points to the right of the v-locus the movement in v would become positive. It follows from the figure that the model is saddlepath stable, which allows us to obtain the optimum trajectory for the tax rate τ (and consequently for v, because $\tau = v$) and for Z (hence W), because the saddlepath

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feature of the problem implies that there is a unique value of v_0 that, in combination with (25.A) and (25.B), will drive v and Z toward their steady state values.

2.6. Obtaining the Transition Path

Because of the nonlinearity of (26.A) and (26.B), the solutions for Z and v unfortunately cannot be obtained by analytical means. Therefore, we will use Mathematica to show how the transition path changes with the parameters of the system. To do this, we use the method of backward integration [Brunner and Strulik (2002)]. The method is suitable because we know where the transition path ends (i.e., in the steady state), whereas we also "know" where the state variable Z starts (i.e., at $Z_0 = 1/B$). Hence by integrating (26.A) and (26.B) backward in time, and waiting till Z hits the Z_0-mark in Figure 1, we also obtain the corresponding initial value of v, i.e., v_0 in Figure 1. The only problem is that if we started integrating backward while *exactly* in the steady state, we would not be able to get away from there, because the speed of adjustment in the steady state is exactly equal to zero. Consequently, we need to move slightly outside the steady state along the transition path and in the direction of v_0, and then start the integration process.[7] However, by picking a point closer to the steady state, it will take longer to move away from the steady state, because the movement in Z and in v will be less in absolute terms, and the time spent during the transition from that point toward the initial coordinates $\{Z_0, v_0\}$ will become correspondingly longer. There is an element of arbitrariness, therefore, in the calculation of the length of the transition period, because the choice of an initial point close to, but not quite in, the steady state, which is necessary for backward integration to work, is itself somewhat arbitrary. Nonetheless, the way in which changes in the structural parameters would lengthen or shorten the transition period is not arbitrary.

In the next section, we will present the results of a sensitivity analysis performed using the backward integration method.

3. SENSITIVITY ANALYSIS

The results obtained using the backward integration method for the transitional dynamics are associated with (variations on) the base-run parameter set listed in Table 1. The parameter values listed here are all consistent with the parameter restrictions outlined in the previous section. They generate moderate but positive growth rates for the number of connected communities and transition paths that take several centuries before reaching the steady state.

The parameters can be divided into three different groups. The first group consists of the production cost parameters α and β. As is clear from (26.B) and (26.C), these cost parameters do not influence the steady state itself, but affect the transitional dynamics only. The other structural parameters, $\sigma, L, \delta, \mu, \theta$, do have an impact on both the steady state and the corresponding transitional dynamics, whereas the fixed-cost parameter $\bar{l}$ does not influence either the steady state or the

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TABLE 1. The base run parameter set

Parameter	Value	Parameter	Value
α	0.5	L	2
β	0.1	θ	2
σ	1.25	δ	0.1
$\bar{l}$	1	μ	0.05

transitional dynamics. It does have an impact on welfare, though, because higher fixed labor costs imply lower numbers of varieties and hence lower welfare, ceteris paribus. Because the results for variations in $\bar{l}$ are relatively uninteresting, we will not be showing them here.

Using the preceding parameter vector, we have performed a sensitivity analysis for all the elements in the vector separately. Each element has been varied over a range $\pm 10\%$ of the central value, except for the parameters α and β, for which the range has been set to $\pm 50\%$. These relative shocks are all still compatible with the parameter constraints. For the parameters α and β, the shocks are relatively large, because otherwise the effects on the transition path would hardly be visible. The corresponding results are depicted in the figures that follow. In these figures, we first see the variables Z,W,v graphed against time, and then the implied graph of Z against v, as in the phase diagram in Figure 1. A further plot shows the development over time of the present value of per capita utility (called PVPCU) along the transition path until the moment the steady state is reached. The last plot shows the growth rate of W (labeled GW) along the transition path. The various striped curves in the plots are drawn with different stripe-lengths. The length of the stripes is increasing as the parameter that is varied moves from its low bound to its high bound. In addition to this, the curves become lighter as the parameter increases.

In Figure 2, the results for a change in the elasticity of substitution between varieties in the utility function have been depicted. Note that σ also equals the (absolute value of the) price elasticity of demand. A higher value of σ would therefore lower profit margins, and hence would enable a community to sustain a smaller number of varieties [see (14)], ceteris paribus. We see that a relatively low price elasticity of demand raises the transition path for Z. Higher values of σ lower the transition path relative to the central value path. However, it should be noted that the paths do actually intersect. This is most clearly seen in the plot for v. In addition, it can be seen from the position of the endpoints of the transition paths in the plot for W that the length of the transition path for a low value of σ is greater than that of the central value for σ. The same holds for the endpoint for the high value of the σ-path, suggesting the relevance of intertemporal trade-offs in a setting like this. As apparent from the plot of GW, the growth rate of W is very close to its steady state value from the beginning in the low-σ case, whereas it is relatively low to start with in the high-σ case but ends higher than in the low-σ

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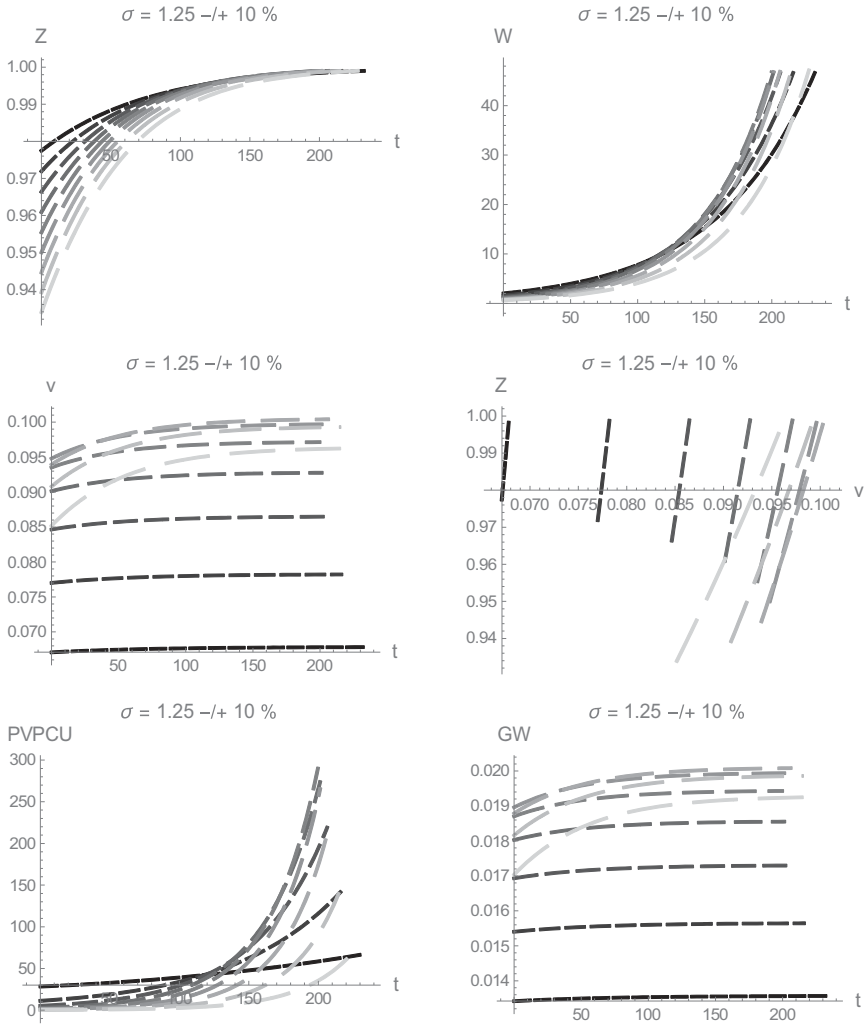


FIGURE 2. Sensitivity results with respect to changes in σ.

case. From the plot holding the outcomes for PVPCU, it follows immediately that a lower value of σ is relatively good news for the consumers that are all connected, at least for a while. There is bad news as well, because the (steady state) growth rate GW is the lowest of all. It follows from a comparison of the plots for PVPCU and GW that having a high GW does not have to be a good thing per se.

Figure 3 shows the results for variations in θ, where $1/\theta$ is the intertemporal elasticity of substitution between (the utilities derived from) consumption at different moments in time. We find that low values of θ are associated with the highest growth rate, because the implied value of the intertemporal elasticity of

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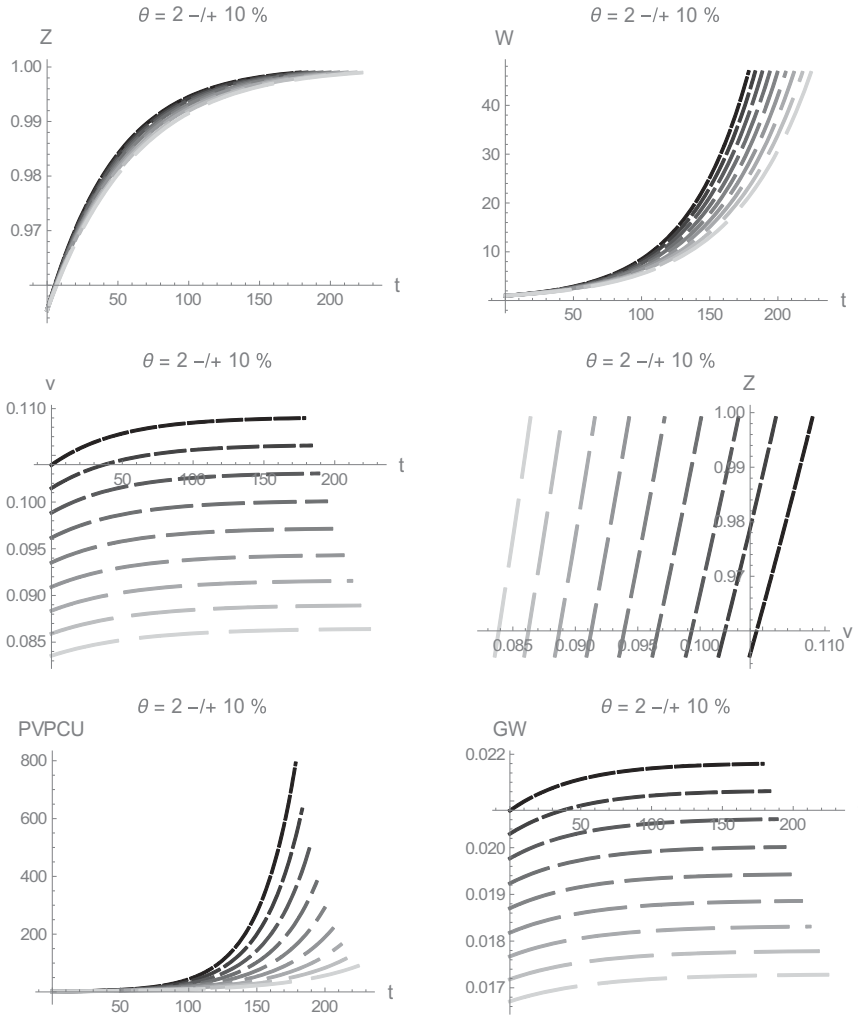


FIGURE 3. Sensitivity results with respect to changes in θ.

substitution is relatively high in that case, and so therefore is the willingness to give up resources now in exchange for higher returns in the future. We see therefore that the transition path for v for the low-θ case is indeed above the path for the high-θ case at the end of the transition period and in the steady state itself. Also, it should be noted that a lower value of the intertemporal elasticity of substitution (hence a higher value of θ) would tend to cause a more uniform distribution of consumption over time, and hence a lower dispersion in transitional growth rates

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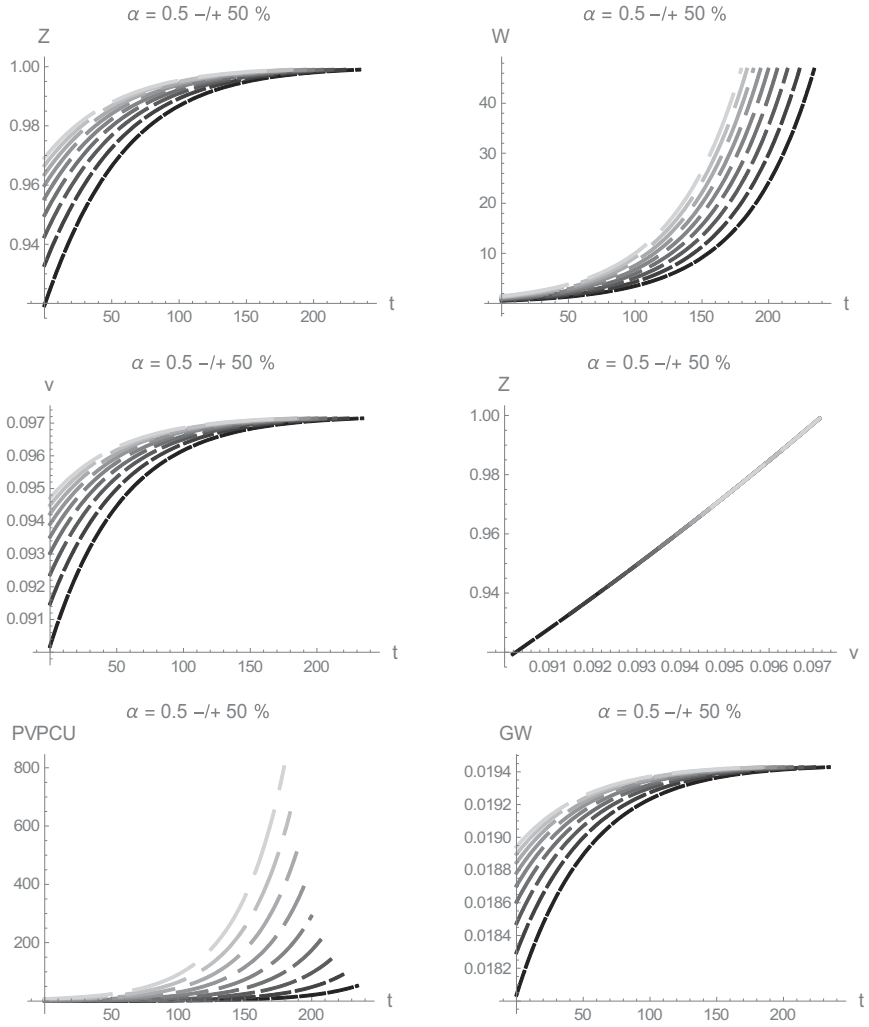


FIGURE 4. Sensitivity results with respect to changes in α.

and lower steady state growth. This is exactly what can be observed from the plot of GW: the transitional growth dispersion falls as θ increases.

Figure 4 shows the results for variations in α, the marginal production cost of each variety. As stated before, variations in α only affect the transitional dynamics, and hence the levels of W, but not the steady state growth rate GW, as can readily be observed from the last plot, where all GW curves converge to the same long-term value of GW. Raising α from its low value to its high value leads to a shortening of the transition period on one hand (see the plot of PVPCU, where the curve

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for the high bound of α hits the steady state well before the curve associated with the low bound of α) and to a rise in the transitional growth rate (the GW growth curve for the low bound is completely below the curve for the high bound). Interestingly, higher values of α raise the contribution of variety to utility relative to the contribution of quantity to utility. Hence for high values of α, the incentive to increase the number of available varieties by raising connectivity increases as well. Consequently, we find higher transitional growth rates as α increases. Higher values of α also shift down the time path for the present value of utility per capita (PVUPC), as one would expect. This is because higher marginal production cost would, for a given level of resources, reduce (ex ante) profits, and hence the number of varieties that can be sustained by a community. At the same time, for a given number of varieties, the volume of each variety must go down as well, reducing per capita utility on both accounts [cf. equation (17)]. Note that, even though the time paths for Z and v are clearly influenced, they are affected to exactly the same extent, so that Z plotted against v for all values of t remains exactly where it was. This is easy to understand, because a change in α (or β) would only affect the value of Z_0, i.e., the position of the horizontal $Z = Z_0 = 1/B$ in Figure 1. Hence, when integrating backward, we would still follow the same trajectory along the stable arm from the steady state and down to the old value for Z_0, and then we would have to extend the stable arm from that point up to the new value of Z_0 (assuming the latter is below the former).

In Figure 5, we see that the results for variations in β, i.e., unit transportation cost, are qualitatively similar, in the sense that only transitional dynamics are affected and not the steady state. However, now we find that an increasing value of β will lead to lower transitional growth and longer transition periods, whereas a rising value in α would tend to have the opposite effect. Still, the effect on utility per head goes in the same direction as for variations in α. The reason for the different impacts of variations in α and β on utility per capita can be found in (17). As β is associated with transportation costs, its impact on utility per capita becomes larger the larger the number of connected communities is. Hence if β rises, the direct impact on per capita utility will be negative, but that impact can be mitigated to some extent by reducing the rate at which W grows.

Figure 6 shows the results for variations in δ. These are relatively spectacular. Note that the (labor) costs of extending the number of connected communities depend inversely on δ. Hence low values of δ imply high costs of extending the number of connected communities, and we consequently see that the duration of the transition period falls as δ increases. We also see that the growth rate of W, i.e., GW, is positively affected, and quite significantly so, if δ increases. As a rise in δ implies a lower cost for creating new connections, the resources allocated to making these new connections actually go up. This follows from the fact that the steady state values of v increase as δ rises. But even though v increases, a rising value of δ has a positive net effect on the present value of per capita utility.

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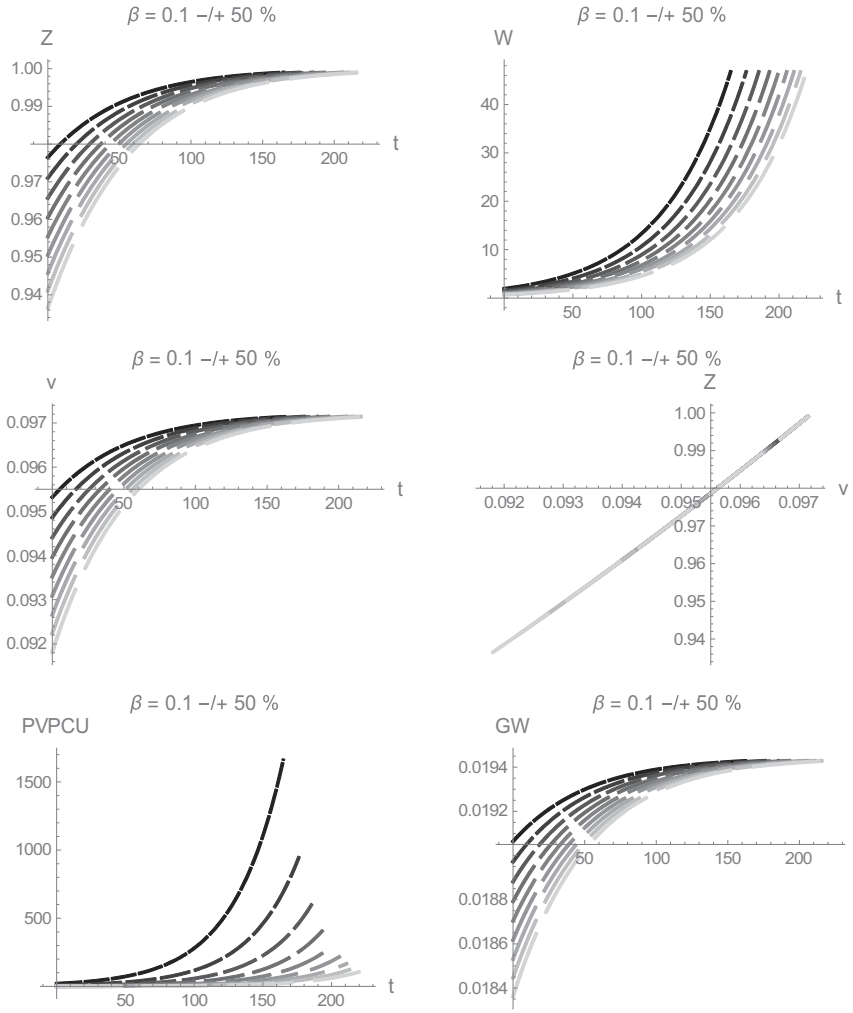


FIGURE 5. Sensitivity results with respect to changes in β.

Figure 7 shows how the model reacts to variations in the rate of discount μ. Basically, the results are opposite to those of variations in δ, as a rise in μ would disfavor the execution of activities with a return that would largely be in the future (such as extending the number of connected communities). Consequently we find lower growth in W and longer transitions as μ rises. As a consequence, the time path for utility per capita also shifts downward as μ increases.

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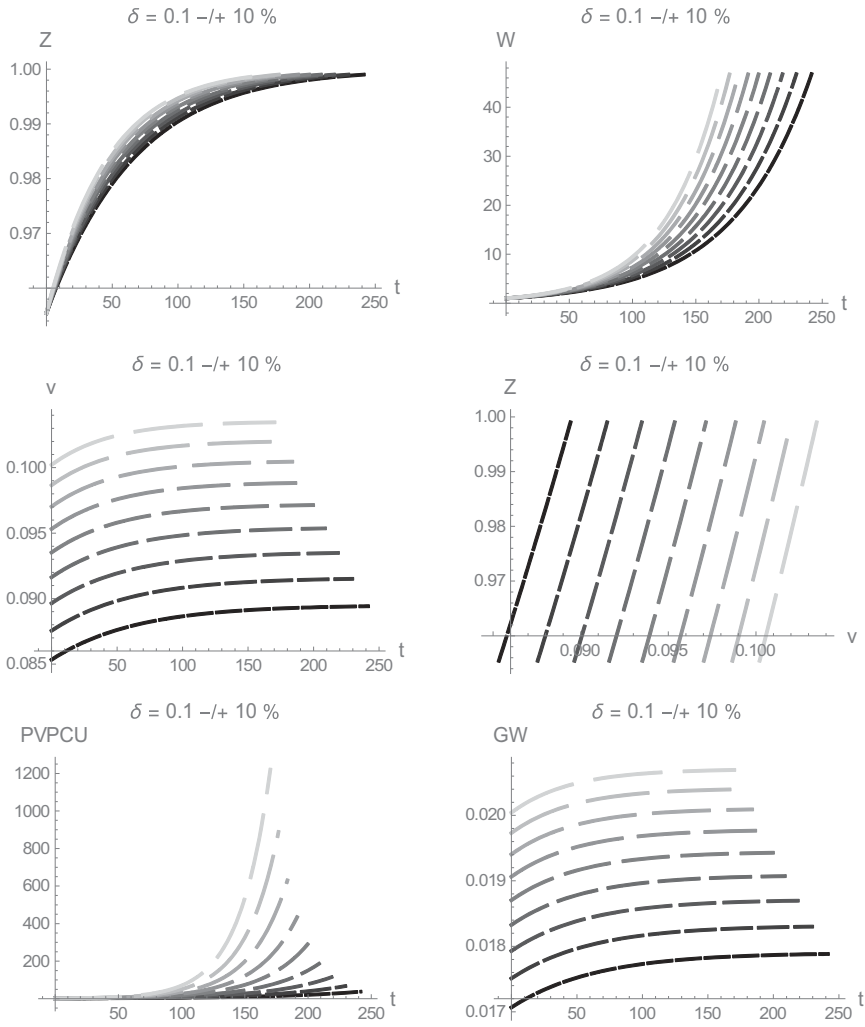


FIGURE 6. Sensitivity results with respect to changes in δ.

Figure 8 shows what happens with variations in the size of the communities. Increases in size would allow more varieties to be produced before profits were squeezed to zero because of free entry. This means that the returns to connecting additional communities would go up. Consequently, we observe a rise in both transitional and steady state growth as L increases, whereas the length of the transition period decreases. Because of the increase in the returns to making new connections, we see that the proportion of the resources allocated to doing that also increases for rising values of L.

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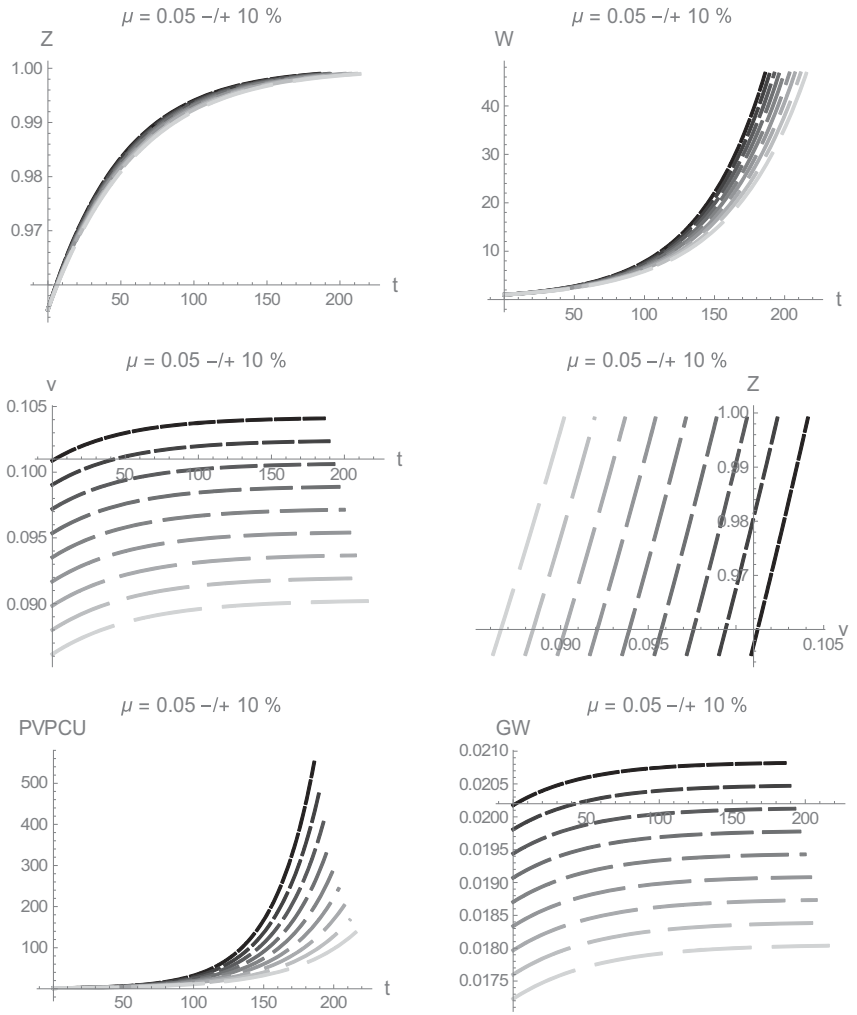


FIGURE 7. Sensitivity results with respect to changes in μ.

4. CONCLUDING REMARKS

In the previous sections we have shown that reductions in production costs, but also in transport and communication costs, have an immediate effect on the rate at which communities would want to become connected and trade with each other. We have shown that increasing production costs lead to faster transitions, whereas increasing transportation costs lengthen the transition toward the steady state. Neither type of costs affect the steady state as such, but because the transition period is quite long, their different impacts on transitional growth point at

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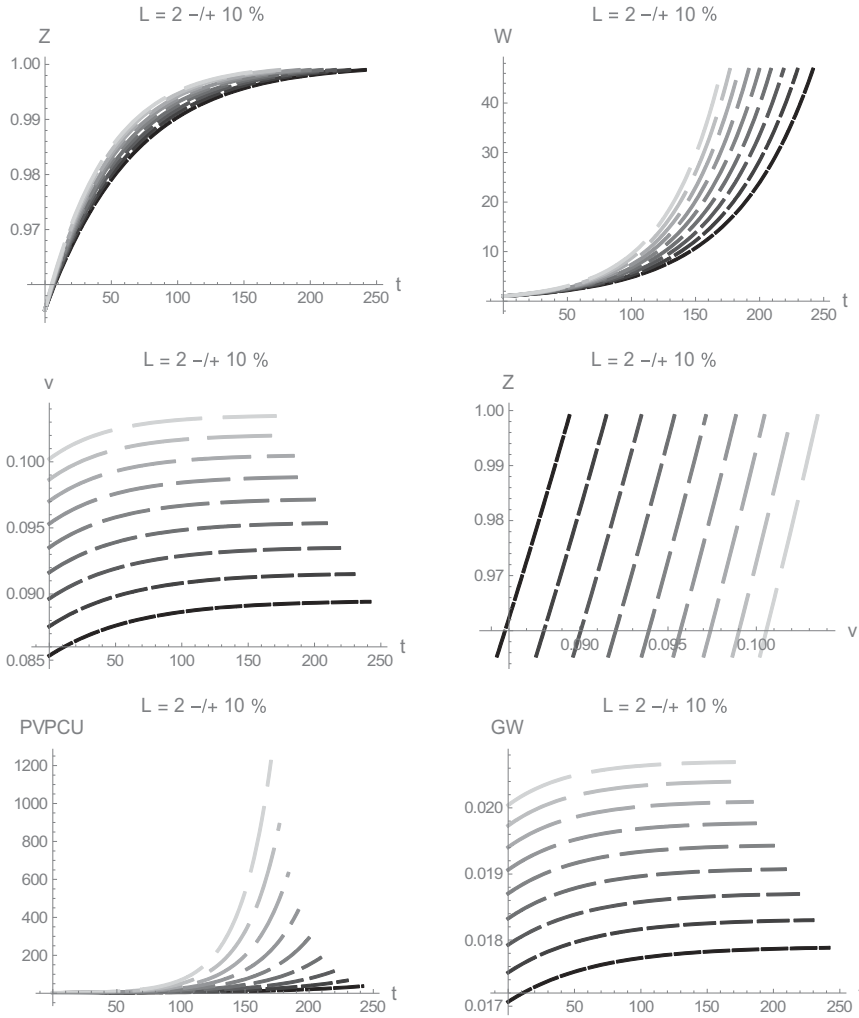


FIGURE 8. Sensitivity results with respect to changes in L.

transportation and communication cost reductions in particular as a suitable vehicle for speeding up growth. Again, this is reminiscent of the semi-endogenous growth reaction to the problems presented by the existence of a scale effect for the seminal endogenous growth models of Romer (1990) and Aghion and Howitt (1992); in the Jones (1995) model, the scale effect was removed, and so was the possibility of steady state growth, to be replaced by transitional growth over long stretches of time.

Very significant effects in growth terms can be observed for the case of reductions in the cost of making new connections. That has a relatively significant

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impact both on the steady state growth rate *and* on transitional growth, while reducing the transitional period equally significantly. This seems to be stressing the importance of the public provision of cheap forms of communication and transportation between thus far disconnected communities, rather than leaving this to private initiative, with the danger of the underprovision of communication and transportation facilities under oligopolistic or even monopolistic conditions. Similarly, in our sensitivity analysis, the cost of using a communication and transportation network also proves to be a strong determinant of growth, albeit (as in a semi-endogenous growth setting) transitional growth over long transition periods. Again, this stresses the importance of the public provision of such facilities to avoid the overpricing of the use of such facilities by private parties and consequently a possibly stunted growth performance.

The removal of the scale effect in the semi-endogenous growth literature underlined the importance of population growth as the ultimate source of steady state growth. By analogy, the vanishing transitional growth effects of the reduction in communication and transportation costs can be counteracted by continuous reductions in the cost of communication and transportation as a result of technical change in ICT and of energy-saving innovations in new transportation systems. The fact that Bill Gates and Steve Jobs could become multimultibillionaires in just a few decades is instructive both of the value of their products to society as a whole and of the nonrealization of a huge consumer welfare potential in the process. Again, this suggests a role for the public provision of at least support for the basic innovations necessary to provide a public knowledge base that would be conducive to a sufficient amount of competition serving to keep the cost of using and expanding communication and transportation networks down.

Changes in the rate of discount are just as important for growth as those in the cost of making new connections, mutatis mutandis. Communities with lower rates of discount would have a higher incentive to connect/become integrated with the rest of the world. This also goes the other way around; if the rate of discount in some community is low, the rest of the world has a relatively strong incentive to become connected to that community, because it will be more willing to share in the common burden of maintaining and extending current and future connections. Hence, good governance and international reliability may induce a reduction in discount rates and so stimulate growth through ongoing integration. Other types of (endogenous) growth models that generate a negative relationship between discount rates and growth would support a similar conclusion, though.

We also showed that the population size of the communities (to be) connected strongly determines both the steady state growth rate and the transitional growth rates of all connected communities. The larger the communities are, the stronger growth will be, which points toward a positive scale effect that arises out of the nature of the communication and transportation network itself, rather than out of a knife-edge assumption about the functional form of the production function underlying the process of connecting communities. To some extent, then, this scale effect may be considered to be more real than the one present in Romer (1990)

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or Aghion and Howitt (1992). However, even in the presence of a positive scale effect, communities that are lagging behind in educational terms would probably not be able to produce as many varieties as other communities of similar size but with a higher average level of education would be able to. This would limit the benefits for other communities from being connected with low-education communities. It follows that to make such connections worthwhile for every community concerned, differences in educational levels should not be too large. This again points to education as an important determinant of the growth performance of an economy (and to the importance of the provision of cheap high-quality education for everyone), through its impact not just on the quality of labor per se [as in the Lucas (1988) model], but on the size of the subspectrum of varieties that could be produced depending on average levels of educational attainment.

For now, we have to leave an extension of our model in which we formally integrate investment in the level of education of a community, as an additional determinant of a community's growth potential through its impact on that communities' attractiveness to other communities as a potential trading partner, for future research.

NOTES

1. Specialization and trading are not necessarily the prerogative of countries, as in the KNS model. Rather, we adopt the view that organized communities of people are doing the specialization and trading, and that there could well be more than one community inside a country.

2. The other part that is not exported will be consumed at home.

3. We drop the subscripts indicating a particular variety, because we assume that the production technologies are symmetric as well.

4. Because of the assumption of symmetry between countries/communities, the export price of the home country coincides with its import price. In both cases, there are transportation and communication costs to be covered.

5. To keep things as simple as possible, we assume that there are only fixed setup costs in doing this, so that the infrastructure is effectively infinitely long-lived.

6. Note that (22) and (17) are the same for $\tau = 0$.

7. In the Appendix, we provide more detail about the backward integration approach.

8. Cf. Barro and Sala-i-Martin (1995, Appendix 1).

9. One of the eigenvalues must be negative for saddlepath stability; cf. Barro and Sala-i-Martin (1995, Appendix 1).

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APPENDIX: THE BACKWARD INTEGRATION METHOD

In order to find an initial value for the backward integration process close to the steady state, we may draw a tiny circle around the steady state with radius ε, and pick the point of intersection of that circle with the transition path as the initial values for v and Z in the backward integration process. This in turn requires that we linearize (25.A) and (25.B) around the steady state and obtain the eigenvalues of the linearized system, which can be written as[8]

$$\begin{pmatrix} \dot{Z} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{(-1+\sigma)((-1+\theta)\mu\sigma+L\delta(-\theta+\sigma))}{(\theta-\sigma)(-1+\theta\sigma)} & 0 \\ -\frac{(-1+\theta)^2\mu(L\delta(\theta-\sigma)+\mu(-1+\sigma))(-1+\sigma)\sigma}{L\delta(\theta-\sigma)^2(-1+\theta\sigma)^2} & \frac{L\delta(\theta-\sigma)+\mu(-1+\sigma)}{-1+\theta\sigma} \end{pmatrix} \cdot \begin{pmatrix} Z-\mathrm{ZSS} \\ v-v\mathrm{SS} \end{pmatrix}, \quad \textbf{(A.1)}$$

which has eigenvalues $\frac{L\delta(\theta-\sigma)+\mu(-1+\sigma)}{-1+\theta\sigma}$ and $-\frac{(-1+\sigma)((-1+\theta)\mu\sigma+L\delta(-\theta+\sigma))}{(\theta-\sigma)(-1+\theta\sigma)}$, of which the first is positive and the second negative,[9] provided that $\mu > \frac{L\delta(\theta-\sigma)}{\sigma(\theta-1)}$. The transversality condition implies that $\mu > \frac{L\cdot\delta\cdot(\theta-\sigma)}{\sigma-1}$, which has the same numerator as the negative eigenvalue condition just given. Because we made the assumption that $\theta > \sigma > 1$, the parameter constraint coming from the negative eigenvalue requirement is weaker than the constraint resulting from the transversality condition. We conclude that the parameter restrictions outlined in Section 2.5 imply the saddlepath stability of the optimization problem and that the negative eigenvalue has a corresponding eigenvector u given by

$$u = \begin{pmatrix} \frac{L\delta\left(L\delta(-1+\theta)(\theta-\sigma)-\theta\mu(-1+\sigma)^2\right)(\theta-\sigma)(-1+\theta\sigma)}{(-1+\theta)^2\mu(L\delta(\theta-\sigma)+\mu(-1+\sigma))(-1+\sigma)\sigma} \\ 1 \end{pmatrix}. \quad \textbf{(A.2)}$$

The slope of the stable arm in the steady state is given by the top element of u, the numerical value of which we will further call s, for reasons of simplicity. Now consider again the tiny circle with radius ε and center coordinates $\{ZSS, vSS\}$ in the Z, v-plane. Also, consider a straight line through that center with slope s. The points of intersection of this line with the circle can be found by solving the simultaneous system $\{(Z-\mathrm{ZSS})^2+(v-v\mathrm{SS})^2=\varepsilon^2, (Z-\mathrm{ZSS})=s\cdot(v-v\mathrm{SS})\}$. The relevant solution for our case with an upward-sloping stable arm lies to the southwest of the steady state (see Figure 1) and is given by

$$Z' = \mathrm{ZSS} - \varepsilon\cdot s/\sqrt{1+s^2}, \quad \textbf{(A.3)}$$

$$v' = v\mathrm{SS} - \varepsilon/\sqrt{1+s^2}. \quad \textbf{(A.4)}$$

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