*Euro. Jnl of Applied Mathematics* (2009), *vol.* 20, *pp.* 319–342. © 2009 Cambridge University Press 
doi:10.1017/S0956792509007815 Printed in the United Kingdom

# Stress–elastic strain relation for a two-phase isotropic strain gradient plastic composite

VIET HA HOANG

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK*
*email*: V.H.Hoang@damtp.cam.ac.uk

(*Received 13 March 2008; revised 11 January 2009; first published online 10 February 2009*)

The stress–elastic strain relationship is studied for a composite under a plastic deformation. The constitutive law of each component is described by a deformation theory of strain gradient plasticity which introduces an internal length scale. The conventional deformation plastic theory is obtained when the internal length scale tends to 0. The Hashin–Shtrikman upper bound for a two-phase composite governed by a power law is derived. It is predicted, by differentiating the bounds, that in most cases, the stress and the elastic strain follow a non-linear relation immediately after the elastic range. However, for some particular values of the ratio of the internal length scale and the micro-scale of the composite, this relation is linear. The prediction is illustrated by various numerical examples.

## 1 Introduction

Bounds and estimates for the overall properties of plastic composite materials have been studied in several works. A number of them (e.g. Ponte Castañeda, 1992; Ponte Castañeda & De Botton, 1992; Suquet, 1993) study conventional plastic materials where bounds and estimates for the effective energy density, and thus for the yield strength, are deduced. The techniques employed are based on variational formulations and the use of comparison linear materials which are much in the same spirit as that of the initial work by Talbot and Willis (1985). Extending these, composites whose components are under the effect of strain gradient are considered in Smyshlyaev and Fleck (1994, 1995), Fleck and Willis (2004) and Aifantis and Willis (2005). The mechanical framework employed is based on the deformation strain gradient theories by Fleck and Hutchinson (1993, 2001) in which the energy is a non-linear function of the strain field and its gradient and depends on an internal length scale $l$. The overall properties now depend on the ratio $l/a$ where $a$ is the material length scale. These works find bounds and estimates for the effective yield strength of the composite, ignoring the elastic strain or assuming that the materials are elastically homogeneous. In particular, when the materials follow a power law, the effective materials are also assumed to follow a power law and explicit bounds for the effective reference strength are established. Though a rigorous mathematical homogenization procedure has not been carried out, in these cases a power law effective material is expected. However, when the materials are elastically heterogeneous, the form of the effective energy potential is not clear. Approaching the

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

problem mathematically, we wish to explore the dependency of the stress on the elastic strain. As shown in this paper, this relationship is complicated and depends essentially on the plastic reference strength, even in the range where the plastic strain is relatively small. Mechanically, a study of the stress–elastic strain relation is of interest when the total strain is not large, i.e. when the elastic strain is significant in comparison to the plastic strain.

We consider a two-phase incompressible plastic composite whose components obey a power law deformation theory with strain gradient effect. The phenomenological constitutive law in each phase is that in Fleck and Willis (2004) which modifies the theory by Fleck and Hutchinson (2001) for easier analysis but keeps the same features. The material properties depend on an internal length scale $l$ which is assumed to be the same for all the components. Although the framework is that of a strain gradient plastic composite, our findings cover conventional plastic composites as when $l \to 0$ the framework becomes conventional plastic deformation theory. By using the upper Hashin–Shtrikman bound for a comparison linear composite, we deduce an upper bound for the energy potential using the procedure introduced by Ponte Castañeda (1991, 1992). As in Fleck and Willis (2004), an approximation for the average stress is obtained by formally differentiating this bound. As expected, the stress and the elastic strain follow a linear relation governed by the upper Hashin–Shtrikman bound for incompressible linear elasticity when the total strain is small, i.e. when the plastic strain is insignificant in comparison to the elastic strain. Deep in the plastic range (when the total strain is large) it is shown that they asymptote another linear relation. As we expect the stress to be a smooth function of the elastic strain, this shows that in the range of interest (i.e half-way between these two), this relation is expected to be non-linear. We also predict that the effective elastic moduli in the small strain range depend only on the elastic moduli of the phases; in the large strain range, it depends essentially on the plastic reference strength. This is not a typical strain gradient effect, it holds for conventional plasticity ($l = 0$), but the stress–elastic strain relation depends on the internal scale $l$. For completeness and comparison, we also deduce the elementary Voigt upper bound and Reuss lower bound. A lower bound in the Hashin–Shtrikman framework cannot be deduced for this non-linear material. Instead, a lower estimate similar to those deduced by Ponte Castañeda and De Botton (1992) and Fleck and Willis (2004) is introduced briefly. Rigorous bounds for the average stress in terms of the strain have been established for elastically homogeneous composites in Hoang (in press). When the composite is elastically heterogeneous, it is more complicated to deduce stress bounds in terms of the strain; this is the subject of a future study.

The paper is organized as follows. In the next section, we briefly introduce the deformation strain gradient theory that is employed in Fleck and Willis (2004). The Voigt and Reuss bounds are deduced in Section 3. In Section 4, we study the Hashin–Shtrikman upper and lower bounds for a linear material. This bound will be used in deriving the Hashin–Shtrikman upper bound and lower estimate for non-linear materials in Section 5. In Section 6, we investigate numerically some examples which illustrate the prediction we make from the Hashin–Shtrikman bound, and the dependence of the stress–elastic strain relation on the plastic reference strength, and the internal scale $l$. Finally we present some technical details and calculations in the appendices.

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

## 2 Strain gradient energy

We consider a plastic composite material whose energy functional depends on the gradient of the plastic strain. The formulation we study is that of Fleck and Willis (2004) which is a modification of the strain gradient plastic energy originally introduced by Fleck and Hutchinson (2001). The energy potential function is defined as

$$U\left(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\right) = \frac{1}{2} L_{ijkl}\left(\varepsilon_{ij} - \varepsilon^P_{ij}\right)\left(\varepsilon_{kl} - \varepsilon^P_{kl}\right) + V\left(\varepsilon^P_{ij}, \varepsilon^P_{ij,k}\right),$$

in which the potential $V = V(E_P)$ where the effective plastic strain measure

$$E_P = \sqrt{\frac{2}{3}}\left(\varepsilon^P_{ij}\varepsilon^P_{ij} + l^2 \varepsilon^P_{ij,k}\varepsilon^P_{ij,k}\right)^{1/2};$$

$l$ is the material scale. The material is a composite so that the elastic moduli $L_{ijkl}$ and the potential $V$ depend on the position $x$ and are rapidly oscillating. The strain $\varepsilon_{ij}$ is defined, as usual, as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In this setting, the displacement $u_i$ and the plastic strain $\varepsilon^P_{ij}$ are treated as dependent variables on equal footing. Assuming for simplicity that the displacement is prescribed on the boundary, the deformation minimizes the energy functional

$$\Psi\left(u_i, \varepsilon^P_{ij}\right) = \int_\Omega U\left(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\right) dx. \tag{1}$$

Introducing the conjugate variables

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = L_{ijkl}\left(\varepsilon_{kl} - \varepsilon^P_{kl}\right), \tag{2}$$

$$s_{ij} = \frac{\partial U}{\partial \varepsilon^P_{ij}} = -L_{ijkl}\left(\varepsilon_{kl} - \varepsilon^P_{kl}\right) + \frac{\partial V}{\partial \varepsilon^P_{ij}}, \tag{3}$$

$$\tau_{ijk} = \frac{\partial U}{\partial \varepsilon^P_{ij,k}} = \frac{\partial V}{\partial \varepsilon^P_{ij,k}}, \tag{4}$$

and setting to zero the first variation of (1), we get the equilibrium equations

$$\sigma_{ij,j} = 0, \tag{5}$$

$$\sigma_{ij} + \tau_{ijk,k} = \frac{\partial V}{\partial \varepsilon^P_{ij}}. \tag{6}$$

When the loading varies slowly in comparison to the micro-structure, the effective response does not contain the strain gradient effect so the effective energy potential can be written as $U^{\text{eff}}(\varepsilon_{ij}, \varepsilon^P_{ij})$. Without the strain gradient effect, the plastic strain can be written as a function of the total strain alone from (2) and (6). Thus $U^{\text{eff}}(\varepsilon_{ij}, \varepsilon^P_{ij})$ can be written as $w^{\text{eff}}(\varepsilon_{ij})$. Assuming that the boundary displacement is $u_i = \varepsilon^0_{ij} x_j$, it is expected that the

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

equivalent homogeneous problem generates a uniform strain field. The infimum becomes

$$w^{\text{eff}}\left(\varepsilon_{ij}^0\right) = \inf_{u_i, \varepsilon_{ij}^P} \int_\Omega U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right) dx,$$

where to make the presentation concise, we will assume that the volume of $\Omega$ is 1. After Fleck and Willis (2004), we write this as

$$w^{\text{eff}}\left(\varepsilon_{ij}^0\right) = \inf_{\varepsilon_{ij}^{P0}} U^{\text{eff}}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) \tag{7}$$

in which the effective potential $U^{\text{eff}}$ is defined as

$$U^{\text{eff}}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) = \inf \int_\Omega U\left(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right) dx, \tag{8}$$

where the infimum is taken over all the displacement fields $u_i$ such that $u_i = \varepsilon_{ij}^0 x_j$ on the boundary of $\Omega$ and all the plastic strain fields $\varepsilon_{ij}^P$ with mean value $\varepsilon_{ij}^{P0}$ over $\Omega$.

We restrict our consideration to incompressible materials, i.e. $\varepsilon_{ii} = \varepsilon_{ii}^P = 0$. The phases are assumed to be perfectly bonded together so that there is no energy built up on the surface between different components as considered in, e.g. Gudmundson (2004) and Aifantis and Willis (2005).

## 3 The elementary Voigt and Reuss bounds

The Voigt and Reuss bounds for the effective plastic energy functions have been deduced in Fleck and Willis (2004) for elastically homogeneous materials. Here, we deal with a statistically homogeneous $M$-phase composite which is elastically heterogeneous. We denote the volume fraction of phase $r$ by $p_r$.

### 3.1 Voigt bound

The fields $\varepsilon_{ij} = \varepsilon_{ij}^0$ and $\varepsilon_{ij}^P = \varepsilon_{ij}^{P0}$ are admissible in the variational formulation (8). Therefore

$$U^{\text{eff}}\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) \leqslant \int_\Omega U\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}, 0\right) dx. \tag{9}$$

Let $\mu_r$ $(r = 1, \ldots, M)$ be the elastic stiffness of phase $r$. Assume further that the potential $V$ in phase $r$ has the form

$$V_r\left(\varepsilon_{ij}^P, \varepsilon_{ij,k}^P\right) = \frac{\sigma_r e_0}{N+1}\left(\frac{E_P}{e_0}\right)^{N+1}, \tag{10}$$

where $\sigma_r$ is the constant reference strength in phase $r$ (which may differ from that in other phases). The Voigt bound deduced from (9) is

$$U_V\left(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\right) = \mu_V\left(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}\right) + \frac{\sigma_V e_0}{N+1}\left(\frac{E_{P0}}{e_0}\right)^{N+1}, \tag{11}$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

where

$$\mu_V = \sum_{r=1}^{M} p_r \mu_r, \quad \sigma_V = \sum_{r=1}^{M} p_r \sigma_r$$

and $E_{P0}$ is the von Mises equivalent plastic strain $E_{P0} = ((2/3)\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0})^{1/2}$.

### 3.2 Reuss bound

We now deduce the Reuss lower bound for elastically heterogeneous materials. For tensor fields $t_{ij}, s_{ij}$ and $\tau_{ijk}$, we define the complementary energy potential as

$$U^*(t_{ij}, s_{ij}, \tau_{ijk}) = \sup_{\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}} \{t_{ij}\varepsilon_{ij} + s_{ij}\varepsilon_{ij}^P + \tau_{ijk}\gamma_{ijk} - U(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk})\}.$$

Letting $\tau_{ijk} = 0$, we then have

$$U(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}) \geqslant t_{ij}\varepsilon_{ij} + s_{ij}\varepsilon_{ij}^P - U^*(t_{ij}, s_{ij}, 0).$$

Therefore from (8) for all constant fields $t_{ij}^0$ and $s_{ij}^0$

$$U^{\text{eff}}(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}) \geqslant t_{ij}^0\varepsilon_{ij}^0 + s_{ij}^0\varepsilon_{ij}^{P0} - U_R^*(t_{ij}^0, s_{ij}^0), \tag{12}$$

where

$$U_R^*(t_{ij}^0, s_{ij}^0) = \int_\Omega U^*(t_{ij}^0, s_{ij}^0, 0)\, dx.$$

With the particular form (10) of the plastic density $V$, we have

$$U^*(t_{ij}^0, s_{ij}^0, 0) = \frac{t_{ij}^0 s_{ij}^0}{4\mu} + \frac{N}{N+1}\sigma^{-1/N}\left(\frac{3}{2}\right)^{\frac{(N+1)}{N}} e_0\left((t_{ij}^0 + s_{ij}^0)(t_{ij}^0 + s_{ij}^0)\right)^{\frac{(N+1)}{2N}}.$$

Thus

$$U_R^*(t_{ij}^0, s_{ij}^0) = t_{ij}^0 s_{ij}^0 \left(\sum_{r=1}^{M} p_r \mu_r^{-1}\right) \Big/ 4 +$$
$$\frac{N}{(N+1)}\left(\frac{3}{2}\right)^{\frac{(N+1)}{N}} e_0 \left(\sum_{r=1}^{M} p_r \sigma_r^{-1/N}\right)\left((t_{ij}^0 + s_{ij}^0)(t_{ij}^0 + s_{ij}^0)\right)^{(N+1)/(2N)}.$$

Taking the maximum with respect to $t_{ij}^0$ and $s_{ij}^0$ for the right-hand side of (12) we then have the Reuss bound

$$U^{\text{eff}}(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}) \geqslant \mu_R(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0})(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}) + \frac{\sigma_R e_0}{N+1}\left(\frac{E_{P0}}{e_0}\right)^{N+1}, \tag{13}$$

where

$$\mu_R = \left(\sum_{r=1}^{M} p_r \mu_r^{-1}\right)^{-1}, \qquad \sigma_R = \left(\sum_{r=1}^{M} p_r \sigma_r^{-1/N}\right)^{-N} \tag{14}$$

and $E_{P0}$ is the von Mises equivalent plastic strain.

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

## 4 Hashin–Shtrikman bound for linear materials

In this section, we deduce the Hashin–Shtrikman bound for a linear material. The result will be employed in the next section to deduce bounds for non-linear materials. The potential $V$ is now of the form $V = \frac{1}{2}bE_P^2$ and the energy potential function $U$ is

$$U\big(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\big) = \mu\big(\varepsilon_{ij} - \varepsilon_{ij}^P\big)\big(\varepsilon_{ij} - \varepsilon_{ij}^P\big) + \frac{1}{2}bE_P^2,$$

where the shear modulus $\mu$ and the stiffness $b$ are functions of $x$ and vary rapidly. We introduce a homogeneous linear comparison medium

$$U_0\big(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\big) = \mu_0\big(\varepsilon_{ij} - \varepsilon_{ij}^P\big)\big(\varepsilon_{ij} - \varepsilon_{ij}^P\big) + \frac{1}{2}b_0E_P^2,$$

where $\mu_0$ and $b_0$ are constants.

To deduce the upper Hashin–Shtrikman bound, we choose $\mu_0$ and $b_0$ such that $\mu_0 \geqslant \mu$ and $b_0 \geqslant b$. The dual formulation is defined as

$$(U - U_0)_*(\tau_{ij}, \alpha_{ij}, \beta_{ijk}) = \inf\big\{\tau_{ij}\varepsilon_{ij} + \alpha_{ij}\varepsilon_{ij}^P + \beta_{ijk}\gamma_{ijk} - (U - U_0)\big(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}\big)\big\},$$

where the infimum is taken over all the tensor fields $\varepsilon_{ij}$, $\varepsilon_{ij}^P$ and $\gamma_{ijk}$. It is simple to see that the infimum is attained at

$$\begin{aligned}
\tau_{ij} &= 2(\mu - \mu_0)(\varepsilon_{ij} - \varepsilon_{ij}^P),\\
\alpha_{ij} &= -2(\mu - \mu_0)(\varepsilon_{ij} - \varepsilon_{ij}^P) + \frac{2}{3}(b - b_0)\varepsilon_{ij}^P,\\
\beta_{ijk} &= \frac{2}{3}l^2(b - b_0)\gamma_{ijk}.
\end{aligned}$$

We then have

$$(U - U_0)_*(\tau_{ij}, \alpha_{ij}, \beta_{ijk}) = \frac{1}{4(\mu - \mu_0)}\tau_{ij}\tau_{ij} + \frac{3}{4(b - b_0)}(\alpha_{ij} + \tau_{ij})(\alpha_{ij} + \tau_{ij}) + \frac{3}{4(b - b_0)l^2}\beta_{ijk}\beta_{ijk}.$$

Therefore

$$\begin{aligned}
U^{\text{eff}}\big(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\big) \leqslant \inf \int_\Omega \Bigg( & U_0\big(\varepsilon_{ij}, \varepsilon_{ij}^P, \varepsilon_{ij,k}^P\big) + \tau_{ij}\varepsilon_{ij} + \alpha_{ij}\varepsilon_{ij}^P + \beta_{ijk}\varepsilon_{ij,k}^P - \frac{1}{4(\mu - \mu_0)}\tau_{ij}\tau_{ij}\\
& - \frac{3}{4(b - b_0)}(\alpha_{ij} + \tau_{ij})(\alpha_{ij} + \tau_{ij}) - \frac{3}{4(b - b_0)l^2}\beta_{ijk}\beta_{ijk}\Bigg)dx. \qquad (15)
\end{aligned}$$

The infimum is taken over all the displacement fields $u$ and plastic strain fields $\varepsilon^P$ such that $u_i = \varepsilon_{ij}^0 x_j$ on $\partial\Omega$ and $\langle\varepsilon^P\rangle = \varepsilon^{P0}$ ($\langle . \rangle$ denotes the average of a quantity in $\Omega$). Setting the first variations with respect to $u_i$ and $\varepsilon_{ij}^P$ to zero, we obtain ($p(x)$ is an equilibrating pressure field accounting for the incompressibility of the material)

$$\mu_0 u_{i,jj} - 2\mu_0\varepsilon_{ij,j}^P + \tau_{ij,j} - p_{,i} = 0, \qquad (16)$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

and

$$\frac{2}{3}b_0 l^2 \varepsilon^P_{ij,kk} - \beta_{ijk,k} - 2\mu_0\left(\varepsilon_{ij} - \varepsilon^0_{ij}\right) + \left(2\mu_0 + \frac{2}{3}b_0\right)\left(\varepsilon^P_{ij} - \varepsilon^{P0}_{ij}\right) + \alpha_{ij} - \langle\alpha_{ij}\rangle = 0. \tag{17}$$

Setting $\beta_{ijk} = 0$ will not decrease the value of the minimum in (15), so for simplicity we only consider $\beta_{ijk} = 0$ from now on. Using these two equations, it is straightforward to compute the value of the infimum in (15). We have

$$\begin{aligned} U^{\text{eff}}\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\right) \leqslant\ & U_0\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}, 0\right) + \frac{1}{2}\int_\Omega \tau_{ij}\left(\varepsilon_{ij} - \varepsilon^0_{ij}\right) dx + \langle\tau_{ij}\rangle\varepsilon^0_{ij} \\ & + \frac{1}{2}\int_\Omega \alpha_{ij}\left(\varepsilon^P_{ij} - \varepsilon^{P0}_{ij}\right) dx + \langle\alpha_{ij}\rangle\varepsilon^{P0}_{ij} \\ & - \int_\Omega \left(\frac{1}{4(\mu - \mu_0)}\tau_{ij}\tau_{ij} + \frac{3}{4(b - b_0)}(\alpha_{ij} + \tau_{ij})(\alpha_{ij} + \tau_{ij})\right) dx. \end{aligned} \tag{18}$$

Solving equations (16) and (17), we get

$$\left(\varepsilon^P_{ij} - \varepsilon^{P0}_{ij}\right)(x) = -\int \left(\Lambda_{ijkl}(x - x')(\alpha_{kl} - \langle\alpha_{kl}\rangle)(x') + \Theta_{ijkl}(x - x')(\tau_{kl} - \langle\tau_{kl}\rangle)(x')\right) dx',$$

and

$$\left(\varepsilon_{ij} - \varepsilon^0_{ij}\right)(x) = -\int \left(K_{ijkl}(x - x')(\tau_{kl} - \langle\tau_{kl}\rangle)(x') + H_{ijkl}(x - x')(\alpha_{kl} - \langle\alpha_{kl}\rangle)(x')\right) dx',$$

where supposing that the correlation length between the phases is small, the integrals over $\Omega$ have been replaced by integrals over the whole space and the tensors are replaced by those for the infinite space (see Fleck & Willis, 2004). The fourth-order tensors $\Lambda$, $\Theta$, $K$ and $H$ are defined in Appendix A. Substituting these into (18), and minimizing the right-hand side with respect to $\alpha_{kl}$ and $\tau_{kl}$, we get the upper bound

$$U^{\text{eff}}\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\right) \leqslant U_0\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}, 0\right) + \frac{1}{2}\langle\tau_{ij}\rangle\varepsilon^0_{ij} + \frac{1}{2}\langle\alpha_{ij}\rangle\varepsilon^{P0}_{ij}. \tag{19}$$

Now we consider an $M$-phase material. Let

$$\mu(x) = \sum_{r=1}^{M} \mu^{(r)}\chi^{(r)}(x), \qquad b(x) = \sum_{r=1}^{M} b^{(r)}\chi^{(r)}(x),$$

where $\chi^{(r)}(x)$ is the characteristic function of the region occupied by material $r$. We assume that the material is isotropic, so that the probability $p_r$ that a point is in material $r$ is independent of the position, and is the volume fraction of material $r$. The probability of finding the material $r$ at point $x$ and the material $s$ at point $x'$ is

$$p_{rs}(x - x') = \left\langle\chi^{(r)}(x)\chi^{(s)}(x')\right\rangle.$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

Making the Hashin–Shtrikman approximation

$$\alpha_{ij}(x) = \sum_{r=1}^{M} \alpha_{ij}^{(r)} \chi^{(r)}(x), \quad \tau_{ij}(x) = \sum_{r=1}^{M} \tau_{ij}^{(r)} \chi^{(r)}(x),$$

we get from (18)

$$\begin{aligned} U^{\text{eff}}\big(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}\big) \leqslant U_0\big(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}, 0\big) &+ \sum_{r=1}^{M} \tau_{ij}^{(r)} p_r \varepsilon_{ij}^0 + \sum_{r=1}^{M} \alpha_{ij}^{(r)} p_r \varepsilon_{ij}^{P0} - \sum_{r=1}^{M} \frac{1}{4(\mu^{(r)} - \mu_0)} \tau_{ij}^{(r)} \tau_{ij}^{(r)} p_r \\ &- \sum_{r=1}^{M} \frac{3}{4(b^{(r)} - b_0)} \big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big)\big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big) p_r \\ &- \frac{1}{2} \sum_{r=1}^{M} \sum_{s=1}^{M} \tau_{ij}^{(r)} \tau_{kl}^{(s)} \int K_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx \\ &- \frac{1}{2} \sum_{r=1}^{M} \sum_{s=1}^{M} \tau_{ij}^{(r)} \alpha_{kl}^{(s)} \int H_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx \\ &- \frac{1}{2} \sum_{r=1}^{M} \sum_{s=1}^{M} \alpha_{ij}^{(r)} \alpha_{kl}^{(s)} \int \Lambda_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx \\ &- \frac{1}{2} \sum_{r=1}^{M} \sum_{s=1}^{M} \alpha_{ij}^{(r)} \tau_{kl}^{(s)} \int \Theta_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx. \end{aligned}$$

Setting the first variations of $\alpha_{ij}^{(r)}$ and $\tau_{ij}^{(r)}$ to zero to minimize the right-hand side, we get

$$\begin{aligned} p_r \varepsilon_{ij}^0 = &\frac{1}{2(\mu^{(r)} - \mu_0)} \tau_{ij}^{(r)} p_r + \frac{3}{2(b^{(r)} - b_0)} \big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big) p_r \\ &+ \sum_{s=1}^{M} \tau_{kl}^{(s)} \int K_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx + \frac{1}{2} \sum_{s=1}^{M} \alpha_{kl}^{(s)} \int H_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx \\ &+ \frac{1}{2} \sum_{s=1}^{M} \alpha_{kl}^{(s)} \int \Theta_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx, \end{aligned}$$

and

$$\begin{aligned} p_r \varepsilon_{ij}^{P0} = &\frac{3}{2(b^{(r)} - b_0)} \big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big) p_r + \frac{1}{2} \sum_{s=1}^{M} \tau_{kl}^{(s)} \int H_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx \\ &+ \sum_{s=1}^{M} \alpha_{kl}^{(s)} \int \Lambda_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx + \frac{1}{2} \sum_{s=1}^{M} \tau_{kl}^{(s)} \int \Theta_{ijkl}(x)(p_{rs}(x) - p_r p_s)\, dx. \end{aligned}$$

Solving these equations, we get the values of $\alpha_{ij}^{(r)}$ and $\tau_{ij}^{(r)}$ that give the optimal upper bound.

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

Now we consider the case of a two-phase composite. Since the material is isotropic

$$p_{11}(x) - p_1 p_1 = p_{22}(x) - p_2 p_2 = -(p_{12}(x) - p_1 p_2) = -(p_{21}(x) - p_1 p_2) = p_1 p_2 h(r),$$

where $r = |x|$. The fourth-order tensors $Q_{ijkl}$, $R_{ijkl}$, $S_{ijkl}$ and $T_{ijkl}$ are defined as

$$\begin{aligned}
p_1 p_2 Q_{ijkl} &= \int \Lambda_{ijkl}(x)(p_{22}(x) - p_2 p_2)\,dx = p_1 p_2 \int \Lambda_{ijkl}(x) h(|x|)\,dx,\\
p_1 p_2 R_{ijkl} &= \int \Theta_{ijkl}(x)(p_{22}(x) - p_2 p_2)\,dx = p_1 p_2 \int \Theta_{ijkl}(x) h(|x|)\,dx,\\
p_1 p_2 S_{ijkl} &= \int H_{ijkl}(x)(p_{22}(x) - p_2 p_2)\,dx = p_1 p_2 \int H_{ijkl}(x) h(|x|)\,dx,\\
p_1 p_2 T_{ijkl} &= \int K_{ijkl}(x)(p_{22}(x) - p_2 p_2)\,dx = p_1 p_2 \int K_{ijkl}(x) h(|x|)\,dx;
\end{aligned}$$

they are isotropic and incompressible. In the concise Hill's notations, these tensors can be written as

$$T = (0, 2\mu_T), \quad S = (0, 2\mu_S), R = (0, 2\mu_R), Q = (0, 2\mu_Q).$$

The equations for $\tau_{ij}^{(r)}$ and $\alpha_{ij}^{(r)}$ then have the form

$$\begin{aligned}
\varepsilon_{ij}^0 &= \frac{\tau_{ij}^{(r)}}{2\big(\mu^{(r)} - \mu_0\big)} + \frac{3\big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big)}{2\big(b^{(r)} - b_0\big)} + T_{ijkl}\big(\tau_{kl}^{(r)} - \langle\tau_{kl}\rangle\big)\\
&\quad + \frac{1}{2} S_{ijkl}\big(\alpha_{kl}^{(r)} - \langle\alpha_{kl}\rangle\big) + \frac{1}{2} R_{ijkl}\big(\alpha_{kl}^{(r)} - \langle\alpha_{kl}\rangle\big),\\
\varepsilon_{ij}^{P0} &= \frac{3\big(\alpha_{ij}^{(r)} + \tau_{ij}^{(r)}\big)}{2\big(b^{(r)} - b_0\big)} + \frac{1}{2} S_{ijkl}\big(\tau_{kl}^{(r)} - \langle\tau_{kl}\rangle\big)\\
&\quad + Q_{ijkl}\big(\alpha_{kl}^{(r)} - \langle\alpha_{kl}\rangle\big) + \frac{1}{2} R_{ijkl}\big(\tau_{kl}^{(r)} - \langle\tau_{kl}\rangle\big).
\end{aligned}$$

In Hill's notation, these equations are

$$\begin{aligned}
\big(0, \bar{a}^{(r)}\big)\tau^{(r)} + \big(0, \bar{b}^{(r)}\big)\alpha^{(r)} &= (0,1)\varepsilon^0 + (0, 2\mu_T)\langle\tau\rangle + (0, \mu_S + \mu_R)\langle\alpha\rangle,\\
\big(0, \bar{b}^{(r)}\big)\tau^{(r)} + (0, \bar{c}^r)\alpha^{(r)} &= (0,1)\varepsilon^{P0} + (0, \mu_S + \mu_R)\langle\tau\rangle + (0, 2\mu_Q)\langle\alpha\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\bar{a}^{(r)} &= \frac{1}{2\big(\mu^{(r)} - \mu_0\big)} + \frac{3}{2\big(b^{(r)} - b_0\big)} + 2\mu_T,\\
\bar{b}^{(r)} &= \frac{3}{2(b^{(r)} - b_0)} + \mu_S + \mu_R,\\
\bar{c}^{(r)} &= \frac{3}{2\big(b^{(r)} - b_0\big)} + 2\mu_Q.
\end{aligned}$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

Solving them we get

$$(0,1)\alpha^{(r)} = \left(0, \frac{\bar{b}^{(r)}}{(\bar{b}^{(r)})^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\varepsilon^0 - \left(0, \frac{\bar{a}^{(r)}}{(\bar{b}^{(r)})^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\varepsilon^{P0}$$
$$+ \left(0, \frac{2\mu_T\bar{b}^{(r)} - (\mu_S+\mu_R)\bar{a}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\langle\tau\rangle + \left(0, \frac{(\mu_S+\mu_R)\bar{b}^{(r)} - 2\mu_Q\bar{a}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\langle\alpha\rangle,$$
$$(0,1)\tau^{(r)} = -\left(0, \frac{\bar{c}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\varepsilon^0 + \left(0, \frac{\bar{b}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\varepsilon^{P0}$$
$$+ \left(0, \frac{-2\mu_T\bar{c}^{(r)} + (\mu_S+\mu_R)\bar{b}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\langle\tau\rangle + \left(0, \frac{2\mu_Q\bar{b}^{(r)} - (\mu_S+\mu_R)\bar{c}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}}\right)\langle\alpha\rangle.$$

Letting

$$A = \left\langle \frac{\bar{a}^{(r)}}{(\bar{b}^{(r)})^2 - \bar{a}^{(r)}\bar{c}^{(r)}} \right\rangle, \quad B = \left\langle \frac{\bar{b}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}} \right\rangle, \quad C = \left\langle \frac{\bar{c}^{(r)}}{\left(\bar{b}^{(r)}\right)^2 - \bar{a}^{(r)}\bar{c}^{(r)}} \right\rangle, \tag{20}$$

the mean values of $\tau$ and $\alpha$ satisfy

$$(0, 2\mu_T B - (\mu_S+\mu_R)A)\langle\tau\rangle + (0, (\mu_S+\mu_R)B - 2\mu_Q A - 1)\langle\alpha\rangle = (0,A)\varepsilon^{P0} - (0,B)\varepsilon^0,$$
$$(0, (\mu_S+\mu_R)B - 2\mu_T C - 1)\langle\tau\rangle + (0, 2\mu_Q B - (\mu_S+\mu_R)C)\langle\alpha\rangle = (0,C)\varepsilon^0 - (0,B)\varepsilon^{P0}.$$

Solving these equations, we get

$$(0,1)\langle\alpha\rangle = \left((B^2-AC)((\mu_S+\mu_R)^2 - 4\mu_T\mu_Q) - 2B(\mu_S+\mu_R) + 2A\mu_Q + 2C\mu_T + 1\right)^{-1}$$
$$\times ((0, 2\mu_T(B^2-AC) - A)\varepsilon^{P0} - (0, (\mu_S+\mu_R)(B^2-AC) - B)\varepsilon^0)$$
$$(0,1)\langle\tau\rangle = \left((B^2-AC)((\mu_S+\mu_R)^2 - 4\mu_T\mu_Q) - 2B(\mu_S+\mu_R) + 2A\mu_Q + 2C\mu_T + 1\right)^{-1}$$
$$\times ((0, 2\mu_Q(B^2-AC) - C)\varepsilon^0 - (0, (\mu_S+\mu_R)(B^2-AC) - B)\varepsilon^{P0}).$$

Therefore from (19)

$$U^{\text{eff}}\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\right) \leqslant \mu_0\left(\varepsilon^0_{ij} - \varepsilon^{P0}_{ij}\right)\left(\varepsilon^0_{ij} - \varepsilon^{P0}_{ij}\right) + \frac{1}{3}b_0\varepsilon^{P0}_{ij}\varepsilon^{P0}_{ij} + (1/2)((B^2-AC)((\mu_S+\mu_R)^2$$
$$- 4\mu_T\mu_Q) - 2B(\mu_S+\mu_R) + 2A\mu_Q + 2\mu_T C + 1)^{-1}$$
$$\times \left((2\mu_T(B^2-AC) - A)\varepsilon^{P0}_{ij}\varepsilon^{P0}_{ij} + (2\mu_Q(B^2-AC) - C)\varepsilon^0_{ij}\varepsilon^0_{ij}\right.$$
$$\left. - 2((\mu_S+\mu_R)(B^2-AC) - B)\varepsilon^0_{ij}\varepsilon^{P0}_{ij}\right). \tag{21}$$

When the function $h(r) = e^{-r/a}$ (see Appendix B)

$$\mu_Q = \frac{3}{10b_0(1+l/a)^2} + \frac{9}{20(3\mu_0+b_0)(1+(l/a)(b_0/(3\mu_0+b_0)^{1/2}))^2},$$
$$\mu_R = \mu_S = \frac{3}{10b_0(1+l/a)^2},$$
$$\mu_T = \frac{1}{10\mu_0} + \frac{3}{10b_0(1+l/a)^2}.$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

For the case of an elastically homogeneous material, i.e. $\mu_1 = \mu_2 = \mu_0$, $A = -\langle 1/\bar{c}^{(r)} \rangle$, $B = C = 0$. We recover the upper bound established by Fleck and Willis (2004).

The lower Hashin–Shtrikman bound has exactly the same structure, but the constants $b_0$ and $\mu_0$ are chosen so that $b_0 \leqslant b_r$ and $\mu_0 \leqslant \mu_r$ for all $r$.

## 5 Bounds for a non-linear material

Now we use the Hashin–Shtrikman bounds for linear materials in the previous section to deduce bounds for non-linear materials. We consider the strain-hardening characteristic given by (10); the potential $V$ is of the form

$$V = \sum_{r=1}^{M} \frac{\sigma_r e_0}{(N+1)} \left( \frac{E_P}{e_0} \right)^{N+1} \chi^{(r)}(x),$$

where the hardening index $N$ is the same in all phases. To deduce an upper bound, we use the approach of Ponte-Castañeda and De Botton (1992) where a comparison linear material is chosen so that the elastic part has the same structure as the non-linear material, but the plastic part is linear; its energy potential is

$$U_L\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\big) = \mu\big(\varepsilon_{ij} - \varepsilon^P_{ij}\big)\big(\varepsilon_{ij} - \varepsilon^P_{ij}\big) + \frac{1}{2} b E_P^2,$$

where $\mu(x) = \sum_{r=1}^{M} \mu^{(r)} \chi^{(r)}(x)$ and $b(x) = \sum_{r=1}^{M} b^{(r)} \chi^{(r)}(x)$, $\mu^{(r)}$ are the elastic moduli of the component materials and the values of $b^{(r)}$ will be chosen later. From (8)

$$U^{\text{eff}}\big(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\big) \leqslant \inf \int_\Omega U_L\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\big)\, dx + \sup \int_\Omega \big(U\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \gamma_{ijk}\big) - U_L\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \gamma_{ijk}\big)\big)\, dx,$$

where the infimum is taken over $u$ and $\varepsilon^P$ such that $u_i = \varepsilon^0_{ij} x_j$ on $\partial\Omega$ and $\langle \varepsilon^P \rangle = \varepsilon^{P0}$ and the supremum is taken over all the elastic strain fields $\varepsilon_{ij}$, all the plastic strain fields $\varepsilon^P_{ij}$ and all the fields $\gamma_{ijk}$. The first term on the right-hand side is $U_L^{\text{eff}}(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij})$ which admits the linear upper bound found in the previous section; we denote it as $U_L^{HS+}(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij})$. We observe that

$$\begin{aligned} U\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\big) - U_L\big(\varepsilon_{ij}, \varepsilon^P_{ij}, \varepsilon^P_{ij,k}\big) &= \sum_{r=1}^{M} \chi^{(r)}(x) \left( \frac{\sigma_r e_0}{(N+1)} \left( \frac{E_P}{e_0} \right)^{N+1} - \frac{1}{2} b_r E_P^2 \right) \\ &\leqslant \sum_{r=1}^{M} \chi^{(r)}(x) \left( \frac{\sigma_r e_0}{(N+1)} \left( \frac{E_P^{(r)}}{e_0} \right)^{N+1} - \frac{1}{2} b_r \big(E_P^{(r)}\big)^2 \right), \end{aligned}$$

where $E_P^{(r)} = (b_r/\sigma_r)^{1/(N-1)} e_0^{N/(N-1)}$. Thus

$$U^{\text{eff}}\big(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\big) \leqslant \inf_{b^{(r)} \leqslant b_0} \left\{ U_L^{HS+}\big(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\big) + \sum_{r=1}^{M} p_r \left( \frac{\sigma_r e_0}{(N+1)} \left( \frac{E_P^{(r)}}{e_0} \right)^{N+1} - \frac{1}{2} b_r \big(E_P^{(r)}\big)^2 \right) \right\}. \tag{22}$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

We now restrict our consideration to a two-phase composite where we assume that $\mu_1 \leqslant \mu_2$. We then choose $b_1 \leqslant b_2$. Letting $b_0 = b_2$ and $\gamma = b_1/b_0$, the upper bound (22) is then written as

$$U^{\text{eff}}\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\right) \leqslant \inf_{0\leqslant\gamma\leqslant 1} \inf_{b_0} U^+,$$

where

$$U^+ = U_L^{HS+}\left(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij}\right)) + \frac{1-N}{2(1+N)} b_0^{\frac{N+1}{N-1}} e_0^{\frac{2N}{N-1}} \left(p_1 \sigma_1^{\frac{2}{1-N}} \gamma^{\frac{N+1}{N-1}} + p_2 \sigma_2^{\frac{2}{1-N}}\right).$$

The upper bound for the energy function $w^{\text{eff}}(\varepsilon^0_{ij})$ is determined by

$$\inf_{\varepsilon^{P0}_{ij}} \inf_{0\leqslant\gamma\leqslant 1} \inf_{b_0} U^+. \tag{23}$$

As $U_L^{HS^+}(\varepsilon^0_{ij}, \varepsilon^{P0}_{ij})$ is a quadratic form of $\varepsilon^0_{ij}$ and $\varepsilon^{P0}_{ij}$, the value of $\varepsilon^{P0}_{ij}$ that minimizes the bound depends on $\varepsilon^0$ linearly and can be found from (21). The upper bound for $w^{\text{eff}}(\varepsilon^0_{ij})$ has the form

$$\inf_{0\leqslant\gamma\leqslant 1} \inf_{b_0} \left\{ \eta \varepsilon^0_{ij} \varepsilon^0_{ij} + \frac{1-N}{2(1+N)} b_0^{\frac{N+1}{N-1}} e_0^{\frac{2N}{N-1}} \left(p_1 \sigma_1^{\frac{2}{1-N}} \gamma^{\frac{N+1}{N-1}} + p_2 \sigma_2^{\frac{2}{1-N}}\right) \right\}, \tag{24}$$

where $\eta$ can be written in terms of $\mu_0$, $b_0$, $A$, $B$, $C$, $\mu_Q$, $\mu_R$, $\mu_S$ and $\mu_T$ in (21). We can approximate the stress by differentiating the bound (24) with respect to $\varepsilon^0_{ij}$. At the optimal values of $b_0$ and $\gamma$, we find that

$$\sigma_{ij} = 2\eta \varepsilon^0_{ij}. \tag{25}$$

Now we consider the range of a small total strain $|\varepsilon^0|$. We expect the optimal value of $b_0$ to be much larger than the elastic stiffness $\mu_0$, i.e. $\mu_0/b_0 \ll 1$ so that the optimal value of $\varepsilon^{P0}_{ij}$ in (23) is insignificant in comparison to the total strain. After some computation, we find that the optimal value of $\varepsilon^{P0}_{ij} = O(\mu_0/b_0)\varepsilon^0_{ij}$. The upper bound (24) then becomes

$$\inf_{b_0} \inf_{0\leqslant\gamma\leqslant 1} \frac{\mu_0(3p_2 + (3p_1+2)\delta)}{3+2p_1+2\delta p_2} \varepsilon^0_{ij}\varepsilon^0_{ij} + 3\varepsilon^0_{ij}\varepsilon^0_{ij}\mu_0 \left(-M\frac{\mu_0}{b_0} + O\left(\frac{\mu_0}{b_0}\right)^2\right)$$
$$+ b_0^{\frac{N+1}{N-1}} \frac{1-N}{2(1+N)} e_0^{\frac{2N}{N-1}} \left(p_1 \sigma_1^{2/(1-N)} \gamma^{(N+1)/(N-1)} + p_2 \sigma_2^{2/(1-N)}\right),$$

where $\delta = \mu_1/\mu_2$ and $M$ is a positive constant. The expression for $M$ is very complicated and has no role in the development that follows so we do not present it here (it can be found by using the software Mathematica). We now compute $b_0$ in terms of the strain $\varepsilon^0_{ij}$. Expecting that $(\mu_0/b_0)^2 \ll \mu_0/b_0$, we seek for $b_0$ such that $\varepsilon^0_{ij}\varepsilon^0_{ij}/b_0$ is of the same order as $b_0^{(N+1)/(N-1)}$. We thus find the value of $b_0$ that minimizes

$$-3M\varepsilon^0_{ij}\varepsilon^0_{ij}\frac{\mu_0^2}{b_0} + b_0^{\frac{N+1}{N-1}} \frac{1-N}{2(1+N)} e_0^{\frac{2N}{N-1}} \left(p_1 \sigma_1^{\frac{2}{1-N}} \gamma^{\frac{N+1}{N-1}} + p_2 \sigma_2^{\frac{2}{1-N}}\right).$$

The upper bound (24) is then written as

$$\frac{\mu_0(3p_2 + (3p_1+2)\delta)}{3+2p_1+2\delta p_2} \varepsilon^0_{ij}\varepsilon^0_{ij} - \left(6\varepsilon^0_{ij}\varepsilon^0_{ij}\mu_0^2 M\right)^{\frac{1+N}{2N}} e_0 \left(p_1 \sigma_1^{\frac{2}{1-N}} \gamma^{\frac{N+1}{N-1}} + p_2 \sigma_2^{\frac{2}{1-N}}\right)^{\frac{N-1}{2N}}.$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

Thus when the total strain is small, the upper bound (24) behaves like

$$\frac{\mu_0(3p_2+(3p_1+2)\delta)}{3+2p_1+2\delta p_2}\varepsilon^0_{ij}\varepsilon^0_{ij}.$$

This is the Hashin–Shtrikman upper bound for a two-phase incompressible linear elastic materials with shear moduli $\mu_1$ and $\mu_2$ (see, e.g. Milton 2002). When the total strain is small, differentiating this bound, an approximation for the stress is given by

$$\sigma_{ij}=2\frac{\mu_0(3p_2+(3p_1+2)\delta)}{3+2p_1+2\delta p_2}\varepsilon^0_{ij}-C\left(\varepsilon^0_{kl}\varepsilon^0_{kl}\right)^{(1-N)/(2N)}\varepsilon^0_{ij},$$

where $C$ is a positive constant. The elastic strain is written as

$$\varepsilon^e_{ij}=\varepsilon^0_{ij}-\varepsilon^{P0}_{ij}=\left(1+O\left(\frac{\mu_0}{b_0}\right)\right)\varepsilon^0_{ij},$$

so we can write the stress in terms of the elastic strain as

$$\sigma_{ij}=2\frac{\mu_0(3p_2+(3p_1+2)\delta)}{3+2p_1+2\delta p_2}\left(1+O\left(\frac{\mu_0}{b_0}\right)\right)\varepsilon^e_{ij}$$
$$-C\left(\left(1+O\left(\frac{\mu_0}{b_0}\right)\right)\varepsilon^e_{kl}\varepsilon^e_{kl}\right)^{(1-N)/(2N)}\left(1+O\left(\frac{\mu_0}{b_0}\right)\right)\varepsilon^e_{ij}.$$

As the optimal value of $b_0$ is of the order $(\varepsilon^0_{ij}\varepsilon^0_{ij})^{(N-1)/(2N)}$ so $\mu_0/b_0=O(1)(\varepsilon^e_{kl}\varepsilon^e_{kl})^{(1-N)/(2N)}$. The stress is

$$\sigma_{ij}=2\frac{\mu_0(3p_2+(3p_1+2)\delta)}{3+2p_1+2\delta p_2}\varepsilon^e_{ij}+O(1)\left(\varepsilon^e_{kl}\varepsilon^e_{kl}\right)^{\frac{1-N}{2N}}\varepsilon^e_{ij}.$$

In the case of a uniaxial strain, this relation has the form

$$\sigma=2\frac{\mu_0(3p_2+(3p_1+2)\delta)}{3+2p_1+2\delta p_2}\varepsilon^e+O(1)(\varepsilon^e)^{1/N}.$$

We now consider the plastic range. Deep in this range, we expect that the optimal value of $b_0$ is of the order of the secant modulus (see Fleck & Willis, 2004) so that an asymptotic expansion in terms of $b_0/\mu_0$ can be found. The constants $A$, $B$ and $C$ in (20) can be written as

$$B=-p_1\frac{2(\delta-1)}{\delta}\mu_0-b_0p_1\frac{(1-\delta)(2\delta+3)}{\delta^2}\frac{2}{25}\frac{1}{1/(\gamma-1)+4\mu^*p_2/3}+O\left(\left(\frac{b_0}{\mu_0}\right)^2\right),$$
$$A=B-b_0p_1\frac{2\delta+3}{\delta}\frac{2}{15}\frac{1}{1/(\gamma-1)+4\mu^*p_2/3},$$

and

$$C=B+b_0p_1\frac{1-\delta}{\delta}\frac{2}{5}\frac{1}{1/(\gamma-1)+4\mu^*p_2/3},$$

where

$$\mu^*=\frac{3}{10(1+l/a)^2}.$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

To the first order in $b_0/\mu_0$, the upper bound (22) can be written as

$$\inf_{0\leqslant\gamma\leqslant 1}\inf_{b_0}\left(\frac{\delta\mu_0}{p_1+\delta p_2}+\frac{2\sqrt{3}(\delta-1)^2(l/a)p_1p_2\mu_0^{1/2}b_0^{1/2}}{5(p_1+\delta p_2)^2}\right)\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)+\frac{1}{3}b_0\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}$$
$$+\frac{b_0}{\left(1/(\gamma-1)+\frac{4}{3}p_2\mu^*\right)(p_1(1-\delta)/\delta+1)}\times\left(\frac{1}{15}p_1\left(\frac{2\delta+3}{\delta}+2p_1\frac{1-\delta}{\delta}\right)\varepsilon_{ij}^{P0}\varepsilon_{ij}^{P0}\right.$$
$$\left.-\frac{1}{5}p_1p_2\frac{1-\delta}{\delta}\varepsilon_{ij}^0\varepsilon_{ij}^0\right)+\frac{1-N}{2(1+N)}b_0^{\frac{N+1}{N-1}}e_0\left(p_1\sigma_1\left(\frac{\sigma_1}{e_0\gamma}\right)^{\frac{(1+N)}{1-N}}\right.$$
$$\left.+p_2\sigma_2\left(\frac{\sigma_2}{e_0}\right)^{\frac{1+N}{1-N}}\right)+b_0O(b_0/\mu_0). \tag{26}$$

The upper bound for the energy function $w^{\text{eff}}(\varepsilon_{ij}^0)$ is found by taking the infimum of the Hashin–Shtrikman upper bound with respect to $\varepsilon_{ij}^{P0}$, $b_0$ and $\gamma$. The value of $\varepsilon_{ij}^{P0}$ that minimizes the Hashin–Shtrikman upper bound (26) is

$$\varepsilon_{ij}^{P0}=\left(\frac{\delta\mu_0}{p_1+\delta p_2}+\frac{2\sqrt{3}(\delta-1)^2(l/a)p_1p_2\mu_0^{1/2}b_0^{1/2}}{5(p_1+\delta p_2)^2}\right)$$
$$\times\left(\frac{\delta\mu_0}{p_1+\delta p_2}+b_0^{1/2}\frac{2\sqrt{3}(\delta-1)^2(l/a)\mu_0^{1/2}p_1p_2}{5(p_1+\delta p_2)^2}\right.$$
$$\left.+b_0\left(\frac{1}{3}+\frac{(1/15)(\gamma-1)p_1(2\delta p_2+2p_1+3)}{(1+(4/3)p_2\mu^*(\gamma-1))(p_1+\delta p_2)}\right)\right)^{-1}\varepsilon_{ij}^0. \tag{27}$$

To the first order in $b_0/\mu_0$, this expression is

$$\varepsilon_{ij}^{P0}=\varepsilon_{ij}^0\left(1-\frac{b_0}{\mu_0}\frac{5(3+4p_2\mu^*(\gamma-1))(p_1+\delta p_2)+3(\gamma-1)p_1(2\delta p_2+3+2p_1)}{15(3+4p_2\mu^*(\gamma-1))\delta}\right). \tag{28}$$

Substituting this into (26), we get

$$\inf_{0\leqslant\gamma\leqslant 1}\inf_{b_0}b_0\varepsilon_{ij}^0\varepsilon_{ij}^0\left(\frac{1}{3}+\frac{(\gamma-1)p_1}{3+4(\gamma-1)\mu^*p_2}\right)$$
$$+\frac{1-N}{2(1+N)}b_0^{\frac{N+1}{N-1}}e_0\left(p_1\sigma_1\left(\frac{\sigma_1}{e_0\gamma}\right)^{(1+N)/(1-N)}+p_2\sigma_2\left(\frac{\sigma_2}{e_0}\right)^{(1+N)/(1-N)}\right)+b_0O(b_0/\mu_0).$$

We approximate the Hashin–Shtrikman bound by eliminating the term $b_0O(b_0/\mu_0)$. The infimum is attained when

$$b_0=\left(2\frac{\varepsilon_{ij}^0\varepsilon_{ij}^0(1/3+(\gamma-1)p_1/(3+4(\gamma-1)\mu^*p_2))}{e_0(p_1\sigma_1(\sigma_1/(e_0\gamma))^{(1+N)/(1-N)}+p_2\sigma_2(\sigma_2/e_0)^{(1+N)/(1-N)})}\right)^{(-1+N)/2}. \tag{29}$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

At this value of $b_0$, the approximated Hashin–Shtrikman upper bound equals

$$\frac{2}{1+N}\left(\frac{e_0}{2}\right)^{(1-N)/2}\left(\varepsilon_{ij}^0\varepsilon_{ij}^0\right)^{(1+N)/2}\left(\frac{1}{3}+\frac{(\gamma-1)p_1}{3+4(\gamma-1)\mu^* p_2}\right)^{(N+1)/2}$$
$$\times\left(p_1\sigma_1(\sigma_1/(e_0\gamma))^{(1+N)/(1-N)}+p_2\sigma_2(\sigma_2/e_0)^{(1+N)/(1-N)}\right)^{(1-N)/2}.$$

Thus, $\gamma$ minimizes

$$\left(\frac{1}{3}+\frac{(\gamma-1)p_1}{3+4(\gamma-1)\mu^* p_2}\right)^{(N+1)/2}\left(p_1\sigma_1(\sigma_1/(e_0\gamma))^{(1+N)/(1-N)}+p_2\sigma_2(\sigma_2/e_0)^{(1+N)/(1-N)}\right)^{(1-N)/2}.$$

Differentiating this bound, the stress is determined from (26) by

$$\sigma_{ij}=2\left(\frac{\delta\mu_0}{p_1+\delta p_2}+\frac{2\sqrt{3}(\delta-1)^2(l/a)p_1p_2\mu_0^{1/2}b_0^{1/2}}{5(p_1+\delta p_2)^2}\right)\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)$$
$$-\frac{b_0(\gamma-1)}{(1+4p_2\mu^*(\gamma-1)/3)(p_1+\delta p_2)}\frac{2}{5}p_1p_2(1-\delta)\varepsilon_{ij}^0.$$

From (28), writing $b_0\varepsilon_{ij}^0$ in terms of $\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}$, we get approximately

$$\sigma_{ij}=2\mu^+\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right),$$

where

$$\mu^+=\frac{\delta\mu_0}{p_1+p_2\delta}\left(1+\frac{3p_1p_2(1-\gamma)(1-\delta)}{5(1+(4/3)p_2\mu^*(\gamma-1))(p_1+p_2\delta)+p_1(2\delta p_2+3+2p_1)(\gamma-1)}\right). \quad (30)$$

On the other hand, from (28), writing $(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0})$ in terms of $b_0\varepsilon_{ij}^0$ we get

$$\sigma_{ij}=\left(\frac{2}{3}\right)^{\frac{N+1}{N}}e_0^{-N}\sigma^+\left(\varepsilon_{kl}^0\varepsilon_{kl}^0\right)^{\frac{N-1}{2}}\varepsilon_{ij}^0,$$

where

$$\sigma^+=\inf_{0\leqslant\gamma\leqslant1}\left\{\left(1+\frac{p_1}{1/(\gamma-1)+(4/3)p_2\mu^*}\right)^{(1+N)/2}\right.$$
$$\left.\times\left(p_1\gamma^{(N+1)/(N-1)}\sigma_1^{2/(1-N)}+p_2\sigma_2^{2/(1-N)}\right)^{(1-N)/2}\right\}. \quad (31)$$

The Hashin–Shtrikman upper bound for the energy potential can be written in an asymptotically equivalent form as

$$\mu^+\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)\left(\varepsilon_{ij}^0-\varepsilon_{ij}^{P0}\right)+\frac{\sigma^+e_0}{N+1}\left(\frac{E_{P0}}{e_0}\right)^{N+1}. \quad (32)$$

When the total strain is small, the stress and the elastic strain follow the linear Hashin–Shtrikman relation while deep in the plastic range, they asymptote a different linear relation. As we expect the stress–elastic strain relation to vary smoothly between these

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

two extreme regimes, this shows that half-way between these two ranges, the stress–elastic strain is non-linear.

The lower Hashin–Shtrikman bound for the linear problem is of exactly the same form as the upper bound except that the roles of $\mu_1$ and $\mu_2$, of $b_1$ and $b_2$ and of $p_1$ and $p_2$ interchange. However for the non-linear composite, we cannot deduce a lower bound using the same approach. As $U = U_L + (U - U_L)$, for a lower bound, we should have

$$U^{\text{eff}} \geqslant \inf \int_\Omega U_L + \inf \int_\Omega (U - U_L),$$

which gives the trivial $-\infty$ bound. However, we can obtain the lower estimate

$$U_L^{HS-}(\varepsilon_{ij}^0, \varepsilon_{ij}^{P0}) + \sup \int_\Omega \big(U(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk}) - U_L(\varepsilon_{ij}, \varepsilon_{ij}^P, \gamma_{ijk})\big)\, dx,$$

where the supremum is taken over all the fields $\varepsilon_{ij}$, $\varepsilon_{ij}^P$ and $\gamma_{ijk}$, and $U_L^{HS-}$ is the lower Hashin–Shtrikman linear bound. The lower estimate is

$$\begin{aligned}
&\inf_{0\leqslant w\leqslant 1} \inf_{b_0} \left( \frac{\mu_1}{p_1 + p_2\delta} + \frac{2\sqrt{3}(\delta-1)^2 (l/a) p_1 p_2 \mu_1^{1/2} b_1^{1/2}}{5(p_1 + \delta p_2)} \right) (\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0})(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}) \\
&\quad + \frac{1}{3} b_0 \varepsilon_{ij}^{P0} \varepsilon_{ij}^{P0} + \frac{b_0}{(w/(1-w) + (4/3)p_1\mu^*)(p_2(\delta-1)+1)} \\
&\quad \times \left( \frac{1}{15} p_2 (2 + 3\delta + 2p_2(\delta-1)) \varepsilon_{ij}^{P0} \varepsilon_{ij}^{P0} - \frac{1}{5} p_1 p_2 (\delta-1) \varepsilon_{ij}^0 \varepsilon_{ij}^0 \right) \\
&\quad + \frac{1-N}{2(1+N)} b_0^{\frac{N+1}{N-1}} e_0 \left( p_1 \sigma_1 \left( \frac{\sigma_1}{e_0} \right)^{\frac{1+N}{1-N}} + p_2 \sigma_2 \left( \frac{\sigma_2 w}{e_0} \right)^{\frac{1+N}{1-N}} \right),
\end{aligned}$$

where $b_0 = b_1$ and $\mu_1 = \delta\mu_2$ ($\delta < 1$) assuming that $b_0/\mu_0$ is small.

In the asymptotic regime when $b_0/\mu_0 \to 0$, the Hashin–Shtrikman lower estimate can be written as

$$\mu_- (\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0})(\varepsilon_{ij}^0 - \varepsilon_{ij}^{P0}) + \frac{\sigma_- e_0}{N+1} \left( \frac{E_{P0}}{e_0} \right)^{N+1}, \tag{33}$$

where

$$\begin{aligned}
\sigma_- = \inf_{0\leqslant w\leqslant 1} \Bigg\{ &\left( 1 + \frac{p_2}{w/(1-w) + (4/3)p_1\mu^*} \right)^{(1+N)/2} \\
&\times \left( p_2 w^{(N+1)/(1-N)} \sigma_2^{2/(1-N)} + p_1 \sigma_1^{2/(1-N)} \right)^{(1-N)/2} \Bigg\},
\end{aligned} \tag{34}$$

and

$$\mu_- = \frac{\mu_1}{p_1 + p_2\delta} \left( 1 + \frac{3p_1p_2(1-1/w)(1-1/\delta)}{5(1+(4/3)p_1\mu^*(1/w-1))(p_2+p_1/\delta)+p_2(2p_1/\delta+3+2p_2)(1/w-1)} \right), \tag{35}$$

where $w$ is the minimizer of (34).

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

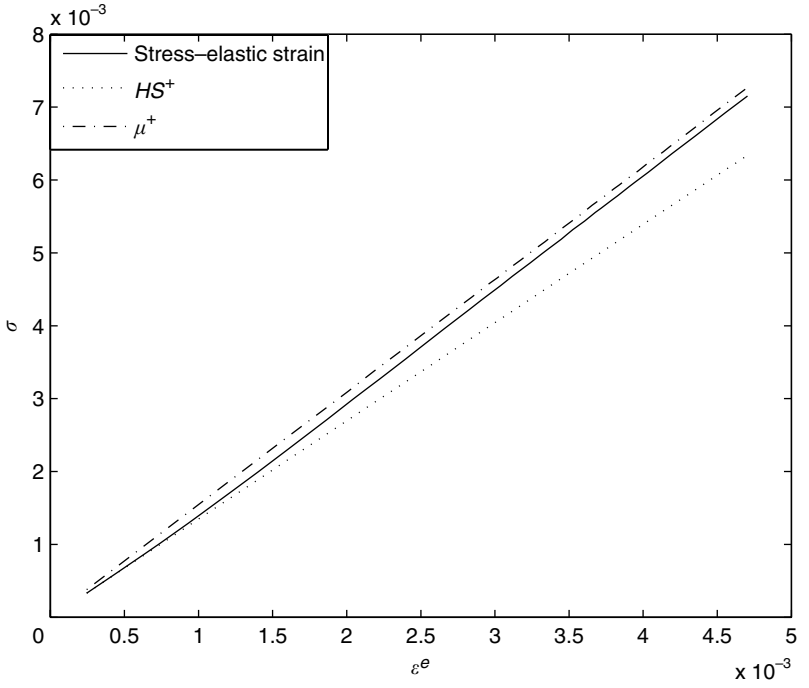



FIGURE 1. Stress versus elastic strain for $\sigma_1 = 0.01$, $\sigma_2 = 0.05$, $l/a = 0.1$.

## 6 Numerical results

In this section, we present some numerical calculations to explore the dependence of the stress–elastic strain relation on the physical parameters. As predicted in the previous section, when the total strain is small, the stress–elastic strain relation follows the upper Hashin–Shtrikman bound and deep in the plastic range, it follows another linear relation. In the range that is of interest, i.e. in the middle of these two regimes, it is thus predicted that this relation is non-linear. The Hashin–Shtrikman upper bound for two linear incompressible elastic materials with shear moduli $\mu_1$ and $\mu_2$ ($\mu_1 < \mu_2$) and with the volume fractions $p_1$ and $p_2$, respectively, is

$$\left(\mu_2 + \frac{5p_1\mu_2(\mu_1 - \mu_2)}{5\mu_2 + 2p_2(\mu_1 - \mu_2)}\right)\varepsilon^0_{ij}\varepsilon^0_{ij} = \mu_{HS^+}\varepsilon^0_{ij}\varepsilon^0_{ij}. \tag{36}$$

In the following figures, we plot the stress versus the elastic strain where the stress is taken from (25) and the elastic strain is $\varepsilon^0_{ij} - \varepsilon^{P0}_{ij}$ for the optimal value of $\varepsilon^{P0}_{ij}$ that minimizes the bound (23). In figure 1, we plot the stress–elastic strain relation for the composite with $N = 0.3$, $\mu_1 = 0.5$, $\mu_2 = 1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.05$, $p_1 = 0.6$, $p_2 = 0.4$. The material length scale and the correlation length satisfies $l/a = 0.1$. It is clear that the stress–elastic strain relation follows the linear Hashin–Shtrikman relation when the elastic strain is small and approaches a linear relation when the elastic strain is larger. In between, the relationship is non-linear.

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

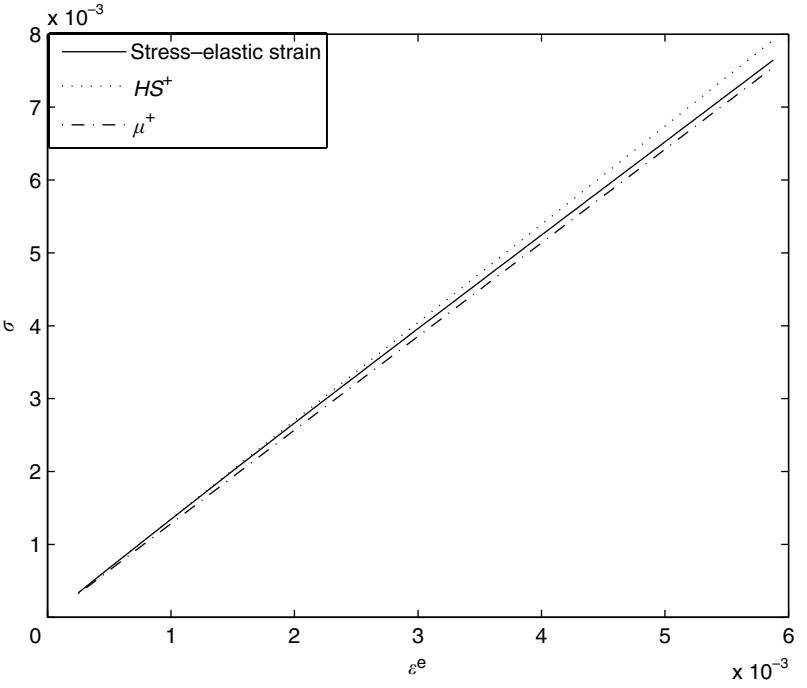



FIGURE 2. Stress versus elastic strain for $\sigma_1 = 0.04$, $\sigma_2 = 0.05$, $l/a = 0.1$.

In figure 2 we plot the stress versus the elastic strain for the case $\sigma_1 = 0.04$ and $\sigma_2 = 0.05$; other parameters are kept as in figure 1. The stress–elastic strain relation follows the Hashin–Shtrikman bound when the elastic strain is small, and asymptotes to a linear relation when the elastic strain is larger. However, unlike the example depicted in figure 1, in this case, the 'asymptotic' elastic moduli $\mu^+$ is smaller than the Hashin–Shtrikman bound so the stress decreases as the elastic strain increases. Mathematically, we can see that from (31), when $\sigma_1$ is close to $\sigma_2$, the optimal value of $\gamma$ is close to 1 so $\mu^+$ approaches the Reuss bound $\mu_R$ in (14) which is smaller than the Hashin–Shtrikman bound $\mu_{HS^+}$. This explains the lower stress. We thus predict that the effective elastic moduli in the small strain range can be approximated by the effective elastic moduli of elastic materials with the elastic moduli $\mu_1$ and $\mu_2$ in the phases, and with volume fractions $p_1$ and $p_2$. This does not depend on the plastic reference strength. As the elastic strain increases, the stress increases when the ratio $\sigma_1/\sigma_2$ is small, and it decreases as the ratio $\sigma_1/\sigma_2$ approaches 1.

However, the stress–elastic strain curve is not always non-linear. Figure 3 shows an example when $\mu^+$ is approximately equal to the Hashin–Shtrikman bound so the stress–elastic strain remains linear. Here $\sigma_1 = 0.025$, $\sigma_2 = 0.05$, $l/a = 0.6$ and other parameters are kept unchanged.

We now study the effect of the material length scale on the stress–elastic strain curve. This effect depends essentially on the values of $\sigma_1$ and $\sigma_2$. Generally, if $\sigma_1/\sigma_2$ is sufficiently small, with the same elastic strain, the stress increases with a smaller value of $l/a$ but is always greater than the Hashin–Shtrikman bound. This is shown in figure 4 where the

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

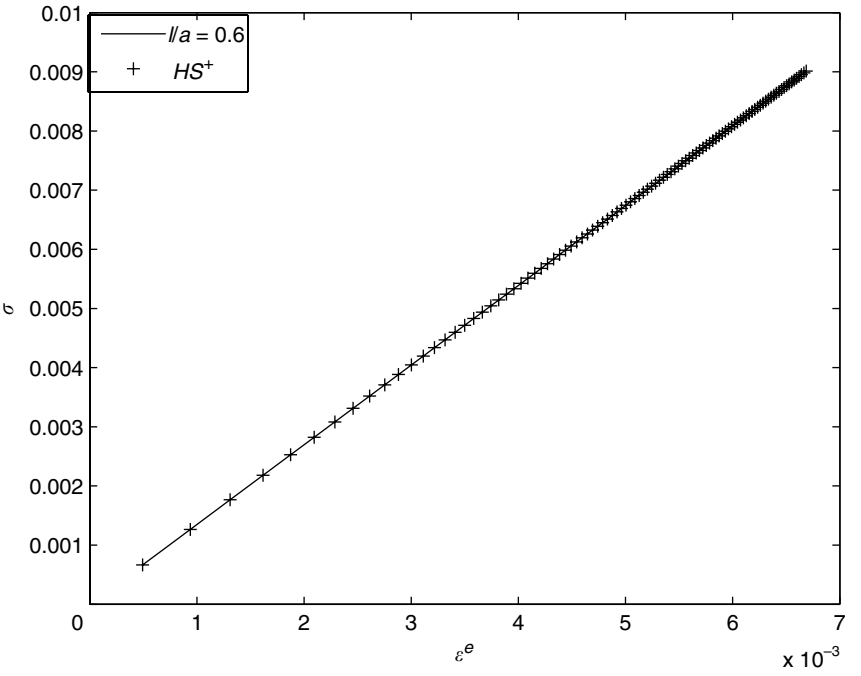



FIGURE 3. Stress versus elastic strain for $\sigma_1 = 0.025$, $\sigma_2 = 0.05$.

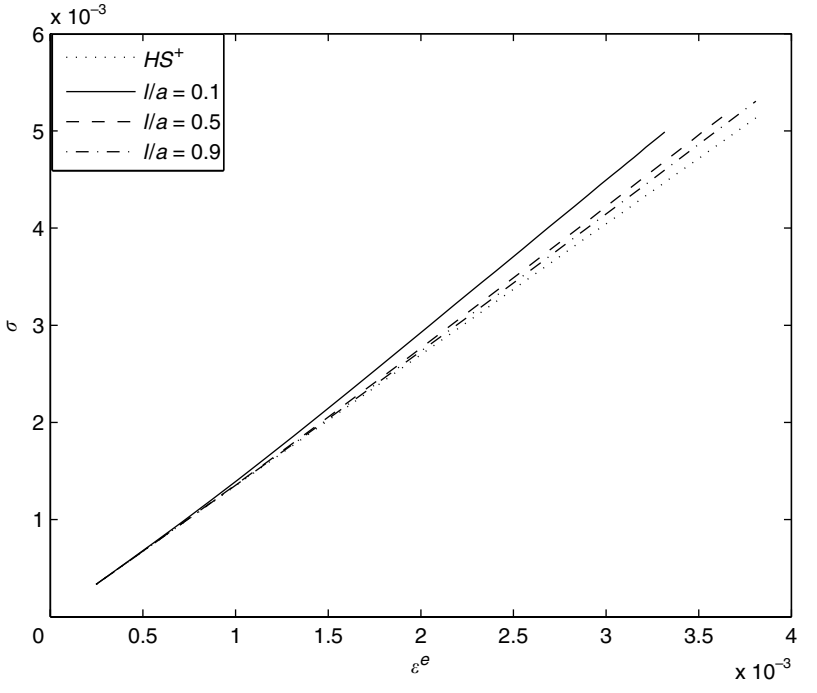



FIGURE 4. Stress versus elastic strain for $\sigma_1 = 0.01$, $\sigma_2 = 0.05$.

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

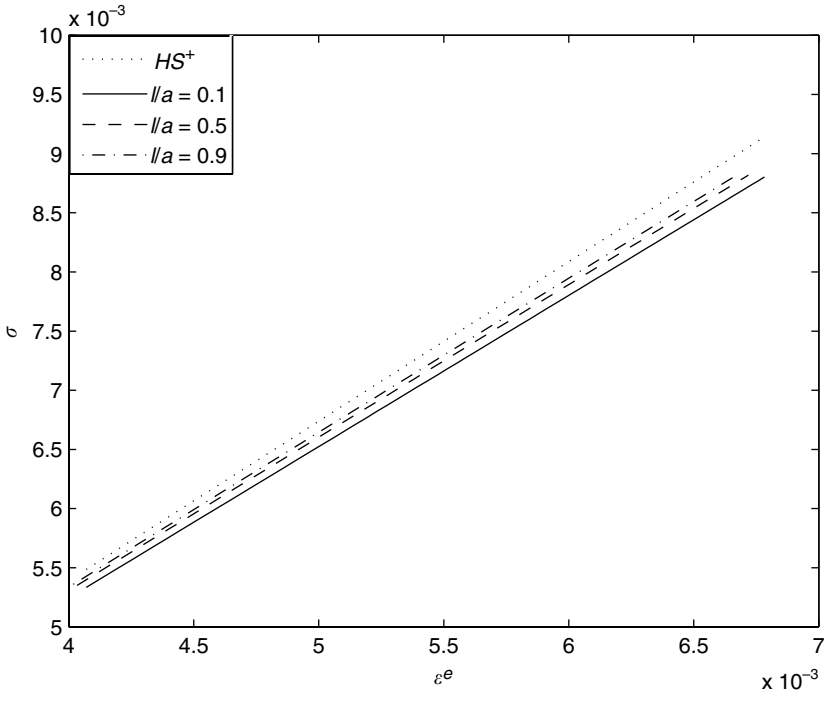



FIGURE 5. Stress versus elastic strain for $\sigma_1 = 0.04$, $\sigma_2 = 0.05$.

values of $\sigma_1$ and $\sigma_2$ are chosen as 0.01 and 0.05 as in figure 1. The other parameters are kept unchanged. When $\sigma_1/\sigma_2$ is relatively large, the behaviour is completely different. The stress increases with a larger value of $l/a$ and is always less than the Hashin–Shtrikman bound for all values of $l/a$. This is shown in figure 5 for $\sigma_1 = 0.04$ and $\sigma_2 = 0.05$. In the case where $\sigma_1/\sigma_2 = 1$, $\gamma = 1$ for all the values of $l/a$. The value of $\mu^+$ is the Reuss bound $\mu_R$. Figure 6 shows the stress–elastic strain curve for some values of $l/a$. We see that an increasing value of $l/a$ now increases the stress but the stress is always less than the Hashin–Shtrikman bound.

Numerical results for other values of the hardening index $N$ show exactly the same behaviour. The value $N = 0.3$ was chosen because of the clarity in the non-linear relationship between the stress and the elastic strain produced by the numerical simulations.

## 7 Conclusions

We explored the relation between the stress and the elastic strain for an incompressible composite plastic material whose components follow a power law. The mechanical framework employed is that of Fleck and Willis (2004) which modifies the strain–gradient plastic theory of Fleck and Hutchinson (2001). From the non-linear Hashin–Shtrikman upper bound, we found that in the range of small strain, the stress and the elastic strain follow the linear Hashin–Shtrikman bound for incompressible elastic composites, while when the strain is large, they asymptote another linear relation. As the stress is expected to depend on the elastic strain smoothly, this shows that in the middle, the stress depends

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

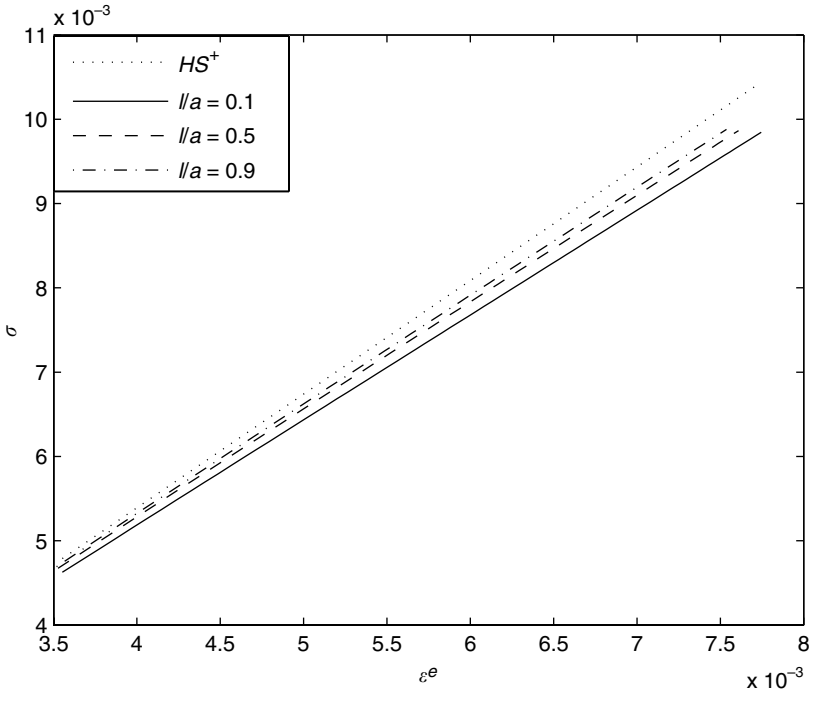



FIGURE 6. Stress versus elastic strain for $\sigma_1 = \sigma_2 = 0.05$.

on the elastic strain non-linearly. We also showed that this relation depends on the ratio $l/a$ of the material length scale $l$ and the correlation length $a$ in a complicated manner, which is essentially determined by the plastic reference strength of the components.

**Acknowledgements**

The research is supported by a Meggitt Fellowship in Mathematics of Emmanuel College, Cambridge. The author is grateful to John R. Willis for many fruitful discussions.

## Appendix A: Solutions to equations (16) and (17)

In this appendix, we solve the equations (16) and (17). Let $G_{ip}(x, x')$ be the Green function satisfying

$$\mu_0 \frac{\partial^2 G_{ip}(x, x')}{\partial x_j \partial x_j} + \delta_{ip}\delta(x - x') = P_{,i},$$

subjecting to the incompressible condition $\partial G_{ip}(x, x')/\partial x_i = 0$ with the homogeneous boundary condition. The solution $u$ for the equation (16) can be written as

$$u_p(x') = u_p^0(x') + \int_\Omega \frac{\partial G_{ip}(x, x')}{\partial x_j}\left(2\mu_0 \varepsilon_{ij}^P - \sigma_{ij}\right)(x)\, dx,$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

where $u_p(x') = \varepsilon^0_{pi} x'_i$. Since

$$\int_\Omega \frac{\partial G_{ip}(x,x')}{\partial x_j}\, dx = 0,$$

we can write $u$ as

$$u_p(x') = u^0_p(x') + \int_\Omega \frac{\partial G_{ip}(x,x')}{\partial x_j}\big(2\mu_0\big(\varepsilon^P_{ij} - \varepsilon^{P0}_{ij}\big) - (\sigma_{ij} - \langle\sigma_{ij}\rangle)\big)\, dx.$$

Differentiating both sides and taking the symmetrization, we get

$$\varepsilon_{ij}(x') = \varepsilon^0_{ij} + \int_\Omega \Gamma_{ijkl}(x,x')\big(2\mu_0\big(\varepsilon^P_{kl} - \varepsilon^{P0}_{kl}\big) - (\sigma_{kl} - \langle\sigma_{kl}\rangle)\big)\, dx \tag{A 1}$$

where the fourth-order tensor $\Gamma$ is defined as

$$\Gamma_{ijkl}(x,x') = \frac{\partial^2 G_{ki}(x,x')}{\partial x_l \partial x'_j} \quad \text{symm } (i,j)(k,l).$$

Assuming that the correlation length is small, we can substitute $\Gamma$ by its infinite body form, which only depends on $x - x'$, i.e. $\Gamma(x,x') = \Gamma(x - x')$.

To solve equation (17), we consider the function $g(x,x')$ which satisfies the equation

$$\nabla^2_x g(x,x') - k^2 g(x,x') + \delta(x - x') = 0,$$

with the Neumann boundary condition; $k^2 = (3\mu_0 + b_0)/(b_0 l^2)$. The solution $\varepsilon^P_{ij}$ of (17) can be written as (here the field $\beta_{ijk}$ is taken as zero)

$$\varepsilon^P_{ij}(x) = \varepsilon^{P0}_{ij} + \frac{3}{2b_0 l^2}\int_\Omega g(x,x')\big\{2\mu_0\big(\varepsilon_{ij} - \varepsilon^0_{ij}\big) - (\alpha_{ij} - \langle\alpha_{ij}\rangle)\big\}dx'. \tag{A 2}$$

In the asymptotic limit, we can substitute $g(x,x')$ by its infinite body form $g(x-x')$ where $g(x) = e^{-kr}/(4\pi r)$ ($r = |x|$). The quantities $\varepsilon_{ij}$ and $\varepsilon^P_{ij}$ are coupled in equations (A 1) and (A 2). To solve them, we take the Fourier transform

$$\hat f(\xi) = \int e^i f(x)\, dx.$$

We then have

$$\begin{aligned}\hat\varepsilon^P_{ij} - \hat\varepsilon^{P0}_{ij} &= \frac{3}{2b_0 l^2}\hat g\big\{2\mu_0\big(\hat\varepsilon_{ij} - \hat\varepsilon^0_{ij}\big) - (\hat\alpha_{ij} - \langle\hat\alpha_{ij}\rangle)\\ &= \frac{3\hat g}{2b_0 l^2}\big\{r\mu_0^2\hat\Gamma_{ijkl}\big(\hat\varepsilon^P_{kl} - \hat\varepsilon^{P0}_{kl}\big) - 2\mu_0\hat\Gamma_{ijkl}(\sigma_{kl} - \langle\hat\sigma_{kl}\rangle - (\hat\alpha_{ij} - \langle\hat\alpha_{ij}\rangle).\end{aligned}$$

On using the relation $2\mu_0\hat\Gamma\,\hat g\hat\Gamma = \hat g\hat\Gamma$, we have

$$\hat\varepsilon^P - \hat\varepsilon^{P0} = \frac{9\hat g^2\mu_0^2}{b_0 l^2(3\mu_0\hat g - b_0 l^2)}\hat\Gamma(\hat\sigma - \langle\hat\sigma\rangle + \hat\alpha - \langle\hat\alpha\rangle) - \frac{3\hat g\mu_0}{b_0 l^2}(\hat\sigma - \langle\hat\sigma\rangle) - \frac{3\hat g}{2b_0 l^2}(\hat\alpha - \langle\hat\alpha\rangle).$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

Substituting

$$\hat{g}(\xi) = \frac{b_0 l^2}{3\mu_0 + b_0 + b_0 l^2|\xi|^2}, \quad \hat{\Gamma}_{ijkl}(\xi) = \frac{1}{\mu_0}\left(\frac{\delta_{ik}\xi_j\xi_l}{|\xi|^2} - \frac{\xi_i\xi_j\xi_k\xi_l}{|\xi|^4}\right),$$

we have

$$\hat{\varepsilon}^P_{ij} - \hat{\varepsilon}^{P0}_{ij} = -\hat{\Lambda}_{ijkl}(\hat{\alpha}_{kl} - \langle\hat{\alpha}_{kl}\rangle) - \hat{\Theta}_{ijkl}(\hat{\sigma}_{kl} - \langle\hat{\sigma}_{kl}\rangle),$$

where

$$\hat{\Lambda}_{ijkl} = \frac{3I'_{ijkl}}{6\mu_0 + 2b_0(1 + l^2|\xi|^2)} + \frac{9\mu_0}{b_0(1 + l^2|\xi|^2)(3\mu_0 + b_0(1 + l^2|\xi|^2))}\left(\frac{\delta_{ik}\xi_j\xi_l}{|\xi|^2} - \frac{\xi_i\xi_j\xi_k\xi_l}{|\xi|^4}\right),$$

symm $(i, j)(k, l)$,

and

$$\hat{\Theta}_{ijkl} = \frac{3}{b_0(1 + l^2|\xi|^2)}\left(\frac{\delta_{ik}\xi_j\xi_l}{|\xi|^2} - \frac{\xi_i\xi_j\xi_k\xi_l}{|\xi|^4}\right), \qquad \text{symm } (i, j)(k, l).$$

From this,

$$\hat{\varepsilon}_{ij} - \hat{\varepsilon}^0_{ij} = -\hat{H}_{ijkl}(\hat{\alpha}_{kl} - \langle\hat{\alpha}_{kl}\rangle) - \hat{K}_{ijkl}(\sigma_{kl} - \langle\hat{\sigma}_{kl}\rangle),$$

where

$$\hat{K}_{ijkl} = \hat{\Gamma}_{ijkl} + 2\mu_0\hat{\Gamma}_{ijkl}\hat{\Theta}_{ijkl}, \qquad \hat{H}_{ijkl} = 2\mu_0\hat{\Gamma}_{ijkl}\hat{\Lambda}_{ijkl}.$$

## Appendix B: Evaluation of $\mu_Q$, $\mu_R$, $\mu_S$ and $\mu_T$

The constant $\mu_Q$ is computed in Fleck and Willis (2004). We use the same routine to compute the other constants here, so we do not repeat all the steps. It is easy to see that

$$10\mu_R = R_{ijij} = \frac{1}{8\pi^3}\int dx \int d\xi(e^{-i\xi.x}\hat{\Theta}_{ijij}(\xi))h(x)\,dx,$$

where

$$\hat{\Theta}_{ijij} = \frac{1}{\mu_0}\left(-\frac{9\hat{g}^2\mu_0^2}{b_0 l^2(3\mu_0\hat{g} - b_0 l^2)} + \frac{3\hat{g}\mu_0}{b_0 l^2}\right).$$

Following exactly the same procedure outlined in Fleck and Willis (2004) to compute the integrals, we deduce that with $h(x) = h(r) = e^{-r/a}$,

$$\mu_R = \frac{3}{10b_0(1 + l/a)^2}.$$

Now

$$H_{ijkl} = 2\mu_0\Gamma_{ijpq} * \Lambda_{pqkl},$$

so

$$\hat{H}_{ijkl} = 2\mu_0\hat{\Gamma}_{ijpq}\hat{\Lambda}_{pqkl}.$$

From this

$$\hat{H}_{ijij} = \frac{3}{b_0(1 + l^2|\xi|^2)},$$

https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press

which implies that

$$\mu_R = \frac{3}{10b_0(1 + l/a)}.$$

Similarly, we have

$$\hat{K}_{ijij} = \frac{1}{\mu_0} + \frac{3}{b_0(1 + l^2|\xi|^2)},$$

so

$$\mu_T = \frac{1}{10\mu_0} + \frac{3}{10b_0(1 + l/a)^2}.$$

## References


Aifantis, K. E. & Willis, J. R. (2005) The role of interfaces in enhancing the yield strength of composites and polycrystals. *J. Mech. Phys. Solids* **53**, 1047–1070.

Fleck, N. A. & Hutchinson, J. W. (1993) A phenomenological theory for strain gradient plasticity. *J. Mech. Phys. Solids* **41**, 1825–1857.

Fleck, N. A. & Hutchinson, J. W. (2001) A reformulation of strain gradient plasticity. *J. Mech. Phys. Solids* **49**, 2245–2272.

Fleck, N. A. & Willis, J. R. (2004) Bounds and estimates for the effect of strain gradients upon the effective plastic properties of an isotropic two phase composite. *J. Mech. Phys. Solids* **52**, 1855–1888.

Gudmundson, P. (2004) A unified treatment of strain gradient plasticity. *J. Mech. Phys. Solids* **52**, 1379–1406.

Hoang, V. H. (in press) Bounds for the effective stress of classical and strain gradient plastic composites.

Milton, G. W. (2002) *The Theory of Composites*, Cambridge University Press, Cambridge, UK.

Ponte Castañeda, P. (1991) The effective mechanical properties of nonlinear isotropic composites. *J. Mech. Phys. Solids* **39**, 45–71.

Ponte Castañeda, P. (1992) New variational principles in plasticity and their application to composite. *J. Mech. Phys. Solids* **40**, 1757–1788.

Ponte Castañeda, P. & De Botton, G. (1992) On the homogenized yield strength of two phase composites. *Proc. Roy. Soc. A* **438**, 419–431.

Smyshlyaev, V. P. & Fleck, N. A. (1994) Bounds and estimates for linear composites with strain gradient effects. *J. Mech. Phys. Solids* **42**, 1851–1882.

Smyshlyaev, V. P. & Fleck, N. A. (1995) Bounds and estimates for the overall plastic behaviour of composites with strain gradient effects. *Proc. Roy. Soc. A* **451**, 795–810.

Suquet, P. M. (1993) Overall potentials and extremal surfaces of power law or ideally plastic composites. *J. Mech. Phys. Solids* **41**, 981–1002.

Talbot, D. R. S. & Willis, J. R. (1985) Variational principles for inhomogeneous nonlinear media. *IMA J. Appl. Math.* **35**, 39–54.


https://doi.org/10.1017/S0956792509007815 Published online by Cambridge University Press