Probability in the Engineering and Informational Sciences, **26**, 2012, 1–15.
doi:10.1017/S0269964811000209

RENEWAL SEQUENCES WITH PERIODIC DYNAMICS

BRIAN FRALIX, JAMES LIVSEY, AND ROBERT LUND
Department of Mathematical Sciences
Clemson University
Clemson, SC 29634-0975
E-mail: bfralix@clemson.edu; jlivsey@clemson.edu; lund@clemson.edu

Discrete-time renewal sequences play a fundamental role in the theory of stochastic processes. This article considers periodic versions of such processes; specifically, the length of an interrenewal is allowed to probabilistically depend on the season at which it began. Using only elementary renewal and Markov chain techniques, computational and limiting aspects of periodic renewal sequences are investigated. We use these results to construct a time series model for a periodically stationary sequence of integer counts.

1. INTRODUCTION

Discrete renewal processes are ubiquitous in stochastic phenomenon and are extensively used in stochastic analyses (see Smith [18], Feller [5], Karlin and Taylor [11], Resnick [16], and Ross [17] for renewal background, history, and applications). This article develops a periodic version of classical discrete-time renewal sequences. The period of the process, denoted by T, is assumed known and fixed throughout. The fundamental difference from an ordinary renewal process is that when a renewal occurs at time t, the time until the next renewal is allowed to depend on the season corresponding to time t. In this manner, one can build processes in which renewals are more likely (or unlikely) during some seasons.

By superimposing or mixing versions of periodic renewal processes, one can construct models for periodic sequences of counts. This is useful in climatology, for example, in which many phenomena have a definitive season of occurrence. Thunderstorms in the southern United States can take place at any time in the year but are most likely during the summer. Hurricanes, tornadoes, and snowstorms are other meteorological count processes with periodic features. Rare disease occurrences, accidental deaths, and animal sightings are nonmeteorological examples of count series with

© Cambridge University Press 2011 0269-9648/12 $25.00

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

periodic structure. Section 6 shows how periodic renewal processes can be used to describe periodically stationary count sequences. Such processes have periodic means and autocovariances (the counts are not independent).

Processes with periodic dynamics have been previously studied. General time series with periodic dynamics are overviewed in Hurd and Miamee [9] and Gardner, Napolitano, and Paura [6] from a second-order point of view. Lund and Basawa [14] studied periodic time series via periodic autoregressive moving-average models. Markov chains with periodic dynamics have been considered in queuing contexts (Harrison and Lemoine [7]; Lemoine [12]; Heyman and Whitt [8]; Asmussen and Thorisson [1]), branching processes (Jagers and Nerman [10]), storage models (Phatarfod [15]; Lund [13]), and general regenerative processes (Çinlar [3], Thorisson [19]). Although the analysis below is in discrete time and is somewhat pedestrian compared to, say, [19], its utility lies with having results carefully stated with explicit formulas supporting the theorems. Our arguments are elementary, developed fully, and our end pursuit—the generation of a time series of counts with periodic features—departs from the stochastic analysis slant of some of the above works.

The rest of this article proceeds as follows. Section 2 rigorously introduces the model and clarifies notation. Section 3 establishes recursions that enable rapid computation of the renewal probabilities. Section 4 studies limiting properties of both the renewal probabilities and the associated age process and Section 5 identifies an initial delay (lifetime) that induces a periodically stationary process from time 0; that is, the renewal probabilities are seasonally constant, depending only on the season of the time index. Here, connections to Markov chains with periodic transition probabilities are made. Section 6 uses this structure to develop an autocorrelated periodic count model that has binomial marginal distributions.

2. PRELIMINARIES

Our periodic discrete renewal process is described as follows. There are T possible seasons, which are indexed in the order $1, 2, \ldots, T$. There are T lifetimes $\{L_k\}_{k=1}^{T}$ that are used as follows: If a renewal occurs in season $\nu \in \{1, 2, \ldots, T\}$, then the time until the next renewal has the same distribution as L_ν and does not depend on past lifetimes in any way. Let $s(t) = t - T\lfloor (t-1)/T \rfloor$ denote the season of time t so that $s(0) = T$, $s(1) = 1, \ldots, s(T-1) = T-1, s(T) = T, s(T+1) = 1$, and so on.

Our periodic renewal process has renewals at the times $R_0 < R_1 < R_2 < \cdots$ and the kth lifetime is I_k, where $I_0 = R_0$ and $I_k = R_k - R_{k-1}$ for $k \geq 1$. Conditional on R_{n-1}, I_n has the same distribution as $L_{s(R_{n-1})}$. The interrenewal lifetimes are governed by T independent and identically distributed (i.i.d.) sequences $\{L_{1,k}\}_{k\geq 1}, \{L_{2,k}\}_{k\geq 1}, \ldots, \{L_{T,k}\}_{k\geq 1}$, with $L_{\nu,k} \stackrel{d}{=} L_\nu$ for each $\nu \in \{1, 2, \ldots, T\}$ and $k \geq 1$. The initial lifetime R_0, which might not be equal in distribution to L_ν for any season ν, is assumed independent of all other lifetimes. For each $n \geq 1$, one has

$$R_n = R_{n-1} + L_{s(R_{n-1}),n}.$$

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

In contrast to classical renewal theory, the interrenewal times are no longer i.i.d. We call the process *pure* if $R_0 = 0$; otherwise, the process is termed *delayed*. Let u_n be the renewal probability at time n, that is,

$$u_n = \sum_{k=0}^{\infty} \Pr(R_k = n).$$

The sequence $\{u_n\}_{n\geq 0}$ is called the renewal probability sequence. One goal of this article is to find an initial distribution of R_0 that makes the renewal probabilities periodically constant: $u_{nT+\nu} = u_\nu$, for all $n \geq 0$. The utility of this will be seen in Section 6. Renewal processes satisfying this condition are called periodically stationary. When $T = 1$, our model reduces to the classical time-homogeneous renewal model.

The age chain $\{A_n\}_{n=0}^{\infty}$ is defined as $A_n = n - \sup\{R_k : R_k \leq n\}$; that is, A_n is the elapsed time since the most recent renewal previous to time n. If a renewal occurs at time n, then $A_n = 0$. If the last renewal occurring at or before time n occurred at time $k < n$, then $A_n = n - k$. Later, the derived limiting behavior of $\{A_n\}_{n=0}^{\infty}$ will show what distribution of the delay R_0 is needed to generate a periodically stationary renewal sequence.

3. COMPUTATION OF RENEWAL PROBABILITIES

As a ground zero issue, one would like to be able to compute $\{u_n\}_{n=1}^{\infty}$. For this, let $u_n^{(\nu)}$ be the probability of a renewal at time $n \geq 1$ when R_0 has the same distribution as L_ν:

$$u_n^{(\nu)} = \Pr(A \textit{ renewal occurs at time } n \textit{ when } R_0 \textit{ is a } L_\nu \textit{ lifetime}).$$

Whereas a nondelayed setup involves the season $\nu = T$ renewal probabilities $\{u_n^{(T)}\}_{n=1}^{\infty}$, it will be convenient to calculate $\{u_n^{(\nu)}\}_{n=1}^{\infty}$ simultaneously for all seasons $\nu \in \{1, 2, \ldots, T\}$.

The renewal probabilities at time 1 are simply $u_1^{(\nu)} = \Pr(L_\nu = 1)$ for each season ν. Conditioning on the time of the first renewal gives the recursion

$$u_n^{(\nu)} = \Pr(L_\nu = n) + \sum_{k=1}^{n-1} \Pr(L_\nu = k) u_{n-k}^{s(\nu+k)}. \tag{1}$$

From (1), it is a simple matter to compute the renewal probabilities in the order $u_1^{(1)}, \ldots, u_1^{(T)}; u_2^{(1)}, \ldots, u_2^{(T)}$, and so on.

As an example of the above, consider the case in which L_ν has a geometric distribution with success probability p_ν. For concreteness, suppose p_ν has the sinusoidal structure

$$p_\nu = A + B\cos\left(\frac{2\pi(\nu - \tau)}{T}\right), \tag{2}$$

where A, B, and τ are parameters rendering $p_\nu \in (0, 1)$ for all seasons ν. Selecting $T = 3$, $A = 1/3$, $B = 3/8$, and $\tau = 3.2$ gives $p_1 = .2941$, $p_2 = .02995$, and

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

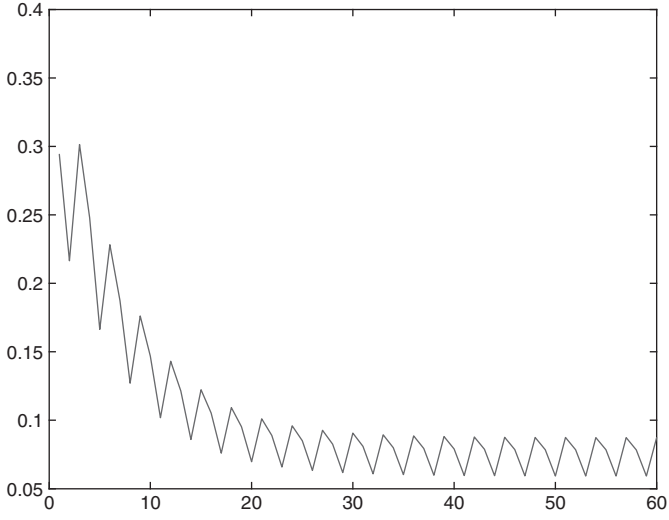


FIGURE 1. Renewal probabilities starting with a season 3 lifetime.

$p_3 = .6759$ to four significant digits. The season 3 renewal probabilities are plotted in Figure 1. Notice the rapid convergence of the renewal probabilities to a periodic limit. In the limit, the season 3 renewal probabilities are the largest and the season 2 probabilities are the smallest.

4. LIMITING PROPERTIES

This section establishes a periodic limiting distribution for $\{A_n\}_{n=0}^{\infty}$ and a deterministic periodic limit for $\{u_n\}_{n=1}^{\infty}$. Since the interest here is only in limiting behavior and the limits will not depend on the initial delay, we assume in this section that $R_0 = 0$, making $A_0 = 0$.

Our first observation shows that $\{A_n\}_{n=0}^{\infty}$ is a Markov chain with periodic transition probabilities.

PROPOSITION 4.1: *$\{A_n\}_{n=0}^{\infty}$ is a Markov chain with periodic transition probabilities. The only nonzero transition probabilities are*

$$\Pr(A_{n+1} = i+1 \mid A_n = i) = \Pr(L_{s(n-i)} > i+1 \mid L_{s(n-i)} > i),$$
$$\Pr(A_{n+1} = 0 \mid A_n = i) = \Pr(L_{s(n-i)} = i+1 \mid L_{s(n-i)} > i).$$

PROOF: Observe that if $A_n = i$, the only possible values for A_{n+1} are $i+1$ or zero. Additionally, if $A_n = i$, then the last renewal previous to time n must have occurred at time $n - i$ and this item is known to have lasted more than i time units. Hence, given $A_n = i$, the item in use at time n has a distribution equivalent to that of $L_{s(n-i)}$.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

Finally, conditional on $A_n = i$, $A_{n+1} = i + 1$ if and only if the item in use at time n lasts at least $i + 1$ time units. This verifies the first equation in the proposition; similar reasoning (or complementation) verifies the second equation. ■

Unfortunately, the one-step-ahead transition probabilities $\Pr(A_{n+1} = j \mid A_n = i)$ change with n in a periodic manner. Thus, although the age process is a Markov chain, it is not a time-homogeneous Markov chain. This observation leads us to the following proposition, which shows that subsequences of the age process on a periodic lattice are time-homogeneous chains.

PROPOSITION 4.2: *The sequence* $\{A_{nT+\nu}\}_{n=0}^{\infty}$ *is a time-homogeneous Markov chain for each fixed season* ν.

PROOF: As any subsequence of a Markov chain retains the Markov property, we need only verify the claimed time-homogeneity of transition probabilities. Given that $A_{nT+\nu} = i$, we know that the item in use at time $nT + \nu$ is drawn from a season $s(nT + \nu - i)$ distribution and is i units old at time $nT + \nu$. Should this item fail within the next T time units, it will be replaced by new item(s) whose distribution only depends on the season of failures(s), not on the cycle index n. Should such replacement(s) result in an age of j at time $(n + 1)T + \nu$ (note that $A_{(n+1)T+\nu}$ can be no larger that $T + i$), then this same replacement sequence would have produced an age of j at time $(n + 2)T + \nu$ should $A_{(n+1)T+\nu} = i$. Invoking periodic i.i.d. renewals proves the time-homogeneity of transition probabilities. ■

We now derive conditions that ensure that $\{A_{nT+\nu}\}_{n=0}^{\infty}$ is irreducible, aperiodic, and positive recurrent. It will be helpful to introduce a Markov chain $\{S_n\}_{n=0}^{\infty}$ that keeps track of the season from which the nth item was generated. Specifically, for $\nu \in \{1, 2, \ldots, T\}$, let $C_\nu = \{\nu, T + \nu, 2T + \nu, \ldots\}$ be the set of all times at which season ν takes place, and for $\nu, \nu' \in \{1, 2, \ldots, T\}$, set

$$p_{\nu,\nu'} = \Pr(S_{n+1} = \nu' \mid S_n = \nu) = \Pr(L_\nu \in C_{s(\nu'-\nu)}).$$

PROPOSITION 4.3: *If* $\{S_n\}_{n=0}^{\infty}$ *is irreducible, aperiodic, and positive recurrent and* $E[L_\nu] < \infty$ *for every season* ν, $\{A_{nT+\nu}\}_{n=0}^{\infty}$ *is irreducible, aperiodic, and positive recurrent for each season* ν *and, hence, has a unique limiting distribution.*

PROOF: It suffices to show that $E_\nu[\tau_\nu] < \infty$ for each season ν, where τ_ν denotes the time spent waiting for a future renewal to take place in season ν and E_ν signifies that a renewal has currently taken place at a time whose season is ν.

We first consider the case in which $p_* := \min_{i,j} p_{i,j} > 0$. Under this assumption, there is some positive probability that each and every new renewal will take place in the season ν lattice set C_ν. A geometric trials argument now shows that τ_ν is stochastically bounded by $\tau_\nu \overset{\text{st}}{\leq} \sum_{j=1}^{N} M_j^*$, where $\{M_j^*\}_{j\geq 1}$ is an i.i.d. sequence, with $M_1^* \overset{d}{=} \max(L_1, \ldots, L_T)$, and N is a geometric random variable with parameter p. Hence, $E_\nu[\tau_\nu] \leq E[M_1^*]E[N] < \infty$ follows from $E[M_1^*] \leq \sum_{j=1}^{T} E[L_j] < \infty$.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

Now, suppose that $\min_{i,j} p_{i,j} = 0$. Since $\{S_n\}_{n=0}^{\infty}$ is a finite state-space chain whose transition matrix is irreducible, aperiodic, and positive recurrent, there exists an integer $k > 1$ such that the k-step-ahead transitions of $\{S_n\}_{n=0}^{\infty}$ are uniformly positive for all i and j: $\min_{i,j} p_{i,j}^{(k)} > 0$ (Billingsley [2]). Given that we are currently experiencing a season ν renewal, let $\tau_\nu^{(k)}$ be the first time until ℓ more renewals occur, where (1) ℓ is a whole multiple of k and (2) the renewal puts the chain in season ν. Applying the above argument shows that $E[\tau_\nu^{(k)}] < \infty$. However, since $\tau_\nu \le \tau_\nu^{(k)}$, $E[\tau_\nu] < E[\tau_\nu^{(k)}] < \infty$ and our work is done. ■

For notation, let $\pi_k(\nu) = \lim_{n\to\infty} \Pr(A_{nT+\nu} = k)$ denote the stationary distribution of $\{A_{nT+\nu}\}_{n=0}^{\infty}$. Our next goal is to compute this distribution for each season ν. The following result reduces this issue to the case in which $k = 0$.

PROPOSITION 4.4: *For each $k \ge 0$ and each season ν,*

$$\pi_k(\nu) = \Pr(L_{s(\nu-k)} > k)\pi_0(s(\nu - k)). \tag{3}$$

PROOF: Simply use that

$$\begin{aligned}
\pi_k(\nu) &= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k) \\
&= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k, A_{nT+\nu-k} = 0) \\
&= \lim_{n\to\infty} \Pr(A_{nT+\nu} = k \mid A_{nT+\nu-k} = 0) \Pr(A_{nT+\nu-k} = 0) \\
&= \Pr(L_{s(\nu-k)} > k)\pi_0(s(\nu - k)).
\end{aligned}$$

■

When the lifetimes $\{L_\nu\}_{\nu=1}^{T}$ are all identically distributed (seasonally nonvarying), the age process is time-homogeneous and there is only one stationary distribution to compute. In this case, (3) gives

$$1 = \sum_{k=0}^{\infty} \pi_k(1) = \sum_{k=0}^{\infty} \Pr(L_1 > k)\pi_0(1) = \pi_0(1)E(L_1),$$

which yields the classical result $\pi_0(1) = E[L_1]^{-1}$.

An analogous approach will be used in our periodic setting but will require slightly more work. For each season ν, $\{\pi_k(\nu)\}_{k=0}^{\infty}$ is a probability measure. Using this and Proposition 4.4 provides, for each season ν,

$$\begin{aligned}
1 = \sum_{k=0}^{\infty} \pi_k(\nu) &= \sum_{j=1}^{\nu} \pi_0(j) \sum_{n=0}^{\infty} \Pr(L_j > nT + \nu - j) \\
&\quad + \sum_{j=\nu+1}^{T} \pi_0(j) \sum_{n=0}^{\infty} \Pr(L_j > (n+1)T + \nu - j).
\end{aligned} \tag{4}$$

Hence, $\vec{\pi}_0 := (\pi_0(1), \pi_0(2), \ldots, \pi_0(T))$ is a solution to a linear system of T equations.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

Our immediate goal is to solve (4) for $\pi_0(1), \ldots, \pi_0(T)$. To help solve this, we offer the following lemma. Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

LEMMA 4.1: *Let X be a positive integer-valued random variable. Then, for each season ν,*

$$\sum_{n=0}^{\infty} \Pr(X > nT + \nu) = E\left[\lfloor X/T \rfloor\right] + \Pr\left(X \in \cup_{\ell=\nu+1}^{T-1} C_\ell\right). \tag{5}$$

PROOF: Since $\Pr(X > nT + \nu) = \Pr(X \geq (n+1)T) + \Pr(X \in \{nT + \nu + 1, \ldots, nT + (T-1)\})$,

$$\sum_{n=0}^{\infty} \Pr(X > nT + \nu) = \sum_{n=0}^{\infty} \Pr(X \geq (n+1)T) + \Pr(X \in B_\nu),$$

where $B_\nu = \cup_{\ell=\nu+1}^{T-1} C_\ell$. Using this and

$$\sum_{n=0}^{\infty} \Pr(X \geq (n+1)T) = \sum_{n=1}^{\infty} \Pr\left(\lfloor X/T \rfloor \geq n\right) = E[\lfloor X/T \rfloor]$$

finishes our work. ■

Applying Lemma 4.1 to (4) for each season ν yields

$$\begin{aligned} 1 = &\sum_{j=1}^{\nu} \pi_0(j) \left[E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{\nu-j+1})\right] \\ &+ \sum_{j=\nu+1}^{T} \pi_0(j) \left[E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{T+\nu-j+1})\right]. \end{aligned} \tag{6}$$

Equation (6) shows that $\vec{\pi}_0$ lies in $\{x \in \mathbb{R}^T : Ax = \vec{1}\}$, where $\vec{1}$ is a T-dimensional column vector containing all ones and $A := [a_{i,j}]_{1 \leq i,j \leq T}$ is a matrix with entries

$$a_{i,j} = \begin{cases} E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{i-j+1}), & i \geq j \\ E(\lfloor L_j/T \rfloor) + \Pr(L_j \in B_{T+i-j+1}), & i < j, \end{cases}$$

where the convention $B_0 = B_T$ is used. We now show that $\vec{\pi}_0$ can be expressed purely in terms of the stationary distribution of the seasonal chain and the first moment of the T lifetime distributions.

Readers should note that the expressions in Theorem 4.1 for $\vec{\pi}_0$ are akin to those found in Theorem 3 of Thorisson [19], who studied periodic renewal processes in continuous time when the interrenewal distributions have some common absolutely continuous component over all seasons.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

THEOREM 4.1: *For each season ν,*

$$\pi_0(\nu) = \lim_{n\to\infty} \Pr(A(nT+\nu) = 0) = \frac{T\gamma_\nu}{\sum_{\ell=1}^{T} E(L_\ell)\gamma_\ell},$$

where $\vec{\gamma} = \{\gamma_\nu\}_{\nu=1}^{T}$ represents the stationary distribution of $\{S_n\}_{n=0}^{\infty}$.

PROOF: Consider the linear system given by (6). Subtracting the second equation from the first equation, the third equation from the second equation, the fourth from the third, and so on, we observe that $\vec{\pi}_0$ must also satisfy a linear system of T equations, the first $T-1$ being of form

$$\pi_0(\nu)\sum_{\nu'\neq\nu} p_{\nu,\nu'} = \sum_{\nu'\neq\nu} \pi_0(\nu')p_{\nu',\nu}, \qquad 2 \le \nu \le T,$$

where $p_{\nu',\nu}$ is the (ν',ν)th entry in the transition matrix of $\{S_n\}_{n=0}^{\infty}$. The last equation is

$$1 = \sum_{\nu=1}^{T} \pi_0(\nu)\left\{E(\lfloor L_\nu/T \rfloor) + \Pr(L_\nu \in B_{T-\nu+1})\right\}.$$

Adding $\pi_0(\nu)p_{\nu,\nu}$ to both sides of the first $T-1$ equations shows that $\vec{\pi}_0$ also satisfies

$$\pi_0(\nu) = \sum_{\ell=1}^{T} \pi_0(\ell)p_{\ell,\nu}, \qquad 2 \le \nu \le T,$$

$$1 = \sum_{\nu=1}^{T} \pi_0(\nu)\left\{E(\lfloor L_\nu/T \rfloor) + \Pr(L_\nu \in B_{T-\nu+1})\right\}.$$

This new system is extremely elegant; in particular, the first $T-1$ equations are the stationary balance equations of $\{S_n\}_{n=0}^{\infty}$. From these first $T-1$ equations, one can show that $\vec{\pi}_0$ must satisfy all balance equations for the stationary distribution of $\{S_n\}_{n=0}^{\infty}$. Since $\{S_n\}_{n=0}^{\infty}$ is irreducible and positive recurrent, all solutions of the balance equations are scalar multiples of the stationary distribution $\vec{\gamma}$ of the seasonal chain (for details, see, e.g., [16]). Thus, there exists a constant $c > 0$ satisfying $\vec{\pi}_0 = c\vec{\gamma}$.

We finish by determining the unknown constant c. Summing all equations in (4) and applying $\vec{\pi}_0 = c\vec{\gamma}$ gives $T = c\sum_{\ell=1}^{T} E[L_\ell]\gamma_\ell$. Hence,

$$c = \frac{T}{\sum_{\ell=1}^{T} E[L_\ell]\gamma_\ell}$$

and our derivation is complete. ■

Our results are expressed in terms of the stationary distribution $\vec{\gamma}$ of $\{S_n\}_{n=0}^{\infty}$, which depends on terms of the form $\Pr(L \in C_\nu)$. Such probabilities are tractable for many

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

classical types of integer-valued random variables. For example, if L is geometric with parameter p, then, for each season ν,

$$\Pr(L \in C_\nu) = \frac{p(1-p)^{\nu-1}}{1-(1-p)^T}, \qquad \nu = 1, 2, \ldots, T. \tag{7}$$

It is also possible to compute such probabilities when L is a negative binomial random variable with parameters $m \geq 2$ and $p \in (0,1)$. One efficient way of doing this is to recognize that L is now an i.i.d. sum of m geometric random variables with parameter p. Setting G to be a matrix with the (i,j)th element

$$g_{i,j} = \frac{p(1-p)^{s(j-i)-1}}{1-(1-p)^T},$$

we see that $\Pr(L \in C_\nu)$ is simply the νth component of $\vec{e}_1 G^m$, where $\vec{e}_1$ is the first basis vector $(1,0,0,\ldots,0)$. These types of probabilities can also be computed explicitly when L is a Poisson random variable. Interested readers will find that $P(L \in C_\nu)$ can, in this case, be expressed rather elegantly in terms of the T roots of unity (possibly complex-valued solutions to $z^T = 1$).

Example 4.1: Suppose that $T = 2$. Then the transition matrix of $\{S_n\}_{n=0}^\infty$ has the (i,j)th element $\Pr(L_i \in C_{s(j-i)})$ for $i,j \in \{1,2\}$. The stationary distribution of $\{S_n\}_{n=0}^\infty$ hence has form

$$\vec{\gamma} = \left(\frac{\Pr(L_2 \in C_1)}{\Pr(L_1 \in C_1) + \Pr(L_2 \in C_1)}, \frac{\Pr(L_1 \in C_1)}{\Pr(L_1 \in C_1) + \Pr(L_2 \in C_1)} \right).$$

Hence,

$$\pi_0(1) = \lim_{n\to\infty} u_{nT+1} = \frac{2\Pr(L_2 \in C_1)}{E[L_1]\Pr(L_2 \in C_1) + E[L_2]\Pr(L_1 \in C_1)}$$

and

$$\pi_0(2) = \lim_{n\to\infty} u_{nT+2} = \frac{2\Pr(L_1 \in C_1)}{E[L_1]\Pr(L_2 \in C_1) + E[L_2]\Pr(L_1 \in C_1)}.$$

Arguing with (3) gives

$$\pi_k(1) = \Pr(L_2 > k)\pi_0(2)1_{\{k \text{ odd}\}} + \Pr(L_1 > k)\pi_0(1)1_{\{k \text{ even}\}}, \qquad k \geq 1$$

and

$$\pi_k(2) = \Pr(L_1 > k)\pi_0(1)1_{\{k \text{ odd}\}} + \Pr(L_2 > k)\pi_0(2)1_{\{k \text{ even}\}}, \qquad k \geq 1.$$

Example 4.2: Consider a setting in which a renewal always occurs during season 1; that is, there is always an annual replacement. If the item fails before season 1, the item is replaced at that time. Such a scenario occurs when the support of L_ν is

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

$\{1, 2, \ldots, T - \nu + 1\}$ for each season ν. We start with a season T lifetime at time 0 so that $S_0 = T$.

The renewal probabilities can be obtained from the renewal equations of Section 3: set $u_0 = 1$, and for each season ν,

$$u_\nu = \sum_{k=0}^{\nu-1} u_k P(L_k = \nu - k).$$

One can easily verify that for each season ν,

$$\pi_0(\nu) = \frac{Tu_\nu}{\sum_{\ell=1}^{T} E[L_\ell]u_\ell}$$

satisfy the limiting equations in Theorem 4.1.

5. A PERIODICALLY STATIONARY INITIAL LIFETIME

What should the law of the initial delay R_0 be in order for our process to be exactly periodically stationary from the onset? Specifically, we seek to identify the distribution of R_0 so that the renewal probabilities are exactly periodic: $u_{nT+\nu} \equiv \pi_0(\nu)$ for all $n \geq 0$. When $T = 1$, it is wellknown that the law of R_0 is simply

$$\Pr(R_0 = n) = \frac{\Pr(L_1 \geq n + 1)}{E[L_1]}, \quad n \geq 0.$$

This is often referred to as the equilibrium distribution, or the first-derived distribution of L_1.

As our next result shows, if the law of A_0 is set to $\vec{\pi}(T)$, then the age chain is put into a periodic state (exactly). Our notation uses P_ν for the one-step-ahead transition probability matrix whose (i,j)th element is $\Pr(A_{\nu+1} = j \mid A_\nu = i)$.

PROPOSITION 5.1: *Interpreted periodically with period T, $\vec{\pi}(\nu + 1) = \vec{\pi}(\nu)P_\nu$, where $\vec{\pi}(\nu) = \{\pi_k(\nu)\}_{k=0}^{\infty}$.*

PROOF: The result follows from the fact that the stationary distribution of the subsequence $\{A_{nT+\nu}\}_{n=0}^{\infty}$ must be unique for each fixed season ν. In fact, the one-step-ahead transition matrix of $\{A_{nT+\nu}\}_{n=0}^{\infty}$ is $\mathbb{Q}_\nu := P_\nu P_{\nu+1} \cdots P_T P_1 \cdots P_{\nu-1}$. Since $\vec{\pi}(\nu) = \vec{\pi}(\nu)\mathbb{Q}_\nu$, we have

$$\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu)\mathbb{Q}_\nu P_\nu = \vec{\pi}(\nu)\left(P_\nu P_{\nu+1} \cdots P_T P_1 \cdots P_{\nu-1}\right) P_\nu. \tag{8}$$

Hence, $\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu)P_\nu\mathbb{Q}_{\nu+1}$. However, since $\vec{\pi}(\nu + 1) = \vec{\pi}(\nu + 1)\mathbb{Q}_{\nu+1}$, uniqueness of stationary distributions shows that $\vec{\pi}(\nu)P_\nu = \vec{\pi}(\nu + 1)$. ■

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

Deriving an expression for the probability mass function of R_0 is now a simple task. Suppose that the law of A_0 is $\pi(T)$ so that the age chain is periodically stationary (exactly) from the onset. Let $\tilde{f}_k = \Pr(R_0 = k)$ be the mass function of the initial delay. Conditioning on R_0 gives the recursion

$$u_n = \tilde{f}_n + \sum_{k=0}^{n-1} u_k \Pr(L_{s(k)} = n - k).$$

Solving this for $\tilde{f}_n$ gives

$$\tilde{f}_n = \pi_0(s(n)) - \sum_{k=0}^{n-1} \pi_0(s(k)) \Pr(L_{s(k)} = n - k) \tag{9}$$

when the fact that R_0 was chosen to induce a periodically stationary renewal process is applied ($u_{nT+\nu} = \pi_0(\nu)$ for all n and seasons ν).

6. A COUNT TIME SERIES MODEL WITH PERIODIC DYNAMICS

Count time series models with periodic properties can be devised with the above methods. For example, suppose one is interested in modeling the number of precipitation days in a week at a fixed locality (a day is called a precipitation day if 0.1 in or more of rain or its snow water equivalent is recorded). Here, a binomial marginal distribution with seven trials plausibly describes the counts in any week. However, because adjacent weeks experience similar weather, weekly counts are likely to exhibit positive correlation. Additionally, some localities should display periodic features. For example, rain rarely occurs in California during the summer, but it is common during the winter. Figure 2 plots the number of precipitation days observed in 728 successive weeks at Coldfoot, Alaska spanning the 14-year period January 1, 1996 – December 31, 2009. Coldfoot is noteworthy, as it claims North America's lowest observed temperature of $-82°$F. Although this record is not officially recognized due to gauge deficiencies, Coldfoot, lying near the Brooks Range, has a seasonal but ephemeral climate. Leap year day (February 29) precipitations and December 31 precipitations have been neglected to induce an "exact period" of $T = 52$ weeks in the counts. This should not influence end inferences greatly. In fact, precipitation was observed on December 31 in only 2 of the 14 years in the record, for example.

A model for series with the above properties is easily devised from our work. Let $\{R_t\}$ be a renewal process that is periodically stationary (exactly); that is, R_t is unity if a renewal occurs at time t and zero otherwise. Then $\Pr(R_{nT+\nu} = 1) = u_\nu$ is the probability of a renewal at time $nT + \nu$. Now, suppose that $\{R_{t,1}\}_{t=1}^{\infty}, \{R_{t,2}\}_{t=1}^{\infty}, \ldots$ are independent copies of $\{R_t\}$. To model the precipitation count N_t of week t, set

$$N_t = \sum_{\ell=1}^{7} R_{t,\ell}. \tag{10}$$

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

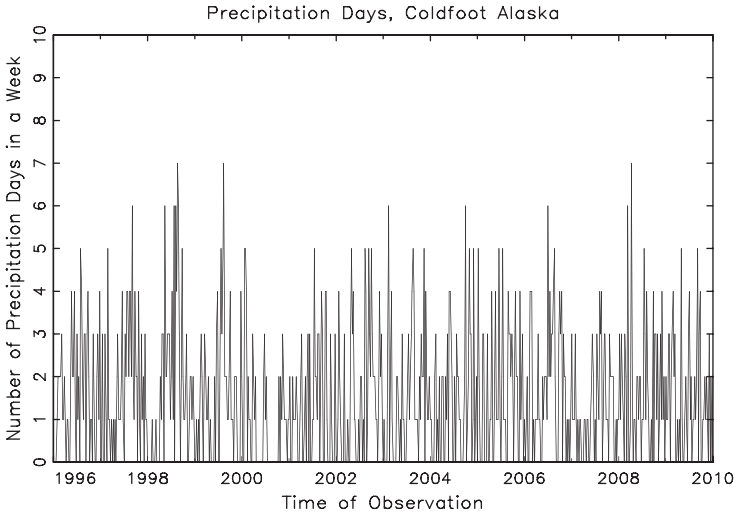


FIGURE 2. Weekly precipitation days at Coldfoot, Alaska.

It is easy to see that $\{N_t\}$ has marginal binomial distributions with seven trials. Additionally, $\{N_t\}$ has a periodic mean and covariance structure. To see this, note that $E[R_{nT+\nu}] = u_\nu$ and, hence, $E[N_{nT+\nu}] = 7\mu_\nu$. By a periodic covariance structure (frequently termed *cyclostationary*, *periodically stationary*, or *periodicially correlated* in the literature), we mean that $\text{Cov}(N_{n+T}, N_{m+T}) = \text{Cov}(N_n, N_m)$ for all integers n and m. This is seen by noting that for $t < s$,

$$\begin{aligned}\text{Cov}(N_t, N_s) &= \sum_{i=1}^{7} \text{Cov}(R_{t,i}, R_{s,i}) \\ &= 7u_t\,[\Pr(R_s = 1|R_t = 1) - u_s] \\ &= 7\pi_0(t)[\Pr(R_s = 1|R_t = 1) - \pi_0(s)]\end{aligned}$$

and applying $\Pr(R_{s+T} = 1|R_{t+T} = 1) = \Pr(R_s = 1|R_t = 1)$ and $u_{t+T} = u_t = \pi_0(t)$.

As an aside, observe that marginal distributions other than Binomial can be devised with the above methods. For example, should one desire periodic Poisson marginals, then (10) is modified to $N_t = \sum_{\ell=1}^{M_t} R_{t,\ell}$, where $\{M_t\}_{t=1}^{\infty}$ is a periodic random sequence having Poisson marginal distributions, say, $E[M_t] = \lambda_t$ with $\lambda_{t+T} = \lambda_t$. Cui and Lund [4] showed how to generate geometric and other classical count structures from the above tactics. The fundamental building block to all constructions is the renewal process, which is simply a correlated sequence of zeros and ones.

Returning to the binomial problem at hand, we briefly attempt to fit a rudimentary statistical model to the Coldfoot series. For the renewal process, we posit that the season ν lifetime L_ν is geometrically distributed with success probability $p_\nu > 0$; moreover, since p_ν is periodic, we explore the first-order Fourier parametrization in

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

(2), where $A > 0$, $B \in [0, 1 - A]$ are unknown amplitude parameters, and $\tau \in [0, T]$ is an unknown phase parameter. We fit this model by minimizing a sum of squared prediction residuals. Specifically, arguing as in [4], one can show that the weekly precipitation counts $\{X_t\}$ obey a periodic Markov structure and that

$$E[N_{t+1}|N_t] = E[N_{t+1}] + \frac{\mathrm{Cov}(N_{t+1}, N_t)}{\mathrm{Var}(N_t)}(N_t - E[N_t]).$$

Using $E[N_t] = 7\pi_0(t)$, $\mathrm{Var}(N_t) = 7\pi_0(t)(1 - \pi_0(t))$, $\Pr[L_{s(t)} = 1] = p_{s(t)}$, and

$$\mathrm{Cov}(N_{t+1}, N_t) = 7\pi_0(t)[\Pr(L_{s(t)} = 1) - \pi_0(t+1)] = 7\pi_0(t)[p_{s(t)} - \pi_0(t+1)],$$

a one-step-ahead prediction of the form

$$\begin{aligned}\hat{N}_{t+1} &= E[N_{t+1}|N_1, \ldots, N_t] \\ &= E[N_{t+1}|N_t] \\ &= 7\pi_0(t+1) + \frac{p_{s(t)} - \pi_0(t+1)}{1 - \pi_0(t)}(N_t - 7\pi_0(t))\end{aligned}$$

is obtained.

A reasonable objective function for selecting A, B, and τ simply minimizes

$$\sum_{\ell=0}^{d-1}\sum_{\nu=1}^{T}(N_{\ell T+\nu} - \hat{N}_{\ell T+\nu})^2 = \sum_{t=1}^{n}(N_t - \hat{N}_t)^2 \tag{11}$$

over feasible values of A, B, and τ. In (11), $n = 728$ is the total number of weeks and $d = n/T = 14$ is the number of years of observed data. Observe that $\hat{N}_t$ is a function of A, B, and τ only and that $\Pr[L_{s(t)} = 1] = p_{s(t)}$.

A numerical minimization routine was used to find parameter values that minimize (11): $\hat{A} = 0.2225$, $\hat{B} = 0.08333$, and $\hat{\tau} = 29.2770$. The minimum sum of squares was 1588.9517. Elaborating, Theorem 4.1 was used to compute $\vec{\pi}_0$ for each feasible triple of A, B, and τ. The values of $\{\gamma_\nu\}_{\nu=1}^{52}$ needed were obtained numerically after explicit expressions for the one-step-ahead transitions of the seasonal chain $\{S_n\}_{n=0}^{\infty}$ were computed. Specifically, L_ν having a geometric distribution with parameter p_ν implies that

$$\Pr(S_{n+1} = j|S_n = i) = \begin{cases} \dfrac{p_i(1-p_i)^{j-i-1}}{1-(1-p_i)^T}, & j > i \\ \dfrac{p_i(1-p_i)^{T+j-i-1}}{1-(1-p_i)^T}, & j \le i. \end{cases}$$

Numerical estimates of the Hessian of this fit indicate that B is significantly positive, implying that periodic features are needed in the model. As an example of what the model fit does, Figure 3 plots the weekly precipitation count segment during 2001 along with one-step-ahead predictions. The one-step-ahead predictions track

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

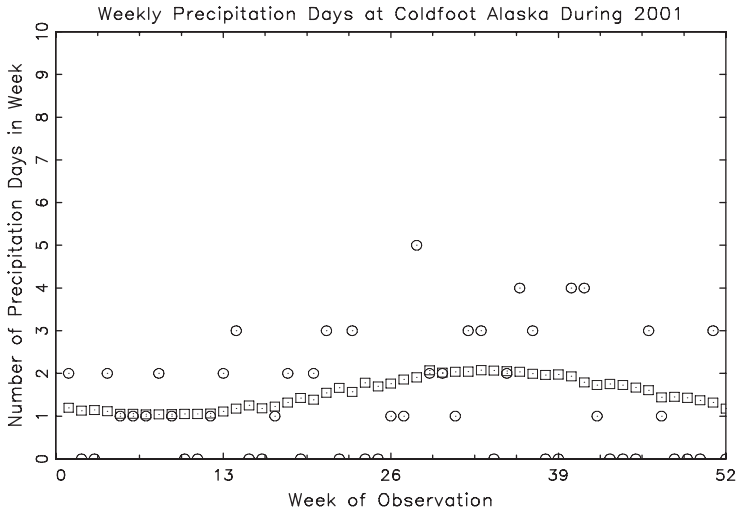


FIGURE 3. Weekly precipitation days during 2001 (circles) and their one-step-ahead predictions (squares).

the observed counts reasonably well. The estimated long-run probabilities of a precipitation day peak during week 29 (summer) at .2950 and are minimal during week 3 (winter) at .1485.

While we leave issues of whether the fit is good and so on, to a statistical inference study, one appreciates that periodic renewal processes have utility in a variety of applications.

References

1. Asmussen, S. & Thorisson, H. (1987). A Markov chain approach to periodic queues. *Journal of Applied Probability* 24: 215–225.
2. Billingsley, P. (1995). *Probability and measure*, 3rd ed., New York: Wiley.
3. Çinlar, E. (1974). Periodicity in Markov renewal theory. *Advances in Applied Probability* 6: 61–78.
4. Cui, Y. & Lund, R. (2009). A new look at time series of counts. *Biometrika* 96: 781–792.
5. Feller, W. (1968). *An Introduction to probability theory and its applications, Volume I*, 3rd ed., New York: Wiley.
6. Gardner, W.A., Napolitano, A., & Paura, L. (2006). Cyclostationarity: Half a century of research. *Signal Processing* 86: 639–697.
7. Harrison, J.M. & Lemoine, A.J. (1977). Limit theorems for queues with periodic input. *Journal of Applied Probability* 14: 566–576.
8. Heyman, D.P. & Whitt, W. (1984). The asymptotic behavior of queues with time-varying arrival rates. *Journal of Applied Probability* 21: 143–156.
9. Hurd, H.L. & Miamee, A. (2007). *Periodically correlated random sequences: Spectral theory and practice*. New York: Wiley.
10. Jagers, P. & Nerman, O. (1985). Branching processes in a periodically varying environment. *Annals of Probability* 13: 254–268.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press

11. Karlin, S. & Taylor, H.M. (1975). *A first course in stochastic processes*. New York: Academic Press.
12. Lemoine, A.J (1981). On queues with periodic Poisson input. *Journal of Applied Probability* 18: 889–900.
13. Lund, R. (1994). A dam with seasonal input. *Journal of Applied Probability* 31: 526–541.
14. Lund, R. & Basawa, I.V. (1999). Modeling for periodically correlated time series, In *Asymptotics, Nonparametrics, and Time Series*. New York: Dekker, pp. 37–62.
15. Phatarfod, R.M. (1980). The bottomless dam with seasonal inputs. *Australian and New Zealand Journal of Statistics* 22: 212–217.
16. Resnick, S.I. (1992). *Adventures in stochastic processes*. Boston: Birkhäuser.
17. Ross, S.M. (1996). *Stochastic processes*, 2nd ed., New York: Wiley.
18. Smith, W. L. (1958). Renewal theory and its ramifications. *Journal of the Royal Statistical Society, Series B* 20: 243–302.
19. Thorisson, H. (1985). Periodic regeneration. *Stochastic Processes and their Applications* 20: 85–104.

https://doi.org/10.1017/S0269964811000209 Published online by Cambridge University Press