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Combinatorial cusp count and clover invariants

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Abstract

We construct efficient topological cobordisms between torus links and large connected sums of trefoil knots. As an application, we show that the signature invariant σ_ω at $\omega = \zeta_6$ takes essentially minimal values on torus links among all concordance homomorphisms with the same normalisation on the trefoil knot.

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1. *Introduction*

The topic of this paper is motivated by the following question, already studied by Lefschetz [7]: how many simple cusps can a complex plane curve of degree d have? Here a simple cusp is locally described by the equation $y^2 = x^3$. The answer is of order about αd^2, with a constant α known to lie in the interval $(29/100, 31/100)$, as explained in the beautiful overview by Greuel and Shustin [6]. Generically, a complex plane curve of degree d with N simple cusps gives rise to a smooth cobordism between the link at infinity - a torus link of type $T(d, d)$ - and the connected sum of N positive trefoil knots $3_1 = T(2, 3)$, the knot associated with the simple cusp. We study the following topological analogue of the

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above question: what is the locally flat topological cobordism of lowest complexity between a torus link of type $T(m, n)$ and the connected sum of N trefoil knots, denoted by 3_1^N? We consider the topological cobordism distance $d_\chi(L, L')$ between two links $L, L' \subset S^3$, defined as the minimal number of 1-handles of a locally flat topological cobordism $C \subset S^3 \times [0, 1]$ between L and L', consisting of connected components intersecting both L and L' (not to be confused with the smooth version of the cobordism distance introduced in [1]). In order to state our main result, we introduce the following variant of the Levine–Tristram signature function $\sigma_\omega(L)$ of a link L (see [8, 11]) at $\omega = e^{\frac{2\pi i}{6}}$:

$$\sigma_6(L) = \lim_{\epsilon \to 0+} \sigma_{e^{\frac{2\pi i}{6}+\epsilon}}(L).$$

Throughout this paper, we use the sign convention that associates a positive signature function to positive torus links. Unlike $\sigma_{e^{\frac{2\pi i}{6}}}(L)$, $\sigma_6(L)$ provides a lower bound on the topological 4-genus of L, even if the Alexander polynomial of L vanishes at $t = e^{\frac{2\pi i}{6}}$. In particular, we have $\sigma_6(3_1) = 2$, an important fact for our purpose.

THEOREM 1·1. *There exist constants $a_0, b_0, c_0 > 0$ with the following property. For all $m, n, N \in \mathbb{N}$ with $N \geq (7/24)mn$:*

$$|d_\chi(T(m, n), 3_1^N) + \sigma_6(T(m, n)) - \sigma_6(3_1^N)| \leq a_0 m + b_0 n + c_0.$$

The value of $\sigma_6(T(m, n))$ is easy to extract from the work of Gambaudo and Ghys on the signature function on braid groups. Indeed, [5, proposition 5·2] implies that the function $n \mapsto \sigma_6(T(m, n))$ is a quasimorphism of slope $5/18$, provided m is divisible by 6. This implies $\sigma_6(T(m, n)) \approx (5/18)mn$, up to an affine error in m and n, for all $m, n \in \mathbb{N}$. In the special case of coprime parameters m, n, i.e. for torus knots $T(m, n)$, the cobordism distance $d_\chi(T(m, n), 3_1^N)$ coincides with twice the minimal genus among all topological locally flat cobordisms in $\mathbb{R}^3 \times [0, 1]$ between the torus knot $T(m, n) \subset \mathbb{R}^3 \times \{1\}$ and the connected sum of N trefoil knots $3_1^N \subset \mathbb{R}^3 \times \{0\}$. This is because the Euler characteristic χ and the genus g of a cobordism $\Sigma \subset \mathbb{R}^3 \times [0, 1]$ with two boundary components are related by $\chi = -2g$. As a consequence, we obtain the following.

COROLLARY 1·2. *There exist constants $a_1, b_1, c_1 > 0$ with the following property. For all $m, n, N \in \mathbb{N}$ with $\gcd(m, n) = 1$ and $N \geq (7/24)mn$, the minimal genus g_4 among all topological locally flat cobordisms in $\mathbb{R}^3 \times [0, 1]$ between the torus knot $T(m, n) \subset \mathbb{R}^3 \times \{1\}$ and the connected sum of N trefoil knots $3_1^N \subset \mathbb{R}^3 \times \{0\}$ satisfies*

$$g_4 = N - \frac{5}{36}mn + E,$$

with an error term $|E| \leq a_1 m + b_1 n + c_1$.

Another important consequence of Theorem 1·1 concerns a large class of additive concordance invariants. Here we call a link invariant additive, if it is additive under all connected sums of links. We define a clover invariant to be an additive link invariant ρ with the following two properties:

(i) $\rho(3_1) = 2$;
(ii) $|\rho(L_1) - \rho(L_2)| \leq d_\chi(L_1, L_2)$, for all links L_1, L_2.

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The second item implies $|\rho(K)| \leq 2g_4(K)$ for all knots K, where $g_4(K) = (1/2)d_\chi(K, O)$ denotes the (locally flat) topological 4-genus of K, i.e. half the cobordism distance between K and the trivial knot O. As a consequence, ρ vanishes on topologically slice knots. Moreover, additivity implies that ρ is a topological concordance invariant. An important family of clover invariants is given by the Levine-Tristram signature invariants $\sigma_{e^{2\pi i\theta}}$ associated with $\theta \in (1/6, 1/2]$, and the limit invariant σ_6 defined above.

COROLLARY 1·3. *There exist constants $a_2, b_2, c_2 > 0$, so that the following inequality holds for all clover invariants ρ, and for all $m, n \in \mathbb{N}$:*

$$\rho(T(m,n)) \geq \frac{5}{18}mn - a_2m - b_2n - c_2.$$

The discussion after Theorem 1·1 shows that the quadratic part of the lower bound, $(5/18)mn$, is sharp, since $\rho = \sigma_6$ is a clover invariant. In summary, the restriction of the invariant $\rho = \sigma_6$ to torus links is essentially dominated by every clover invariant.

It is easy to extract explicit values for the constants appearing in Theorem 1·1 and Corollaries 1·2 and 1·3. A careful inspection of the proofs shows that the constants a_k, b_k can be chosen to be about 20, while the constants c_k can be chosen to be about 200.

The proof of Theorem 1·1 consists of two major steps, which we present in the following two sections. First, a rather involved construction of minimal cobordisms between 6-strand torus links and large connected sums of trefoil knots. This is motivated by a result on the cobordism distance between closed positive 3-braids and connected sums of trefoil knots [**3**]. Second, a cabling construction which yields almost minimal cobordisms between general torus links and large connected sums of trefoil knots. The second step makes essential use of McCoy's twisting method [**9**]. The proof of Corollary 1·3 is short and simple; we present it in the last section.

2. *Torus links with 6 strands*

In this section we derive an almost precise expression for the topological cobordism distance between 6-strand torus links and large connected sums of trefoil knots. Here and throughout this paper, we make use of the fact that the cobordism distance $d_\chi(L_1, L_2)$ is bounded below by the difference $|\sigma_6(L_1) - \sigma_6(L_2)|$. This is true, since σ_6 is a limit of Levine-Tristram signature invariants σ_ω, and the lower bound holds for all σ_ω associated with non-algebraic numbers $\omega \in S^1$ [**10**].

PROPOSITION 2·1. *For all $m, n \in \mathbb{N}$ with $n \geq (5/3)m$:*

$$d_\chi(T(6,m), 3_1^n) = \sigma_6(3_1^n) - \sigma_6(T(6,m)) + E(m,n),$$

where E(m, n) is a globally bounded error term.

A direct application of Proposition 5.2 (for $\theta = 1/6$) and Remark 1 in [**5**] shows $\sigma_6(T(6,m)) = (5/3)m + E(m)$, where $E(m) \leq 12$. Therefore, in order to prove Proposition 2·1, we need to construct a connected cobordism with Euler characteristic of absolute value about $2n - (5/3)m$ between the two links $T(6, m)$ and 3_1^n. This cobordism will in fact be a sequence of smooth saddle moves and smooth concordances, so that Proposition 2·1 remains true in the smooth category.

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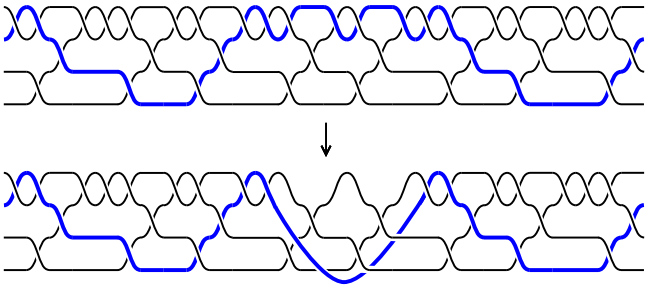


Fig. 1. $(a^2cba^3cb)^4 = (abc)^{12}$.

As a preparation, we derive an algebraic statement about the third power of the central element $(abc)^4$ in the braid group B_4. Here, for simplicity, we denote the standard generators of B_4 by a, b, c instead of the commonly used $\sigma_1, \sigma_2, \sigma_3$. Let $\alpha, \beta \in B_4$ be braids represented by words in the generators a, b, c. We say that β is related to α by a negative t_3-move, if α is obtained from β by removing the third power of any of the standard generators, anywhere in the braid word β. As observed in [**3**, lemma 2·2], the link $\hat{\beta}$ and the connected sum of links $\hat{\alpha}\#3_1$ are then related by a single saddle move, in particular

$$d_\chi(\hat{\beta}, \hat{\alpha}\#3_1) = 1.$$

LEMMA 2·2. *The braid $\beta = a^{-3}c^{-3}(abc)^{12} \in B_4$ can be transformed into the trivial braid by a sequence of* 10 *negative t_3-moves.*

The proof just below also implies the following, more natural, statement, which was already known to Coxeter [**4**]: the braid $(abc)^{12} \in B_4$ can be transformed into the trivial braid by a sequence of 12 negative t_3-moves. However, we will need the more specific formulation of Lemma 2·2 in the proof of Proposition 2·1.

Proof of Lemma 2·2. We use the following algebraic identity, which is a variation of the well-known equality $(abc)^{12} = (a^2cb)^9$ in B_4 stated in [**4**]:

$$(abc)^{12} = (a^2cba^3cb)^4 = \gamma.$$

Figure 1 shows an isotopy between the braid $(a^2cba^3cb)^4$ and a 4-braid which is easy to identify as the third power of a full twist on four strands, i.e. $(abc)^{12}$.

After applying 4 negative t_3-moves to γ, we obtain the braid

$$(a^2(cb)^2)^4 = c^2(a^2bc^3)^3a^2bc.$$

Another 3 negative t_3-moves transform the latter into

$$c^2(a^2b)^3a^2bc = c^2(a^3b)^3c = \delta.$$

Here we use the identity $(a^2b)^4 = (a^3b)^3$. Another 3 negative t_3-moves (removing the second and third instance of a^3, then b^3) transform δ into $c^2a^3c = c^3a^3$. We have just seen that the positive braid $(abc)^{12}$ can be transformed into the positive braid c^3a^3 by a sequence of $4 + 3 + 3 = 10$ negative t_3-moves. Therefore, the braid $\beta = a^{-3}c^{-3}(abc)^{12} \in B_4$ can be transformed into the trivial braid by a sequence of 10 negative t_3-moves.

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Proof of Proposition 2·1. We may assume $m = 6k$, since every positive 6-strand torus link is related to $T(6, 6k)$ by a sequence of at most 15 saddle moves, thus by a smooth cobordism of Euler characteristic at most 15. This operation does not change the value $\sigma_6(T(6, m))$ by more than 15. Furthermore, we need only consider the case $n = 10k$, for the following reason: for all $n' > n$,

$$d_\chi(3_1^{n'}, 3_1^n) = 2(n' - n) = \sigma_6(3_1^{n'}) - \sigma_6(3_1^n).$$

Indeed, the two knots 3_1^n, $3_1^{n'}$ are related by $n' - n$ crossing changes, thus by a smooth cobordism of Euler characteristic $2(n' - n)$. In the first step, we construct a smooth cobordism of small Euler characteristic between the link $T(6, 6k)$ and the closure of the braid

$$(dced(bacb)^5a^3c^3)^{k-3},$$

where a, b, c, d, e denote the standard generators of the braid group B_6. For this, we view $T(6, 6k)$ as a 2-cable of $T(3, 3k)$. In [**2**], a special positive braid representing the link $T(3, 3k)$ is derived, which depends on the parity of k. We only present the odd case $k = 2l + 1$ here; the even one is virtually the same. The link $T(3, 6l + 3)$ is isotopic to the closure of the 3-braid

$$(ba^4ba^3(ba^5)^{l-1})^2.$$

By replacing $a, b \in B_3$ by $bacb, dced \in B_6$, respectively, and introducing the correct framing of the 2-cable in front, we obtain the following 6-braid representing the link $T(6, 6k) = T(6, 12l + 6)$:

$$(ace)^{4l+2}(dced(bacb)^4dced(bacb)^3(dced(bacb)^5)^{l-1})^2.$$

The easiest way to check that the framing $(ace)^{4l+2}$ is correct is by computing the total number of crossings, which must coincide with the crossing number $c(T(6, 12l + 6)) = 60l + 30$. This is indeed enough: all positive 6-braid representatives of the link $T(6, 12l + 6)$ have the same crossing number cr, determined by the Euler characteristic χ of the canonical fibre surface via the formula $\chi = 6 - \mathrm{cr}$. The precise location of the framing is not relevant; in particular, we may slide it along the core link $T(3, 6l + 3)$ and distribute it right after the brackets $(bacb)^5$. As a result, after smoothing a bounded number of crossings by saddle moves (90, to be precise), the above braid can be transformed into the braid

$$\beta = (dced(bacb)^5a^3c^3)^{2l-2}.$$

Now comes the second step: The braid β is easily identified as

$$(dced(bacb)^{-1}(bacb)^6a^3c^3)^{2l-2} = (dced(bacb)^{-1}a^{-3}c^{-3}(abc)^{12})^{2l-2},$$

since the 4-braid $(bacb)^6$ is a 2-cable of the 2-braid a^6.

Thanks to Lemma 2·2, the braid β can be reduced to the braid

$$\alpha = (dced(bacb)^{-1})^{2l-2}$$

by a sequence of $10 \cdot (2l - 2)$ negative t_3-moves. As stated just before Lemma 2·2, the two links $\hat{\beta}$ and $\hat{\alpha}\#3_1^{20l-20}$ are thus related by a sequence of $20l - 20$ saddle moves. Moreover, the link $\hat{\alpha}$ can be transformed into a smoothly slice knot by a constant number of saddle moves, about ten in number. Indeed, after five suitable saddle moves, the link $\hat{\alpha}$ turns into

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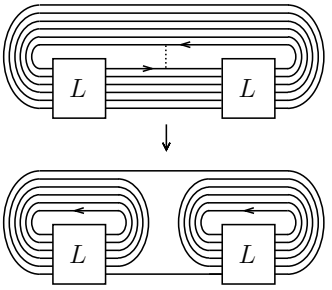


Fig. 2. Five saddle moves.

the connected sum of links $L\#L$, where L is the closure of the braid $(dced(bacb)^{-1})^{l-1}$, see Figure 2. The latter is isotopic to its mirror image, so $L\#L$ is smoothly concordant to the trivial link with six components. Another five saddle moves transform this trivial link into the trivial knot. As a consequence, the original link $T(6, 12l+6)$ can be transformed into the connected sum of trefoil knots 3_1^{20l} by a sequence of about $20l$ saddle moves and link concordances, up to a bounded error. Keeping in mind $m = 6k = 12l + 6$ and $n = 10k = 20l + 10$, we get indeed

$$\begin{aligned} d_\chi(T(6,m), 3_1^n) &= 20l + C(m,n) \\ &= 2n - \frac{5}{3}m + 10 + C(m,n) \\ &= \sigma_6(3_1^n) - \sigma_6(T(6,m)) + E(m,n), \end{aligned}$$

with globally bounded error terms $C(m, n)$, $E(m, n)$.

The above proof produces an explicit upper bound smaller than 200 on the error term $E(n, m)$; this is far from optimal since we tried to keep the argument short.

3. *Twisting torus links*

The proof of Theorem 1·1 relies on McCoy's twisting method [9]. A null-homologous twist is an operation on oriented links that takes place around a disc that intersects an even number of strands of a link transversely, with equally many strands going in either direction. A positive (resp. negative) twist inserts a positive (resp. negative) full twist into these strands. As an example, the torus link $T(2k, 2k)$ is related to the disjoint union of two torus links of type $T(k, 2k)$ by a single negative twist. A special case of Theorem 1·1 in [9] states that if an oriented knot K can be transformed into the trivial knot by a sequence of t positive and t negative null-homologous twists, then $g_4(K) \leq t$. It is the combination of the positive and negative twists that allows us to prove the following lemma, which is the second key ingredient in the proof of Theorem 1·1.

LEMMA 3·1 *For all $k, l \in \mathbb{N}$ coprime and $t \geq (1/2)(k-1)(l-1)$:*

$$d_\chi(T(6k, 6l), T(6, 6kl)\#3_1^t) \leq 2t + 10.$$

There is an ambiguity in the meaning of the direct sum $T(6, 6kl)\#3_1^t$ in the above statement; we use the convention where all the trefoil summands are attached to the same component of the link $T(6, 6kl)$.

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Proof of Lemma 3·1. We start by observing that the link $T(6k, 6l)$ is a 6-cable of the torus knot $T(k, l)$ with framing kl. Indeed, all components of $T(6k, 6l)$ have pairwise linking number kl. The knot $T(k, l)$ can be transformed into the trivial knot by a sequence of $t = (1/2)(k-1)(l-1)$ negative crossing changes. As a consequence, the link $T(6k, 6l)$ can be transformed into the kl-framed (6,0)-cable of the trivial knot, i.e. into the torus link $T(6, 6kl)$, by a sequence of t negative null-homologous twists (compare [**9**, section 5]). In order to apply McCoy's 4-genus bound, we need to consider knots rather than links. Let K be the 0-framed (6,1)-cable of the knot $T(k, l)$. By definition, the knot K is represented by the braid

$$(abcde)^{-1-6kl}\delta \in B_{6k},$$

where $\delta \in B_{6k}$ is the standard braid representing the torus link $T(6k, 6l)$, and a, b, c, d, e denote the first five standard generators of the braid group B_{6k}. Moreover, the knot K can be transformed into the trivial knot by a sequence of t negative null-homologous twists. In turn, the knot $K\#3_1^{-t}$ can be transformed into the trivial knot by a sequence of t negative and t positive null-homologous twists, since we can remove one negative trefoil summand with each positive twist. As a consequence $g_4(K\#3_1^{-t}) \leq t$, hence

$$d_\chi(K, 3_1^t) \leq 2t.$$

We are nearly done, since the link $T(6k, 6l)$ and the link $T(6, 6kl)\#K$ are related by a sequence of just 10 saddle moves:

$$\begin{aligned} d_\chi(T(6k, 6l), T(6, 6kl)\#3_1^t) &\leq d_\chi(T(6, 6kl)\#K, T(6, 6kl)\#3_1^t) + 10 \\ &= d_\chi(K, 3_1^t) + 10 \\ &\leq 2t + 10. \end{aligned}$$

□

Before we prove Theorem 1·1, we invoke again the formula of Gambaudo and Ghys for $\sigma_6(T(m, n))$ [**5**, proposition 5·2]. Their formula holds in fact for a homogenised version of the Levine-Tristram invariant denoted by $\mathrm{Sign}_{e^{\frac{2\pi i}{6}}}$. By [**5**, remark 1], the restriction of the latter to the braid group B_m differs from the invariant $\sigma_{e^{\frac{2\pi i}{6}}}$, and thus from our limit invariant σ_6, by a bounded error of size at most $2m$ (two times the braid index). We obtain the following estimate from their formula, valid for all m divisible by six:

$$|\sigma_6(T(m, n)) - \frac{5}{18}mn| \leq 2m.$$

Since we allow for an affine error in m and n, we may use the approximate formula $\sigma_6(T(m, n)) \approx (5/18)mn$ for all $m, n \in \mathbb{N}$.

Proof of Theorem 1·1. Let $m, n \in \mathbb{N}$. We may replace the link $T(m, n)$ by a link of the form $T(6k, 6l)$ with $|m - 6k| \leq 3$, $|n - 6l| \leq 3$. This changes the value of $\sigma_6(T(m, n))$ and $d_\chi(T(m, n), 3_1^N)$ by $3(m + n)$, at most. Therefore, in order to prove Theorem 1·1, we need to construct a connected cobordism with Euler characteristic of absolute value about $2N - (5/18)mn = 2N - 10kl$ between the two links $T(6k, 6l)$ and 3_1^N, for all $N \geq (7/24)mn = (21/2)kl$. For simplicity, we assume that k, l are coprime. The general case is just a variation on this: if k, l are not coprime, we can transform the link $T(k, l)$ into a positive braid knot by smoothing at most k crossings. As a consequence, the link $T(6k, 6l)$ can

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be transformed into a 6-cable of a positive braid knot by a sequence of at most $36k$ saddle moves.

We are finally in the position to put together the two main steps of the argument. First, by Lemma 3·1,

$$d_\chi(T(6k,6l),T(6,6kl)\#3_1^t)\leq 2t+10,$$

for all $t\geq(1/2)(k-1)(l-1)$. Second, by Proposition 2·1,

$$d_\chi(T(6,6kl),3_1^n)\approx\sigma_6(3_1^n)-\sigma_6(T(6,6kl))\approx 2n-10kl,$$

up to a globally bounded error term, for all $n\geq 10kl$. Putting these two bounds together, and setting $N=t+n$ with $t\geq(1/2)kl$ and $n\geq 10kl$, we obtain

$$\begin{aligned} d_\chi(T(6k,6l),3_1^N) &\leq d_\chi(T(6k,6l),T(6,6kl)\#3_1^t)+d_\chi(T(6,6kl)\#3_1^t,3_1^N)\\ &= d_\chi(T(6k,6l),T(6,6kl)\#3_1^t)+d_\chi(T(6,6kl),3_1^n)\\ &\leq 2t+10+2n-10kl\approx 2N-10kl, \end{aligned}$$

up to a globally bounded error term, for all $N\geq(21/2)kl$, as required.

4. *A lower bound on clover invariants*

We consider a clover invariant, i.e. an additive link invariant ρ satisfying $\rho(3_1)=2$ and $|\rho(L_1)-\rho(L_2)|\leq d_\chi(L_1,L_2)$, for all links L_1,L_2. The second property together with Theorem 1·1 implies for all $N\geq(7/24)mn$:

$$\begin{aligned} |\rho(T(m,n))-\rho(3_1^N)| &\leq d_\chi(T(m,n),3_1^N)\\ &\leq 2N-\sigma_6(T(m,n))+a_0m+b_0n+c_0\\ &\leq 2N-\frac{5}{18}mn+a_2m+b_2n+c_2, \end{aligned}$$

for suitable constants $a_2,b_2,c_2>0$. The last inequality holds thanks to the formula by Gambaudo and Ghys discussed in the paragraph after Theorem 1·1. This concludes the proof of Corollary 1·3, since the normalisation $\rho(3_1^N)=2N$ implies

$$\rho(T(m,n))\geq\frac{5}{18}mn-a_2m-b_2n-c_2.$$

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