Lyness cycles

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1. *A question from* 1942

In 1942, R. C. Lyness noted that some recurrence relations generate cycles, irrespective of the initial values. For example, the order 2 recurrence relation

$$u_{n+2} = \frac{1 + u_{n+1}}{u_n}$$

generates a cycle of period 5 for almost all values of u_1 and u_2 [1].

The globally periodic nature of sequences generated by this recurrence relation can be seen by setting $u_1 = x$ and $u_2 = y$. The sequence is then

$$\left(x, y, \frac{1 + y}{x}, \frac{1 + x + y}{xy}, \frac{1 + x}{y}, x, y, \ldots\right).$$

Lyness gave other examples of such recurrence relations but had been unable to find one with period 7 and challenged readers of the *Gazette* to find such a recurrence relation or prove it to be impossible.

No answer to this challenge was forthcoming. However, since Lyness's time, interest in these cycles has been maintained due to links with cross-ratios and elliptic curves. In recent years, Jonny Griffiths has done much to popularise these cycles [2].

The recurrence relation given above is described in [2] without the use of suffix notation, by the equation

$$z = \frac{1 + y}{x}.$$

This notation is convenient to use and will be adopted throughout this paper. However, it will often prove advantageous to give the equations in multiplied-out form, equal to zero. This example would then be stated as

$$1 + y - xz = 0.$$

This defining equation is linear in each of its arguments. As noted in [2], this property is important in terms of enabling the equation to be solved easily for each variable. We shall formally define a *Lyness cycle of order n* to be any globally periodic cycle determined by a multilinear equation in $n + 1$ variables with rational coefficients.

The aim of this paper is to settle the long-standing question posed by Lyness about an order 2 Lyness cycle of period 7. However, the methods used have more general applicability and can be used to determine fully *all* Lyness cycles of orders 1 and 2.

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2. *Some simple constructions*

It can be useful to have some examples of Lyness cycles in mind when considering general theorems. The following table is taken from [2] and gives examples of cycles of orders 1, 2 and 3 and of various periods.

The table gives the function f, where the order 1 equations are of the form $y = f(x)$, the order 2 equations are of the form $z = f(x, y)$ and the order 3 equations are of the form $t = f(x, y, z)$.

Period	*Order* 1	*Order* 2	*Order* 3
1	x		
2	$-x$	x	
3	$\dfrac{x-3}{x+1}$	$-x-y$	x
4	$\dfrac{x-1}{x+1}$	$\dfrac{-(x+1)(y+1)}{y}$	$-x-y-z$
5		$\dfrac{y+1}{x}$	$\dfrac{-(z+1)(xy+x+1)}{xyz+xy+xz+x+z}$
6	$\dfrac{2x-1}{x+1}$	$\dfrac{y}{x}$	$\dfrac{-(xy+yz+x+y+z+1)}{z}$
7			
8		$\dfrac{x-1}{x+1}$	$\dfrac{y+z+1}{x}$
9			$\dfrac{x-3}{x+1}$
10			$\dfrac{yz+1}{xyz}$
11			
12		$\dfrac{2x-1}{x+1}$	$\dfrac{xy+y+z}{x-z}$
13			
14			
15			
16			
17			
18			$\dfrac{2x-1}{x+1}$

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For cycles of order n, Lyness cycles of periods n, $n + 1$ and $n + 2$ can be constructed in a straightforward manner. Appropriate defining equations are as follows.

Period n $\qquad x_{n+1} = x_1$

Period $n + 1$ $\qquad x_1 + x_2 + \ldots + x_{n+1} = 0$

Period $n + 2$ $\qquad 1 + x_1 + x_1x_2 + \ldots + x_1x_2\ldots x_{n+1} = 0$

For period $n + 1$, the equation for x_{n+2} is

$$x_2 + x_3 + \ldots + x_{n+2} = 0$$

and, by subtraction from the equation for x_{n+1}, we obtain $x_1 = x_{n+2}$, as required.

For period $n + 2$, the equation for x_{n+2} is

$$1 + x_2 + x_2x_3 + \ldots + x_2x_3\ldots x_{n+2} = 0.$$

Multiplication by x_1 gives

$$x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2x_3\ldots x_{n+2} = 0.$$

Then, by subtraction from the equation for x_{n+1}, we obtain $x_1x_2x_3\ldots x_{n+2} = 1$. Comparison with $x_2x_3\ldots x_{n+3} = 1$ gives $x_1 = x_{n+3}$, as required.

A Lyness cycle of one order can be used to produce Lyness cycles of higher orders in a straightforward way. Consider, for example, the equation $1 + x + xy = 0$ which we know will generate a cycle of period 3. This cycle is

$$\left(x, -\frac{1+x}{x}, -\frac{1}{1+x}\right).$$

However, changing the defining equation to

$$1 + x + xz = 0$$

produces the order 2 cycle of period 6

$$\left(x, y, -\frac{1+x}{x}, -\frac{1+y}{y}, -\frac{1}{1+x}, -\frac{1}{1+y}\right).$$

It is clear that simply replacing y by z in the defining equation of an order 1 cycle of period p will always produce an order 2 cycle of period $2p$. Such a cycle will be termed *degenerate*. More generally we can state the following result.

Lemma 1: For any positive integer k, each Lyness cycle of order n and period p can be used to construct a degenerate Lyness cycle of order nk and period pk.

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Proof: Let the cycle of order n and period p have defining equation

$$F(x_1, x_2, \ldots, x_{n+1}) = 0.$$

Simply replace each x_i by $x_{1+k(i-1)}$. The new equation then defines a Lyness cycle of order nk and period pk, as required.

3. *Symmetry and super-symmetry*

Consider the equation

$$2 + x - z + xy + yz = 0.$$

For initial terms u and v, it is easy to check that this generates a cycle of period 4

$$\left(u, v, \frac{2 + u + uv}{1 - v}, \frac{-2 + v - uv}{1 + u}\right).$$

Interchanging x and z gives $2 - x + z + xy + yz = 0$ and, for initial terms v and u, this generates the cycle

$$\left(v, u, \frac{-2 + v - uv}{1 + u}, \frac{2 + u + uv}{1 - v}\right).$$

This is, of course, the original cycle in reverse. In general, suppose a Lyness cycle of order n is defined by a multilinear equation

$$F(x_1, x_2, \ldots, x_{n+1}) = 0.$$

Then, since the order is n, at least one term of F involves x_1. Interchanging each x_i and x_{n+2-i} in F then gives a multilinear equation which defines the original Lyness cycle in reverse.

In some cases, F will be invariant under an interchange of each x_i and x_{n+2-i}. In such a case, F and the cycle it generates will be termed *symmetrical*. The equation for the opening order 2 example of this paper,

$$1 + y - xz = 0,$$

is symmetrical. This means that the recurrence relation defining the sequence also defines the sequence in reverse. The symmetry of the definition can, of course, be seen in the terms of the cycle

$$\ldots, \frac{1 + x + y}{xy}, \frac{1 + x}{y}, x, y, \frac{1 + y}{x}, \frac{1 + x + y}{xy}, \ldots.$$

Each left-hand term, of the form $F(x, y)$, is matched by a right-hand term, $F(y, x)$.

The significance of symmetry for Lyness cycles is that slightly different techniques prove useful for the analysis of symmetrical and non-symmetrical cycles. Note that, for a Lyness cycle of order 2, a multilinear equation of the form

$$a + bx + cy + dz + exy + fxz + gyz + hxyz = 0$$

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is symmetrical if, and only if, $b = d$ and $e = g$.

Ideas of symmetry are also important for generating Lyness cycles. Consider, for example, the order 3 sequence defined by

$$t = \frac{x + y + z - xz}{y - 1}.$$

Given the definition in this form, it is not obvious that this does generate a Lyness cycle. However, the multilinear form

$$x + y + z + t - xz - yt = 0$$

is invariant under the cyclic interchange $x \to y, y \to z, z \to t, t \to x$. Applying these transformations to the original definition of t produces the equation

$$x = \frac{y + z + t - yt}{z - 1}$$

and we therefore have a cycle of period 4

$$\left(x, y, z, \frac{x + y + z - xz}{y - 1}\right).$$

In general, a defining equation and the cycle it determines will be termed *super-symmetrical* if it is invariant under the cyclic interchange

$$x_1 \to x_2, x_2 \to x_3, \ldots, x_{n+1} \to x_1.$$

Lemma 2: A super-symmetrical multilinear equation of order n determines a Lyness cycle of period $n + 1$.

Proof: Suppose a super-symmetrical equation defines x_{n+1} as

$$F(x_1, \ldots, x_{n-1}, x_n).$$

Then by cyclic interchange,

$$x_1 = F(x_2, \ldots, x_n, x_{n+1})$$

$$x_2 = F(x_3, \ldots, x_{n+1}, x_{n+2})$$

$$\ldots$$

and so on. We therefore have a Lyness cycle, $(x_1, \ldots, x_n, x_{n+1})$, of period $n + 1$.

Corollary: Equations of the form $a + bx + by + dxy = 0$ and

$$a + bx + by + bz + exy + exz + eyz + hxyz = 0$$

define Lyness cycles of periods 2 and 3 respectively.

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4. *Möbius transformations*

Apparently very different cycles can be simple transformations of each other. Consider, for example, the order 2 equation

$$x - y + z - xyz = 0.$$

This produces a cycle of period 6:

$$\left(x, y, \frac{-x + y}{1 - xy}, -x - y, \frac{x - y}{1 - xy}\right).$$

The equation $xz = y$ also produces a cycle of period 6 but this has quite a different appearance:

$$\left(x, y, \frac{y}{x}, \frac{1}{x}, \frac{1}{y}, \frac{x}{y}\right).$$

In fact, if we apply the transformation

$$(x, y, z) \to \left(\frac{1 + x}{1 - x}, \frac{1 + y}{1 - y}, \frac{1 + z}{1 - z}\right)$$

to $xz = y$, then

$$\frac{1 + x}{1 - x} \cdot \frac{1 + z}{1 - z} = \frac{1 + y}{1 - y}$$

$$\Rightarrow x - y + z - xyz = 0.$$

Thus these two, apparently very different, Lyness cycles are simple transformations of each other and are therefore, in effect, the same. As we shall prove, there are surprisingly few 'different' Lyness cycles of orders 1 and 2.

The types of transformations that we shall employ are called Möbius transformations. These have the form

$$x \to \frac{\alpha + \beta x}{\gamma + \delta x}, \text{ for } \alpha\delta - \beta\gamma \neq 0.$$

The important feature of these transformations is that the multilinear nature of a defining equation, whether or not it is symmetrical, and the periodicity of a cycle it determines are not affected by a Möbius transformation.

This can be seen most easily by considering any Möbius transformation to be a combination of simple transformations, as follows. For simplicity of explanation, we will consider this in terms of a multilinear equation of order 2:

$$a + bx + cy + dz + exy + fxz + gyz + hxyz = 0.$$

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- *Replacing* (x, y, z) *by* $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

  This can be viewed as interchanging $a \leftrightarrow h, b \leftrightarrow g, \ldots$ and so on.

- *Replacing* (x, y, z) *by* $(\beta x, \beta y, \beta z)$

- *Replacing* (x, y, z) *by* $(x + \alpha, y + \alpha, z + \alpha)$

  This gives

$$a' + b'x + c'y + d'z + e'xy + f'xz + g'yz + hxyz = 0$$

where

$$a' = a + (b + c + d)\alpha + (e + f + g)\alpha^2 + h\alpha^3$$

$$b' = b + (e + f)\alpha + +h\alpha^2$$

$$c' = c + (e + g)\alpha + +h\alpha^2$$

$$d' = d + (f + g)\alpha + +h\alpha^2$$

$$e' = e + h\alpha, f' = f + h\alpha, g' = g + h\alpha.$$

In addition to the Möbius transformations, it can be useful to carry out one or both of the following operations.

- *Dividing all coefficients by the same constant*

- *Replacing* (x, y, z) *by* (z, y, x).

  This can be considered as simply reversing the direction of the cycle or, alternatively, can be viewed as interchanging $b \leftrightarrow d$ and $e \leftrightarrow g$.

Even an elementary application of these transformations can be useful in terms of replacing the consideration of general multilinear equations with the consideration of a few canonical forms.

Extending the use of Möbius transformations to ones with complex number parameters enables further simplifications to be made to the defining equations of Lyness cycles. However, this comes at the price of further analysis having to take place in extension fields of the rationals.

5. *Unlocking Lyness cycles*

One result, above all others, would appear to be crucial to the analysis and understanding of Lyness cycles.

By way of illustration, consider the order 3 equation

$$xy - yz + zt + xyt + xzt = 0.$$

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The sequence $x, y, z, t, \ldots$ determined by this equation is actually a cycle of period 12, although this is far from obvious from the equation.

If we restrict the equation to just the terms of lowest degree, we obtain the equation

$$xy - yz + zt = 0.$$

The sequence determined by this equation is much easier to work out:

$$x, y, z, \frac{y(z-x)}{z}, \frac{-xz}{z-x}, \frac{y(z-x)}{x}, \ldots .$$

These first 6 terms then repeat with each term multiplied by -1 and so the whole sequence has period 12, just the same as the sequence determined by the 'parent' equation. A numerical comparison using a spreadsheet can easily be carried out for various initial values and shows that the two sequences of period 12 can differ markedly from each other. In particular, the sequence determined by the terms of lowest degree is relatively simple. For example, with initial values 3, 4 and 2, the following sequences are obtained.

	Terms of degree 2 only	Parent equation
x_1	3	3
x_2	4	4
x_3	2	2
x_4	−2	−0.2
x_5	6	−1.2
x_6	−1.3333	−0.16
x_7	−3	−0.42857
x_8	−4	−0.4444
x_9	−2	−0.4
x_{10}	2	0.3333
x_{11}	−6	−0.85714
x_{12}	1.3333	0.23529
x_{13}	3	3
x_{14}	4	4
x_{15}	2	2

Being able to simplify in this way will prove crucial in our analysis of Lyness cycles. We must, of course, prove that the example above does

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illustrate a general feature of these cycles. We shall now do this when the terms of lowest degree are of degree 1.

Theorem 1: Let a globally-periodic cycle of order n and period p be determined by a multilinear equation

$$L(x_1, \ldots, x_{n+1}) + M(x_1, \ldots, x_{n+1}) = 0 \tag{1}$$

where M consists of terms of degree at least 2 and where

$$L(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} a_i x_i, \text{ with } a_1 \text{ and } a_{n+1} \text{ not both } 0.$$

Then

$$L(x_1, \ldots, x_{n+1}) = 0 \tag{2}$$

also determines a globally-periodic cycle of order n and period p.

Proof: Firstly, if necessary, reverse the sequence so that $a_{n+1} \neq 0$.

Let (v_i) be the sequence (possibly not cyclic) generated, for given initial terms $(v_1, v_2, \ldots, v_n)$, by (2). Then, for any small number δ, the initial values $(v_1\delta, v_2\delta, \ldots, v_n\delta)$ generate the sequence $(v_i\delta)$.

Now let (x_i) be the cyclic sequence generated by (1) for the initial values $(v_1\delta, v_2\delta, \ldots, v_n\delta)$ and suppose that $x_i = v_i\delta + O(\delta^2)$ for $1 \leqslant i \leqslant k$, a result which is certainly true for $k = n$. Then

$$\begin{aligned} L(x_{k-n+1}, \ldots, x_{k+1}) + M(x_{k-n+1}, \ldots, x_{k+1}) &= 0 \\ \Rightarrow \quad a_{n+1}x_{k+1} &= O(\delta) \\ \Rightarrow \quad x_{k+1} &= O(\delta) \\ \Rightarrow \quad L(x_{k-n+1}, \ldots, x_{k+1}) &= O(\delta^2). \end{aligned}$$

Comparison with

$$L(v_{k-n+1}\delta, \ldots, v_{k+1}\delta) = 0$$

gives $x_{k+1} = v_{k+1}\delta + O(\delta^2)$ and so, by induction, $x_i = v_i\delta + O(\delta^2)$ for all i. For any i, $x_{i+p} = x_i$, and so $v_{i+p}\delta = v_i\delta + O(\delta^2)$. Therefore $v_{i+p} = v_i$.

We can conclude that (2) determines a globally-periodic cycle of period q, where q is a divisor of p.

Now suppose that, for all i, $x_{i+q} = x_i + O(\delta^m)$, a result which is certainly true for $m = 2$. Then, for any i,

$$\begin{aligned} L(x_i, \ldots, x_{i+n}) + M(x_i, \ldots, x_{i+n}) &= 0 \\ L(x_{i+q}, \ldots, x_{i+n+q}) + M(x_{i+q}, \ldots, x_{i+n+q}) &= 0. \end{aligned}$$

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Then

$$M\left(x_{i+q}, \ldots, x_{i+n+q}\right) - M\left(x_i, \ldots, x_{i+n}\right) = O\left(\delta^{m+1}\right)$$

$$\Rightarrow \qquad L\left(x_{i+q} - x_i, \ldots, x_{i+n+q} - x_{i+n}\right) = O\left(\delta^{m+1}\right).$$

L determines a cycle of period q and so, up to $O\left(\delta^{m+1}\right)$,

$$x_{i+q} - x_i = x_{i+2q} - x_{i+q} = x_{i+3q} - x_{i+2q} = \ldots .$$

Then

$$\begin{aligned} 0 &= x_{i+p} - x_i \\ &= \sum_{j=1}^{p/q} \left(x_{i+jq} - x_{i+(j-1)q}\right) \\ &= \frac{p}{q}\left(x_{i+q} - x_i\right) + O\left(\delta^{m+1}\right). \end{aligned}$$

Therefore

$$x_{i+q} - x_i = O\left(\delta^{m+1}\right).$$

Repeating this argument forces $x_{i+q} = x_i$ and so (1) and (2) determine globally-periodic cycles of the same period, as required.

Since the cycle for (2) is periodic, the value of x_1 affects the cycle and therefore (2) is of order n. In particular, both a_1 and a_{n+1} are non-zero.

Theorem 1 is a very powerful tool in the analysis of Lyness cycles. It requires the equation defining the cycle to have no constant term but this restriction is not as serious as might first be thought, since equations can be transformed relatively easily into this form.

6. *Linear difference equations of order* 2

Theorem 1 shows the importance of linear difference equations in determining the structure of Lyness cycles. We shall require the following result.

Theorem 2: Suppose that, for a, b, c and d, elements of $\mathbb{C}$,

$$a + bx + cy + dz = 0$$

determines a globally-periodic cycle of order 2 and period 7. Then:

(i) $bx + cy + dz = 0$ also determines a globally-periodic cycle of period 7;

(ii) there is a primitive 7th root of unity, λ, such that $b + c\lambda + d\lambda^2 = 0$;

(iii) if b, c and d are real, then $b = d$ and $\dfrac{c}{b}$ is irrational.

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Proof:

(i) If $b + c + d = 0$, let $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ be a cycle of period 7. Then

$$7a + (b + c + d)\sum_{i=1}^{7} x_i = 0 \Rightarrow a = 0.$$

Otherwise, (x, y, z) can be replaced by $(x + \alpha, y + \alpha, z + \alpha)$ where $\alpha = -\dfrac{a}{b + c + d}$.

(ii) To produce a cycle of period 7, the general solution of

$$bx_i + cx_{i+1} + dx_{i+2} = 0$$

must have the form

$$x_i = A\lambda^i + B\mu^i,$$

where λ and μ are roots of $b + ct + dt^2 = 0$, such that $\lambda \neq \mu$, $\lambda^7 = \mu^7 = 1$ and not both λ and μ can be 1.

(iii) Let $\lambda = u + jv$, where u and v are real and $u^2 + v^2 = 1$. Then

$$b + cu + jcv + d(u^2 - v^2 + 2juv) = 0$$

$$\Rightarrow \qquad c = -2du \text{ and } b = d(u^2 + v^2) = d.$$

Furthermore, u has the form $\cos\dfrac{2\pi n}{7}$ and is irrational.

7. *Lyness's challenge*

We shall now settle this challenge by proving that there is no such cycle of period 7. It will prove useful to separate the proof into symmetrical and non-symmetrical cases.

Theorem 3: No Lyness cycle of order 2 is globally periodic with period 7.

Proof: Suppose that the equivalent equations, with rational coefficients,

$$a + bx + cy + dz + exy + fxz + gyz + hxyz = 0 \qquad (3)$$

$$h + gx + fy + ez + dxy + cxz + byz + axyz = 0 \qquad (3')$$

determine Lyness cycles of period 7.

Note that if $a = h = 0$, then Theorems 1 and 2 force $b = d = e = g = 0$. Then (3) becomes

$$cy + fxz = 0$$

and so $cf \neq 0$ and the Lyness cycle is of period 6:

$$\left(x, y, \frac{ky}{x}, \frac{k^2}{x}, \frac{k^2}{y}, \frac{kx}{y}\right), \text{ for } k = -\frac{c}{f}.$$

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So, without loss of generality, we can suppose $h \neq 0$. Dividing through by h we can therefore suppose $h = 1$.

The symmetrical case, $b = d$ *and* $e = g$

In this case, (3) is

$$a + bx + cy + bz + exy + fxz + eyz + xyz = 0.$$

If $b = c$ and $e = f$ then this equation would be super-symmetric and the cycle would have period 3. So, without loss of generality, we can suppose that $e \neq f$.

Then replacing (x, y, z) by $(x + \alpha, y + \alpha, z + \alpha)$, with $\alpha = \dfrac{c - b}{f - e}$, makes the coefficients of x and y equal and so we can suppose $c = b$ giving

$$a + bx + by + bz + exy + fxz + eyz + xyz = 0, e \neq f.$$

If $a = 0$, then $b = 0$ by Theorems 1 and 2. Then (3′) becomes

$$1 + ex + fy + ez = 0$$

and we have a contradiction from Theorem 2.

Therefore $a \neq 0$. Replacing (x, y, z) by $\left(\dfrac{ax}{a - bx}, \dfrac{ay}{a - by}, \dfrac{az}{a - bz}\right)$ makes the coefficients of x, y and z equal to zero. We can therefore suppose $b = 0$ and that (3) is

$$a + exy + fxz + eyz + xyz = 0.$$

If $ef \neq 0$

The sequence $0, r, s, t, u, \ldots$ starting with zero and with general second term, r, has

$$s = -\frac{a}{er} \text{ and } \left(ae + ar - efr^2\right)t = 0.$$

So, in general, $t = 0$ and then $u = \left(\dfrac{e}{f}\right)r$. Then the 8th term, which should be 0, is $\left(\dfrac{e}{f}\right)^2 r$.

If $f = 0$

Replacing (x, y, z) by $\left(\dfrac{e}{x}, \dfrac{e}{y}, \dfrac{e}{z}\right)$, (3′) becomes

$$1 + x + z + a'xyz = 0.$$

For $a' \neq -1$, there is a periodic sequence $(0, 0, -1, -1)$ and so the period of a Lyness cycle would have to be divisible by 4. For $a' = -1$, a cycle of period 5 is generated:

$$\left(u, v, \frac{u + 1}{uv - 1}, uv - 1, \frac{v + 1}{uv - 1}\right).$$

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If $e = 0$

Then $a + fxz + xyz = 0$. Replacing (x, y, z) by (fx, fy, fz) gives

$$a' + xz + xyz = 0, \qquad a' = \frac{a}{f^3}.$$

Let α be a real solution of

$$a' + \alpha^2 + \alpha^3 = 0. \tag{4}$$

Then $\alpha \neq 0$. Replace (x, y, z) by $(x + \alpha, y + \alpha, z + \alpha)$ and apply Theorem 1. Then a Lyness cycle of period 7 is defined by

$$(\alpha + \alpha^2)x + \alpha^2 y + (\alpha + \alpha^2)z = 0.$$

By Theorem 2, there is a primitive 7th root of unity, λ, such that

$$\alpha + \alpha^2 + \alpha^2\lambda + (\alpha + \alpha^2)\lambda^2 = 0$$

$$\Rightarrow \qquad \alpha(1 + \lambda + \lambda^2) + 1 + \lambda^2 = 0.$$

Substituting for α in (4) forces the polynomial

$$1 + \left(3 + \frac{1}{a'}\right)\lambda + 6\lambda^2 + \left(7 + \frac{2}{a'}\right)\lambda^3 + 6\lambda^4 + \left(3 + \frac{1}{a'}\right)\lambda^5 + \lambda^6$$

to be zero. However, this polynomial is not a multiple of the cyclotomic polynomial

$$1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6$$

which is irreducible over $\mathbb{Q}$. This contradiction completes the proof for symmetrical cycles.

The non-symmetrical case

Consider equation (3),

$$a + bx + cy + dz + exy + fxz + gyz + hxyz = 0.$$

Replace (x, y, z) by $(x + \alpha, y + \alpha, z + \alpha)$ where α is a real solution of

$$a + (b + c + d)\alpha + (e + f + g)\alpha^2 + \alpha^3 = 0.$$

Then, by Theorems 1 and 2, $b' = d'$, where

$$b' = b + (e + f)\alpha + \alpha^2, \qquad d' = d + (f + g)\alpha + \alpha^2,$$

$$\Rightarrow \qquad b + e\alpha = d + g\alpha.$$

If $e = g$, then $b = d$ and (3) would be symmetrical. So $\alpha = \dfrac{b - d}{g - e} \in \mathbb{Q}$ and we can therefore suppose that $a = 0$ in (3). Then, by Theorem 2, $b = d = 0$.

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Replace (x, y, z) by $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$, then

$$1 + gx + fy + ez + cxz = 0.$$

If $c = 0$, Theorem 2 yields the contradiction $e = g$ and then the equation would be symmetrical. So $c \neq 0$ and (3) can be simplified into the form

$$h' + x + f'y + xz = 0, \text{ where } f' \neq 0 \text{ and } h' \text{ are rational.}$$

Then replace (x, y, z) by $(x + \alpha, y + \alpha, z + \alpha)$, where

$$h' + (f' + 1)\alpha + \alpha^2 = 0.$$

Then, applying Theorem 2,

$$1 + \alpha + f'\lambda + \alpha\lambda^2 = 0$$

where λ is a primitive 7th root of unity.

As a vector space over $\mathbb{Q}$, $\mathbb{Q}(\alpha, \lambda)$ is then spanned by $1, \alpha, \lambda, \lambda\alpha$. Since the cyclotomic polynomial

$$1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6$$

is irreducible over $\mathbb{Q}$, a basis for $\mathbb{Q}(\alpha, \lambda)$ must have at least 6 elements. This contradiction completes the proof for non-symmetrical cycles.

8. *A classification of Lyness cycles*

Lyness's challenge has thus been resolved negatively. The methods used in this paper would appear to be powerful tools in the analysis of Lyness cycles and should enable the interested reader to further explore this topic. The following results can be obtained by these methods and proofs are available by e-mail from the author.

Theorem 4: For rational a, b, c and d, the Lyness cycles of order 1 are

- $y = x$ (Period 1)
- $y = -\dfrac{a + bx}{b + dx}$, $ad \neq b^2$ (Period 2)

or a Möbius transformation of the equation

- $1 + qx + qxy$ (Period 3 if $q = 1$)
  (Period 4 if $q = 2$)
  (Period 6 if $q = 3$)

Theorem 5: Lyness cycles of order 2 are either:

- Degenerate cycles (Period 2, 4, 6, 8 and 12)
- Super-symmetrical (Period 3)

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or defined by a Möbius transformation of one of the following equations

- $1 + x + y + xy + yz = 0$ (Period 4, non-symmetrical)
- $1 + y - xz = 0$ (Period 5, symmetrical)
- $x - y + z + Axyz = 0$ (Period 6, symmetrical)
- $3 - 2x + 2z + xy - xz + yz = 0$ (Period 6, non-symmetrical)

One aim of research into Lyness cycles has been to limit the possible period p of a cycle in terms of its order.

Since we now know *all* Lyness cycles of orders 1 and 2, the following results are immediate consequences of Theorems 4 and 5.

Corollary

(i) The period p of a Lyness cycle of order 1 satisfies $\phi(p) \leqslant 2$.

(ii) The period p of a Lyness cycle of order 2 satisfies $\phi(p) \leqslant 6$.

Corollary (i) was proved, using a different approach to that adopted in this paper, by Cull, Flahive and Robson [3]. Corollary (ii) was proved for some special cases of Lyness cycles of order 2 by Kulenovic and Ladas [4].

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