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Infinite-step nilsystems, independence and complexity

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Abstract. An ∞-step nilsystem is an inverse limit of minimal nilsystems. In this article, it is shown that a minimal distal system is an ∞-step nilsystem if and only if it has no non-trivial pairs with arbitrarily long finite IP-independence sets. Moreover, it is proved that any minimal system without non-trivial pairs with arbitrarily long finite IP-independence sets is an almost one-to-one extension of its maximal ∞-step nilfactor, and each invariant ergodic measure is isomorphic (in the measurable sense) to the Haar measure on some ∞-step nilsystem. The question if such a system is uniquely ergodic remains open. In addition, the topological complexity of an ∞-step nilsystem is computed, showing that it is polynomial for each non-trivial open cover.

1. *Introduction*

In this paper, we introduce the notion of an ∞-step nilsystem and study its relationship with the concept of independence. We also study its topological complexity. In this section, first we discuss the motivations for this subject and then we state the main results of the article.

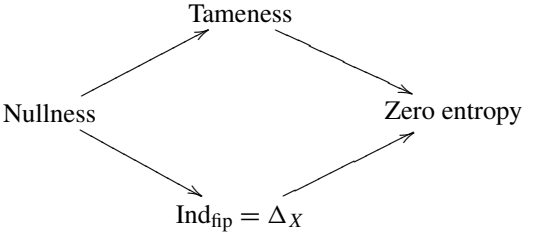
1.1. *Motivations.* By a *topological dynamical system* (TDS), we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism.

There are several motivations for studying this subject. The first one comes from the so called *local entropy theory*, for a survey see [**12**]. Each TDS admits a maximal

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zero topological entropy factor, and this factor is induced by the smallest closed invariant equivalence relation containing *entropy pairs* [**5**]. In [**22**], entropy pairs are characterized as those pairs that admit an *interpolating set* of positive density. Later on, the notions of *sequence* entropy pairs [**20**] and *untame* pairs (called scrambled pairs in [**17**]) were introduced. In [**24**], the concept of *independence* was extensively studied and used to unify the aforementioned notions. Let (X, T) be a TDS and $\mathcal{A} = (A_1, \ldots, A_k)$ be a tuple of subsets of X. We say that a subset $F \subseteq \mathbb{Z}_+$ is an *independence set* for $\mathcal{A}$ if for any non-empty finite subset $J \subseteq F$ and any $s = (s(j) : j \in J) \in \{1, \ldots, k\}^J$ we have $\bigcap_{j\in J} T^{-j} A_{s(j)} \neq \emptyset$. It is shown that a pair of points x, y in X is a sequence entropy pair if and only if each $\mathcal{A} = (A_1, A_2)$, with A_1 and A_2 neighborhoods of x and y, respectively, has arbitrarily long finite independence sets. Also, the pair is an untame pair if and only if each $\mathcal{A} = (A_1, A_2)$ as before has infinite independence sets. It is known that each TDS admits a maximal zero sequence entropy factor, i.e., a null factor [**20**], which is induced by the smallest closed invariant equivalence relation containing sequence entropy pairs, and a maximal tame factor [**24**], which is induced by the smallest closed invariant equivalence relation containing untame pairs. It was shown in [**10, 20, 24**] that a minimal null (respectively tame) system is an almost 1–1 extension of an equicontinuous TDS and is uniquely ergodic. For a similar study see [**27**]. Moreover, in the equicontinuous case the uniquely ergodic measure is measure theoretically isomorphic to the Haar measure of the underlying Abelian group.

To get a better understanding of the role of the notion of independence in a TDS, in [**18, 19**] the authors systematically investigate the independence for a given collection of subsets of $\mathbb{Z}_+$. For a finite subset $\{p_1, \ldots, p_m\}$ of $\mathbb{N}$, the *finite IP-set* generated by $\{p_1, \ldots, p_m\}$ is the set $\{\epsilon_1 p_1 + \cdots + \epsilon_m p_m : \epsilon_i \in \{0, 1\}, 1 \leq i \leq m\}\backslash\{0\}$. The notion of $\mathrm{Ind}_{\mathrm{fip}}$-pair is introduced and studied in [**19**]: a pair of points (x, y) in X is an $\mathrm{Ind}_{\mathrm{fip}}$-pair if and only if each $\mathcal{A} = (A_1, A_2)$, with A_1 and A_2 neighborhoods of x and y, respectively, has arbitrarily long finite IP-independence sets. Among other results, it is shown that the $\mathrm{Ind}_{\mathrm{fip}}$-pair relation has the lifting property, i.e., if $\pi : (X, T) \longrightarrow (Y, S)$ is a factor map between two TDSs, then $\pi \times \pi(\mathrm{Ind}_{\mathrm{fip}}(X, T)) = \mathrm{Ind}_{\mathrm{fip}}(Y, S)$, where $\mathrm{Ind}_{\mathrm{fip}}(X, T)$ is the set of all $\mathrm{Ind}_{\mathrm{fip}}$-pairs of (X, T). It is clear that



So it is interesting to understand the dynamical properties of a minimal TDS without $\mathrm{Ind}_{\mathrm{fip}}$-pairs.

A second motivation comes from the study of the dynamics of nilsystems. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system.

In [**15**], to study the convergence of some non-conventional ergodic averages in this system, the authors proved that the characteristic factors for such averages in $L^2(X, \mathcal{B}, \mu)$

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are d-step nilsystems for some integer $d \geq 1$ (see also [**34**]). Then, in the topological setting, in [**16**] the authors defined the notion of a regionally proximal relation of order d associated to a TDS (X, T), $\mathbf{RP}^{[d]}$, and showed that if the system is minimal and distal, then $\mathbf{RP}^{[d]}$ is an equivalence relation and $(X/\mathbf{RP}^{[d]}, T)$ is the maximal d-step nilfactor of the system. In a recent preprint [**29**], this result was generalized to arbitrarily minimal TDSs. When studying minimal distal systems carefully, one finds that if $(x, y) \in \mathbf{RP}^{[d]}$ for some integer d, then each $\mathcal{A} = (A_1, A_2)$, with A_1 and A_2 neighborhoods of x and y, respectively, has a finite IP-independence set of length $n(d)$ such that $\lim_{d\to\infty} n(d) = \infty$. This means that if $(x, y) \in \mathbf{RP}^{[\infty]} = \bigcap_{d\geq 1} \mathbf{RP}^{[d]}$, then $(x, y) \in \mathrm{Ind}_{\mathrm{fip}}(X, T)$. This is the main reason leading us to define ∞-step nilsystems and to study properties of minimal TDSs without $\mathrm{Ind}_{\mathrm{fip}}$-pairs.

The third motivation comes from the theory of *local complexity* in topological dynamics. In [**4**], the authors introduced the notion of topological complexity for a TDS using open covers, and showed that a TDS is equicontinuous if and only if each non-trivial open cover has a bounded complexity. For a further development, see [**21**]. An interesting, but more difficult, question is to understand TDSs with polynomial complexity or extensions of such systems. It appears (this is proved in §7) that inverse limits of nilsystems are special systems with polynomial complexity.

1.2. *Main results of the paper.* In this paper, we study ∞-step nilsystems, which are TDSs with trivial $\mathbf{RP}^{[\infty]}$, i.e., $\mathbf{RP}^{[\infty]} = \bigcap_{d=1}^{\infty} \mathbf{RP}^{[d]} = \Delta$, the diagonal. First, we prove that a minimal system is an ∞-step nilsystem if and only if it is an inverse limit of minimal nilsystems. Then we study the relation of ∞-step nilsystems and independence pairs. It is proved that any minimal system without non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs is an almost one-to-one extension of its maximal ∞-step nilfactor. Moreover, a minimal distal system is an ∞-step nilsystem if and only if it has no non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs. We observe that there are plenty of minimal distal systems which are not ∞-step nilsystems, though it is not easy to construct explicit examples.

In addition, we show for any minimal system without non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs that each invariant ergodic measure is measure theoretically isomorphic to the Haar measure on some ∞-step nilsystem. We conjecture that such classes of systems are uniquely ergodic.

Finally, we prove that the topological complexity of an ∞-step nilsystem is polynomial for each non-trivial open cover.

1.3. *Organization of the paper.* We organize the paper as follows. In §2, we introduce basic notions and facts we will meet in this paper. Then we define ∞-step nilsystems in §3. In §4, we study the relationship between ∞-step nilsystems and independence pairs, and in §5, we give some examples. In §6, we study the conjecture concerning unique ergodicity, and state some further ideas and questions. Finally, in §7, we discuss the complexity of an ∞-step nilsystem. Moreover, we give proofs of some results stated in §3 in Appendix A.

2. *Preliminaries*

2.1. *Topological dynamical systems.* A *transformation* of a compact metric space X is a homeomorphism of X to itself. A *topological dynamical system* (TDS) or just a *system*,

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is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a transformation. We use $\rho(\cdot, \cdot)$ to denote the metric on X. In the following, and if there is no confusion, in any TDS we will always use T to indicate the transformation.

We will also make use of a more general definition of a system, that is, instead of just considering a single transformation T, we will consider commuting homeomorphisms $T_1, \ldots, T_k$ of X. We recall some basic definitions and properties of systems in the classical setting of one transformation. Extensions to the general case are straightforward.

A system (X, T) is *transitive* if there exists $x \in X$ whose orbit $\mathrm{orb}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X and such a point is called a *transitive point*. The system is *minimal* if the orbit of any point is dense in X. This property is equivalent to saying that X and the empty set are the unique closed invariant subsets of X.

Let (X, T) be a system and $\mathcal{M}(X)$ be the set of Borel probability measures in X. A measure $\mu \in \mathcal{M}(X)$ is *T-invariant* if for any Borel set B of X, $\mu(T^{-1}B) = \mu(B)$. Denote by $\mathcal{M}(X, T)$ the set of invariant probability measures. A measure $\mu \in \mathcal{M}(X, T)$ is *ergodic* if for any Borel set B of X satisfying $\mu(T^{-1}B \triangle B) = 0$, we have $\mu(B) = 0$ or $\mu(B) = 1$. Denote by $\mathcal{M}^e(X, T)$ the set of ergodic measures. The system (X, T) is *uniquely ergodic* if $\mathcal{M}^e(X, T)$ consists of only one element.

A *homomorphism* between the TDS (X, T) and (Y, T) is a continuous onto map $\pi : X \to Y$ which intertwines the actions; one says that (Y, T) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, T). One also refers to π as a *factor map* or an *extension*, and one uses the notation $\pi : (X, T) \to (Y, T)$. The systems are said to be *conjugate* if π is a bijection. An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\} = (\pi \times \pi)^{-1}\Delta_Y \subset X \times X$, where Δ_Y is the diagonal on Y. An extension $\pi : (X, T) \to (Y, T)$ is *almost one-to-one* if the G_δ set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is dense.

2.2. *Distality and proximality.* Let (X, T) be a TDS. A pair $(x, y) \in X \times X$ is a *proximal* pair if

$$\inf_{n \in \mathbb{Z}} \rho(T^n x, T^n y) = 0,$$

and is a *distal* pair if it is not proximal. Denote by $P(X, T)$ or P_X the set of proximal pairs of (X, T). The TDS (X, T) is *distal* if (x, y) is a distal pair whenever $x, y \in X$ are distinct.

An extension $\pi : (X, T) \to (Y, S)$ is *proximal* if $R_\pi \subset P(X, T)$ and is *distal* if $R_\pi \cap P(X, T) = \Delta_X$. Observe that when Y is trivial (reduced to one point), the map π is distal if and only if (X, T) is distal.

2.3. *Independence.* The notion of *independence* was first introduced and studied in [**24**, Definition 2.1]. It corresponds to a modification of the notion of *interpolating* studied in [**11, 22**] and was considerably discussed in [**18, 19**].

Definition 2.1. Let (X, T) be a TDS. Given a tuple $\mathcal{A} = (A_1, \ldots, A_k)$ of subsets of X, we say that a subset $F \subset \mathbb{Z}_+$ is an *independence set* for $\mathcal{A}$ if for any non-empty finite subset $J \subset F$ and any $s = (s(j) : j \in J) \in \{1, \ldots, k\}^J$, we have

$$\bigcap_{j \in J} T^{-j} A_{s(j)} \neq \emptyset.$$

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We will denote the collection of all independence sets for $\mathcal{A}$ by $\mathrm{Ind}(A_1, \ldots, A_k)$ or $\mathrm{Ind}\mathcal{A}$.

A finite subset F of $\mathbb{Z}_+$ is called a *finite IP-set* if there exists a finite subset $\{p_1, p_2, \ldots, p_m\}$ of $\mathbb{N}$ such that

$$F = FS(\{p_i\}_{i=1}^m) = \{p_{i_1} + \cdots + p_{i_k} : 1 \le i_1 < \cdots < i_k \le m\}.$$

Now we define $\mathrm{Ind}_{\mathrm{fip}}$-pairs.

Definition 2.2. Let (X, T) be a TDS. A pair $(x_1, x_2) \in X \times X$ is called an $\mathrm{Ind}_{\mathrm{fip}}$*-pair* if for any neighborhoods U_1, U_2 of x_1 and x_2, respectively, $\mathrm{Ind}(U_1, U_2)$ contains arbitrarily long finite IP-sets. Denote by $\mathrm{Ind}_{\mathrm{fip}}(X, T)$ the set of all $\mathrm{Ind}_{\mathrm{fip}}$-pairs of (X, T).

2.4. *Parallelepipeds.* Let X be a set, $d \ge 1$ be an integer, and write $[d] = \{1, 2, \ldots, d\}$. We view $\{0, 1\}^d$ in one of two ways, either as a sequence $\epsilon = \epsilon_1 \ldots \epsilon_d := (\epsilon_1, \ldots, \epsilon_d)$ of zeros and ones, or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\mathbf{0} = (0, 0, \ldots, 0) \in \{0, 1\}^d$ is the same as $\emptyset \subset [d]$.

If $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$\mathbf{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i.$$

If we consider ϵ as $\epsilon \subset [d]$, then $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

We denote X^{2^d} by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon : \epsilon \in \{0, 1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_\emptyset = x_{\mathbf{0}}$ is the first coordinate of $\mathbf{x}$. As examples, points in $X^{[2]}$ are written

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}),$$

and points in $X^{[3]}$ look like

$$\begin{aligned}&(x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111})\\ &\quad = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}, x_{\{3\}}, x_{\{1,3\}}, x_{\{2,3\}}, x_{\{1,2,3\}}).\end{aligned}$$

For $x \in X$, we write $x^{[d]} = (x, x, \ldots, x) \in X^{[d]}$. The diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, we denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0, 1\}^{d-1})$ and $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0, 1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d - 1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\epsilon : \epsilon \ne \emptyset) \in X_*^{[d]}$.

Identifying $\{0, 1\}^d$ with the set of vertices of the Euclidean unit cube, a Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $\mathbf{x} \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$.

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2.5. *Dynamical parallelepipeds.* We follow definitions from [**16**].

Let (X, T) be a TDS and $d \geq 1$ be an integer.

We define the set of (dynamical) *parallelepipeds of dimension d*, $\mathbf{Q}^{[d]}(X)$ (or just $\mathbf{Q}^{[d]}$), as the closure in $X^{[d]}$ of elements of the form

$$(T^{\mathbf{n}\cdot\epsilon}x = T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}x : \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d),$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. It is important to note that $\mathbf{Q}^{[d]}$ is invariant under the Euclidean permutations of $X^{[d]}$.

As examples, $\mathbf{Q}^{[2]}$ is the closure in $X^{[2]} = X^4$ of the set

$$\{(x, T^m x, T^n x, T^{n+m} x) : x \in X, m, n \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}$ is the closure in $X^{[3]} = X^8$ of the set

$$\{(x, T^m x, T^n x, T^{m+n} x, T^p x, T^{m+p} x, T^{n+p} x, T^{m+n+p} x) : x \in X, m, n, p \in \mathbb{Z}\}.$$

Let $\phi : (X, T) \to (Y, T)$ be a factor map. Define $\phi^{[d]} : X^{[d]} \to Y^{[d]}$ by $(\phi^{[d]}\mathbf{x})_\epsilon = \phi x_\epsilon$ for every $\mathbf{x} \in X^{[d]}$ and every $\epsilon \subset [d]$. The *diagonal transformation* of $X^{[d]}$ is the map $T^{[d]}$. We define *face transformations* inductively as follows. Let $T^{[0]} = T$, $T_1^{[1]} = \mathrm{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is already defined, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in \{1, 2, \ldots, d-1\},$$
$$T_d^{[d]} = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $j \in [d]$, the face transformation $T_j^{[d]} : X^{[d]} \to X^{[d]}$ can be defined, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \subset [d]$, by

$$T_j^{[d]}\mathbf{x} = \begin{cases} (T_j^{[d]}\mathbf{x})_\epsilon = Tx_\epsilon, & j \in \epsilon, \\ (T_j^{[d]}\mathbf{x})_\epsilon = x_\epsilon, & j \notin \epsilon. \end{cases}$$

The *face group* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For $\mathcal{G}^{[d]}$ and $\mathcal{F}^{[d]}$, we use similar notations to that used for $X^{[d]}$, namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0, 1\}^d)$. In particular, $\mathcal{F}^{[d]} = \{S \in \mathcal{G}^{[d]} : S_\emptyset = \mathrm{id}\}$.

For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$ instead of $\overline{\mathrm{orb}}(\mathbf{x}, \mathcal{F}^{[d]})$.

It is easy to verify that $\mathbf{Q}^{[d]}$ is the closure in $X^{[d]}$ of

$$\{Sx^{[d]} : S \in \mathcal{F}^{[d]}, x \in X\}.$$

If (X, T) is a transitive system and x a transitive point, then $\mathbf{Q}^{[d]}$ is the closed orbit of $x^{[d]}$ under the group $\mathcal{G}^{[d]}$. Moreover, it is easy to obtain

$$\mathbf{Q}^{[d]} = \overline{\{(T^{\mathbf{n}\cdot\epsilon}x)_{\epsilon\in\{0,1\}^d} : \mathbf{n} \in \mathbb{N}^d, x \in X\}}.$$

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2.6. *Nilmanifolds and nilsystems.* Let G be a group. For $g, h \in G$ and $A, B \subset G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h and $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$ are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that G is *d-step nilpotent* if G_{d+1} is the trivial subgroup.

Let G be a d-step nilpotent Lie group and Γ be a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a *d-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$ of X. Then (X, μ, T) is called a *d-step nilsystem*. In the topological setting we omit the measure and just say that (X, T) is a d-step nilsystem.

We will need to use inverse limits of nilsystems, so we recall the definition of a sequential inverse limit of systems. If $(X_i, T_i)_{i\in\mathbb{N}}$ are systems with $\operatorname{diam}(X_i) \leq 1$ and $\pi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \pi_i(x_{i+1}) = x_i\}$, and we denote it by $\varprojlim(X_i, T_i)_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} 1/2^i \rho_i(x_i, y_i)$, where ρ_i is the metric in X_i. We note that the maps T_i induce naturally a transformation T on the inverse limit.

The following structure theorem characterizes inverse limits of nilsystems using dynamical parallelepipeds.

THEOREM 2.3. (Host–Kra–Maass) [**16**, Theorem 1.2] *Assume that (X, T) is a transitive topological dynamical system and let $d \geq 2$ be an integer. The following properties are equivalent.*

(1) *If $\mathbf{x}, \mathbf{y} \in \mathbf{Q}^{[d]}$ have $2^d - 1$ coordinates in common, then $\mathbf{x} = \mathbf{y}$.*
(2) *If $x, y \in X$ are such that $(x, y, \ldots, y) \in \mathbf{Q}^{[d]}$, then $x = y$.*
(3) *X is an inverse limit of $(d-1)$-step minimal nilsystems.*

A transitive system satisfying one of the equivalent properties above is called a *system of order* $(d-1)$.

3. ∞-step nilsystems

3.1. *Regionally proximal relation of order d.* First we recall a fundamental relation introduced in [**16**] allowing us to characterize maximal nilfactors in [**16**] (for minimal distal systems) and in [**29**] (for general minimal systems).

Definition 3.1. Let (X, T) be a system and let $d \in \mathbb{N}$. The points $x, y \in X$ are said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \quad \text{for any non-empty } \epsilon \subset [d].$$

In other words, there exists $S \in \mathcal{F}^{[d]}$ such that $\rho(S_\epsilon x', S_\epsilon y') < \delta$ for every $\epsilon \neq \emptyset$. The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}$ (or by $\mathbf{RP}^{[d]}(X, T)$ in case of ambiguity), and is called *the regionally proximal relation of order d*.

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It is easy to see that $\mathbf{RP}^{[d]}$ is a closed and invariant relation. Observe that

$$\mathbf{P}(X, T) \subseteq \cdots \subseteq \mathbf{RP}^{[d+1]} \subseteq \mathbf{RP}^{[d]} \subseteq \cdots \subseteq \mathbf{RP}^{[2]} \subseteq \mathbf{RP}^{[1]} = \mathbf{RP}(X, T).$$

The following theorems, proved in [**16**] (for minimal distal systems) and in [**29**] (for general minimal systems), tell us conditions under which (x, y) belongs to $\mathbf{RP}^{[d]}$ and the relation between $\mathbf{RP}^{[d]}$ and d-step nilsystems.

THEOREM 3.2. *Let (X, T) be a minimal system and let $d \in \mathbb{N}$. Then:*

(1) *$(x, y) \in \mathbf{RP}^{[d]}$ if and only if $(x, y, \ldots, y) \in \mathbf{Q}^{[d+1]}$ if and only if $(x, y, \ldots, y) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;*

(2) *$\mathbf{RP}^{[d]}$ is an equivalence relation; and*

(3) *(X, T) is a system of order d if and only if $\mathbf{RP}^{[d]} = \Delta_X$.*

3.2. *∞-step nilsystems.* The regionally proximal relation of order d allows us to construct the maximal d-step nilfactor of a system, that is, any factor of order d (inverse limit of d-step minimal nilsystems) factorizes through this system.

THEOREM 3.3. [**29**] *Let $\pi : (X, T) \to (Y, T)$ be a factor map between minimal systems and let $d \in \mathbb{N}$. Then:*

(1) *$\pi \times \pi(\mathbf{RP}^{[d]}(X, T)) = \mathbf{RP}^{[d]}(Y, T)$; and*

(2) *(Y, T) is a system of order d if and only if $\mathbf{RP}^{[d]}(X, T) \subset R_\pi$.*

In particular, the quotient of (X, T) under $\mathbf{RP}^{[d]}(X, T)$ is the maximal d-step nilfactor of X (i.e., the maximal factor of order d).

It follows that for any minimal system (X, T),

$$\mathbf{RP}^{[\infty]} = \bigcap_{d \geq 1} \mathbf{RP}^{[d]}$$

is a closed invariant equivalence relation (we write $\mathbf{RP}^{[\infty]}(X, T)$ in case of ambiguity). Now we formulate the definition of ∞-step nilsystems or systems of order ∞.

Definition 3.4. A minimal system (X, T) is an *∞-step nilsystem* or *a system of order ∞*, if the equivalence relation $\mathbf{RP}^{[\infty]}$ is trivial, i.e., coincides with the diagonal.

Remark 3.5. Similar to Theorem 3.3, one can show that the quotient of a minimal system (X, T) under $\mathbf{RP}^{[\infty]}$ is the maximal ∞-step nilfactor of (X, T).

Let (X, T) be a minimal system. It is easy to see that if (X, T) is an inverse limit of minimal nilsystems, then (X, T) is an ∞-step nilsystem. Conversely, if (X, T) is a minimal ∞-step nilsystem, then $\mathbf{RP}^{[\infty]} = \Delta_X$. For any integer $d \geq 1$, let (X_d, T) be the quotient of (X, T) under $\mathbf{RP}^{[d]}$. Then $(X, T) = \varprojlim(X_d, T)_{d \in \mathbb{N}}$ as $\Delta_X = \mathbf{RP}^{[\infty]} = \bigcap_{d \geq 1} \mathbf{RP}^{[d]}$. In fact, we can show more as the following theorem shows.

THEOREM 3.6. *A minimal system is an ∞-step nilsystem if and only if it is an inverse limit of minimal nilsystems.*

Proof. It remains to prove that if (X, T) is a minimal ∞-step nilsystem, then it is an inverse limit of minimal nilsystems. First we may assume that $(X, T) = \varprojlim(X_d, T)_{d \in \mathbb{N}}$,

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where $X_d = X/\mathbf{RP}^{[d]}$ for any $d \geq 1$. By Theorem 2.3, for any $d \geq 1$, (X_d, T) is an inverse limit of minimal d-step nilsystems.

We need the following claim.

CLAIM. *Let (Y, S) be a minimal system, and let (Y_i, S) be factors of (Y, S) which are k_i-step nilsystems, where $1 \leq i \leq n$ and $\max\{k_i : 1 \leq i \leq n\} = k$. Then there exists a k-step nilsystem (Z, S) which is a factor of (Y, S) and an extension of (Y_i, S) for $1 \leq i \leq n$.*

Proof of claim. Let π_i be the factor map between (Y, S) and (Y_i, S) and assume (Y_i, S) has the form of $(H_i/\Gamma_i, h_i)$, where $h_i \in H_i$ and S is the left translation by h_i on Y_i. Set $G = H_1 \times \cdots \times H_n$, $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $g = (h_1, \ldots, h_n)$. Then G is a k-step nilpotent Lie group and Γ is a discrete uniform subgroup of G. Let $S : G/\Gamma \to G/\Gamma$ be the transformation $x \mapsto gx$. Choose any point $y \in Y$ and let $Z = \overline{\{g^n(\pi_1(y), \ldots, \pi_n(y)) : n \in \mathbb{Z}\}} \subset G/\Gamma$ be the orbit closure of $(\pi_1(y), \ldots, \pi_n(y))$ under S. Since nilsystems are distal, (Z, S) is minimal. Moreover, it is a k-step nilsystem [**25**], and, of course, (Z, S) is a factor of (Y, S) and it is an extension of (Y_i, S) for $1 \leq i \leq n$.

Now we show that (X, T) is an inverse limit of minimal nilsystems using the previous claim. As (X_1, T) is a system of order 1, it is an inverse limit of some 1-step nilsystems $((X_1)_i, T)_{i\in\mathbb{N}}$ by Theorem 2.3. Similarly, as (X_2, T) is a system of order 2, it is an inverse limit of some 2-step nilsystems $((X_2)_i, T)_{i\in\mathbb{N}}$. Note that all $((X_1)_i, T)_{i\in\mathbb{N}}$ and $((X_2)_i, T)_{i\in\mathbb{N}}$ are factors of X_2. By the above claim, we may reconstruct 2-step nilsystems $((X_2)_i, T)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $((X_2)_{i+1}, T)$ is an extension of $((X_1)_1, T), ((X_1)_2, T), \ldots, ((X_1)_{i+1}, T)$ and $((X_2)_1, T), \ldots, ((X_2)_i, T)$.

Similarly and inductively, for any given $k \in \mathbb{N}$, (X_k, T) can be written as the inverse limit of some k-step nilsystems $((X_k)_i, T)_{i\in\mathbb{N}}$ satisfying that for all $i \in \mathbb{N}$, $((X_{k+1})_i, T)$ is an extension of $((X_k)_i, T)$.

Since (X, T) is the inverse limit of $(X_k, T)_{k\in\mathbb{N}}$, $\bigcup_{k\in\mathbb{N}} C(X_k)$ is dense in $C(X)$, and as (X_k, T) is the inverse limit of $((X_k)_i, T)_{i\in\mathbb{N}}$, we have that $\bigcup_{k\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} C((X_k)_i)$ is dense in $C(X)$. So we may choose a sequence of $(k_n, i_n)_{n\in\mathbb{N}} \subset \mathbb{N} \times \mathbb{N}$ with $k_n < k_{n+1}$, $i_n < i_{n+1}$ such that $\bigcup_{n\in\mathbb{N}} C((X_{k_n})_{i_n})$ is dense in $C(X)$. Thus (X, T) is the inverse limit of $((X_{k_n})_{i_n}, T)_{n\in\mathbb{N}}$, that is, (X, T) is an inverse limit of minimal nilsystems. This completes the proof of the theorem. □

Since minimal nilsystems are uniquely ergodic, it is easy to see that minimal ∞-step nilsystems are also uniquely ergodic.

3.3. *About* $\mathbf{RP}^{[\infty]} = \mathbf{RP}^{[d]}$. Observe that if $\mathbf{Q}^{[d+1]} = \mathbf{Q}^{[d]} \times \mathbf{Q}^{[d]}$, then $\mathbf{RP}^{[d-1]} = \mathbf{RP}^{[d]}$. Indeed, if $(x, y, \ldots, y)$, $(y, \ldots, y) \in \mathbf{Q}^{[d]}$, then $(x, y, \ldots, y, y, \ldots, y) \in \mathbf{Q}^{[d+1]}$. Moreover, the system is weakly mixing and thus $\mathbf{RP}^{[d]} = X \times X$ for any $d \geq 1$, as the following proposition shows.

PROPOSITION 3.7. *Let (X, T) be a minimal system. If $\mathbf{Q}^{[d+1]} = \mathbf{Q}^{[d]} \times \mathbf{Q}^{[d]}$ for some $d \in \mathbb{N}$, then X is weakly mixing and hence $\mathbf{Q}^{[d]} = X^{[d]}$ for any $d \in \mathbb{N}$.*

Proof. Let $x, y, a \in X$. By minimality, $(x, y, x, y, \ldots, x, y) \in \mathbf{Q}^{[d]}$, and by hypothesis, the point $\mathbf{x} = (x, y, x, y, \ldots, x, y, a, \ldots, a) \in \mathbf{Q}^{[d+1]}$. If $d = 1$, $(x, y, a, a) \in \mathbf{Q}^{[2]}$ and then $(x, y) \in \mathbf{RP}^{[1]}$. For any integer $d > 1$, applying Euclidean permutations, we get

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that $\mathbf{y} = (x, y, \ldots, x, y, a, \ldots, a, x, y \ldots, x, y, a, \ldots, a) \in \mathbf{Q}^{[d+1]}$, too. Considering the first half of $\mathbf{y}$ and iterating the process, we finish in the case $d = 1$. We conclude $\mathbf{RP}^{[1]} = X \times X$ and the result follows. □

Now, if $\mathbf{RP}^{[d-1]} = \mathbf{RP}^{[d]}$ for some d, the following theorem states that all regionally proximal relations of higher order coincide and thus $\mathbf{RP}^{[\infty]} = \mathbf{RP}^{[d-1]}$. This result is natural but its proof is somewhat involved, so we leave it for Appendix A. For the definition of Z_d, see §7.

THEOREM 3.8.

(1) *Let (X, T) be a minimal system. If $\mathbf{RP}^{[d]} = \mathbf{RP}^{[d+1]}$ for some $d \in \mathbb{N}$, then $\mathbf{RP}^{[n]} = \mathbf{RP}^{[d]}$ for all $n \geq d$.*

(2) *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system. If $Z_d = Z_{d+1}$ for some $d \in \mathbb{N}$, then $Z_n = Z_d$ for each $n \geq d$.*

(3) *Let (X, T) be a minimal system and μ be an ergodic Borel probability measure on X. If Z_n is isomorphic (with respect to the corresponding invariant measure) to $X_n = X/\mathbf{RP}^{[n]}$ for some $n \in \mathbb{N}$, then Z_k is isomorphic to $X_k = X/\mathbf{RP}^{[k]}$ for all $k \leq n$.*

4. *The structure of minimal systems without non-trivial* $\mathrm{Ind}_{\mathrm{fip}}$*-pairs*

In this section, we discuss the structure of minimal systems without non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs. We will show that such systems are almost one-to-one extensions of their maximal ∞-step nilfactors.

4.1. *A criterion to be an* $\mathrm{Ind}_{\mathrm{fip}}$*-pair.*

First we characterize $\mathrm{Ind}_{\mathrm{fip}}$-pairs using dynamical parallelepipeds.

Let (X, T) be a transitive system. It is easy to check that $\mathbf{x} = (x_\epsilon : \epsilon \subset [d]) \in \mathbf{Q}^{[d]}$ if and only if for any neighborhood U_ϵ of x_ϵ, respectively, there exist positive integers $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ such that $\bigcap_{\epsilon \subset [d]} T^{-\mathbf{n}\cdot\epsilon} U_\epsilon \neq \emptyset$. Moreover, the point in $\bigcap_{\epsilon \subset [d]} T^{-\mathbf{n}\cdot\epsilon} U_\epsilon$ can be chosen to be in the orbit of a transitive point.

LEMMA 4.1. *Let (X, T) be a transitive TDS and $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$. Then, $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$ for all integers $d \geq 1$ if and only if (x_1, x_2) is an* $\mathrm{Ind}_{\mathrm{fip}}$*-pair.*

Proof. Let $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$. First, we assume $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$ for all integer $d \geq 1$.

Let U_1, U_2 be neighborhoods of x_1 and x_2, respectively, and fix $d \in \mathbb{N}$. We show there exist positive integers $\{p_1, p_2, \ldots, p_d\}$ such that

$$F = FS(\{p_i\}_{i=1}^d) = \{p_{i_1} + \cdots + p_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}$$

belongs to $\mathrm{Ind}(U_1, U_2)$.

Since $\{x_1, x_2\}^{[d]}$ has 2^{2^d} elements, we write it as $\{\mathbf{x}_\eta : \eta \subset [2^d]\}$. Now, for any $\epsilon \subset [d]$ and $\eta \subset [2^d]$, let $x_{\epsilon\eta} = (\mathbf{x}_\eta)_\epsilon$ and construct the point $\mathbf{x} = (x_{\epsilon\eta} : \epsilon \subset [d], \eta \subset [2^d]) \in \{x_1, x_2\}^{[2^d+d]}$. Clearly, if we choose an identification of coordinates, we can write $\mathbf{x} = (x_\rho : \rho \subset [2^d + d])$. For any $\rho \subset [2^d + d]$, let $U_\rho = U_i$ if $x_\rho = x_i$, $i = 1, 2$. Then the product set $\bigotimes_{\rho \subset [2^d+d]} U_\rho$ is a neighborhood of $\mathbf{x}$. Since, by hypothesis, $\mathbf{x} \in \mathbf{Q}^{[2^d+d]}$, then

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there exist $x \in X$, $p_0 \in \mathbb{N}$ and $\mathbf{p} = \{p_1, \ldots, p_{2^d+d}\} \subset \mathbb{N}$ such that for any $\rho \subset [2^d + d]$, $T^{p_0+\mathbf{p}\cdot\rho}x \in U_\rho$.

We show that the finite IP set F generated by $\{p_1, p_2, \ldots, p_d\}$ belongs to $\mathrm{Ind}(U_1, U_2)$. For any $s \in \{1, 2\}^{2^d} = \{1, 2\}^{[d]}$, since $\{x_{s(\epsilon)} : \epsilon \subset [d]\} \in \{x_1, x_2\}^{[d]}$, there exists $\eta \subset [2^d]$ such that $\mathbf{x}_\eta = \{x_{s(\epsilon)} : \epsilon \subset [d]\}$, i.e., for any $\epsilon \subset [d]$, $x_{\epsilon\eta} = x_{s(\epsilon)}$. Let $y = T^{p_0+\sum_{i\in\eta} p_{i+d}}x$. Then, for any $\epsilon \subset [d]$, $T^{\sum_{i\in\epsilon} p_i} y = T^{p_0+\mathbf{p}\cdot(\epsilon\eta)}x \in U_{\epsilon\eta} = U_{s(\epsilon)}$. So

$$\bigcap_{\epsilon\subset[d]} T^{-\sum_{i\in\epsilon} p_i} U_{s(\epsilon)} \neq \emptyset,$$

and F belongs to $\mathrm{Ind}(U_1, U_2)$.

Now assume that (x_1, x_2) is an $\mathrm{Ind}_{\mathrm{fip}}$-pair, that is, for any neighborhood $U_1 \times U_2$ of (x_1, x_2), any $d \in \mathbb{N}$ and any $s \in \{1, 2\}^{[d]}$, there are positive integers $p_1, \ldots, p_d$ such that

$$\bigcap_{\epsilon\subset[d]} T^{-\sum_{i\in\epsilon} p_i} U_{s(\epsilon)} \neq \emptyset.$$

Let

$$x \in \bigcap_{\epsilon\subset[d]} T^{-\sum_{i\in\epsilon} p_i} U_{s(\epsilon)}.$$

Then $T^{\sum_{i\in\epsilon} p_i} x \in U_{s(\epsilon)}$ for any $\epsilon \subset [d]$. This implies that $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$. □

The following lemma is a useful application of the previous lemma.

LEMMA 4.2. *Let (X, T) be a transitive system, $x_1 \in X$ be a transitive point and $d \geq 1$ be an integer. Suppose that $(x_2, x_1, \ldots, x_1) \in \mathbf{Q}^{[d]}$ for some $x_2 \in X$ and that $\pi_1 : A \to X$ is semi-open, where $A = \overline{\mathrm{orb}((x_1, x_2), T \times T)}$ and π_1 is the projection to the first coordinate. Then $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$.*

Proof. If X is finite, then the lemma holds. Thus we assume that X is infinite. We first prove the following claim.

CLAIM. *If $\mathbf{x} = (x_1, \mathbf{a}_*) \in \{x_1, x_2\}^{[d]} \cap \mathbf{Q}^{[d]}$, then $(x_2, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$.*

Let U_1 and U_2 be neighborhoods of x_1 and x_2, respectively. Since π_1 is semi-open and X is infinite, then $V_1 = \mathrm{int}(\pi_1(U_1 \times U_2 \cap A)) \neq \emptyset$ and infinite. Set $V_2 = U_2$.

Let $s = (s(\epsilon) : \epsilon \subset [d]) \in \{1, 2\}^{[d]}$ such that $\mathbf{x} = (x_{s(\epsilon)} : \epsilon \subset [d])$. From the hypothesis, $\mathbf{x} = (x_1, \mathbf{a}_*) \in \{x_1, x_2\}^{[d]} \cap \mathbf{Q}^{[d]}$, then there exist positive integers $n_0, n_1, \ldots, n_d$ such that $T^{n_0+\mathbf{n}\cdot\epsilon}x_1 \in V_{s(\epsilon)}$ for each $\epsilon \subset [d]$, i.e., $\bigcap_{\epsilon\subset[d]} T^{-\mathbf{n}\cdot\epsilon} V_{s(\epsilon)} \neq \emptyset$, where $\mathbf{n} = (n_1, \ldots, n_d)$.

Let $W_1 = \bigcap_{\epsilon\subset[d]} T^{-\mathbf{n}\cdot\epsilon} V_{s(\epsilon)} \subset V_1$ and $W_2 = V_2 = U_2$. Since $W_1 \times W_2 \cap A \neq \emptyset$, there exists a positive integer M such that $(T \times T)^M(x_1, x_2) \in W_1 \times W_2$, and, since $(x_2, x_1, \ldots, x_1) \in \mathbf{Q}^{[d]}$, there exist positive integers $m_0, m_1, \ldots, m_d$ such that $T^{m_0}x_1 \in T^{-M}W_2 = T^{-M}U_2$, and $T^{m_0+\mathbf{m}\cdot\eta}x_1 \in T^{-M}W_1$ for all $\eta \subset [d]\backslash\{\emptyset\}$, where $\mathbf{m} = (m_1, \ldots, m_d)$. It follows that $T^{-m_0-M}U_2 \cap \bigcap_{\eta\subset[d]\backslash\{\emptyset\}} T^{-m_0-M-\mathbf{m}\cdot\eta} W_1 \neq \emptyset$, that is,

$$U_2 \cap \bigcap_{\eta\subset[d]\backslash\{\emptyset\}} T^{-\mathbf{m}\cdot\eta} \bigcap_{\epsilon\subset[d]} T^{-\mathbf{n}\cdot\epsilon} V_{s(\epsilon)} \neq \emptyset.$$

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Thus

$$U_2 \cap \bigcap_{\eta \subset [d] \setminus \{\emptyset\}} T^{-\mathbf{m}\cdot\eta} \bigcap_{\epsilon \subset [d]} T^{-\mathbf{n}\cdot\epsilon} U_{s(\epsilon)} \neq \emptyset.$$

Set $p_i = n_i + m_i$ for $1 \le i \le d$, and $\mathbf{p} = (p_1, \ldots, p_d)$. Then we have

$$U_2 \cap \bigcap_{\eta \subset [d] \setminus \{\emptyset\}} T^{-\mathbf{p}\cdot\eta} U_{s(\eta)} \neq \emptyset,$$

and so we get that $(x_2, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$. The proof of the claim is complete.

For any $\mathbf{x} \in \{x_1, x_2\}^{[d]}$, let $l(\mathbf{x})$ be the number of x_2s appearing in $\mathbf{x}$. We prove this lemma by induction on $l(\mathbf{x})$.

If $l(\mathbf{x}) = 0$, then obviously $\mathbf{x} = x_1^{[d]} \in \mathbf{Q}^{[d]}$. Suppose the lemma holds when $l(\mathbf{x}) \le k$, i.e., $\mathbf{x} \in \mathbf{Q}^{[d]}$ if $l(\mathbf{x}) \le k$. Now, for $l(\mathbf{x}) = k + 1$, without loss of generality, we write $\mathbf{x} = (x_2, \mathbf{a}_*)$. Since $l((x_1, \mathbf{a}_*)) = k$, we have, by hypothesis, that $(x_1, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$. Thus, from the claim, we get that $\mathbf{x} = (x_2, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$. The proof of this lemma is complete. □

The following corollary extends Corollaries 4.2 and 4.3 from [**16**], and the comment right after, which were only proved in the distal case.

COROLLARY 4.3. *Let (X, T) be a minimal TDS, $x_1, x_2 \in X$, and $d \ge 1$ an integer. If $(x_1, x_2) \in \mathbf{RP}^{[d]}$ and (x_1, x_2) is a $T \times T$-minimal point, then $\{x_1, x_2\}^{[d+1]} \subset \mathbf{Q}^{[d+1]}$.*

Proof. Let $A = \overline{\mathrm{orb}((x_1, x_2), T \times T)}$. Since $(A, T \times T)$ and (X, T) are minimal, the projection $\pi_1 : A \to X$ is semi-open. By Theorem 3.2, $(x_2, x_1, \ldots, x_1) \in \mathbf{Q}^{[d+1]}$, so $\{x_1, x_2\}^{[d+1]} \subset \mathbf{Q}^{[d+1]}$ by Lemma 4.2.

By the above discussion, we get the following criterion to be a $\mathrm{Ind}_{\mathrm{fip}}$-pair.

COROLLARY 4.4. *Let (X, T) be a minimal system and $(x_1, x_2) \in \mathbf{RP}^{[\infty]} \setminus \Delta_X = \bigcap_{d \ge 1} \mathbf{RP}^{[d]} \setminus \Delta_X$. If (x_1, x_2) is $T \times T$-minimal or the projection*

$$\pi_1 : \overline{\mathrm{orb}((x_1, x_2), T \times T)} \to X$$

is semi-open, then (x_1, x_2) is an $\mathrm{Ind}_{\mathrm{fip}}$-pair.

Proof. If (x_1, x_2) is $T \times T$-minimal, by Corollary 4.3, we have $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$ for every integer $d \ge 1$. Now, if the projection $\pi_1 : \overline{\mathrm{orb}((x_1, x_2), T \times T)} \to X$ is semi-open, then, since $(x_1, x_2) \in \mathbf{RP}^{[\infty]}$, we have $(x_2, x_1, \ldots, x_1) \in \mathbf{Q}^{[d]}$ for every $d \ge 1$. Hence, by Lemma 4.2, we also have $\{x_1, x_2\}^{[d]} \subset \mathbf{Q}^{[d]}$ for every integer $d \ge 1$.

By Lemma 4.1, the proof is complete. □

4.2. *The structure of minimal systems without non-trivial* $\mathrm{Ind}_{\mathrm{fip}}$*-pairs.* The following is the main result of this section.

THEOREM 4.5. *Let (X, T) be a minimal system. If X does not contain any non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pair, then it is an almost one-to-one extension of its maximal ∞-step nilfactor.*

To prove this theorem, we need some preparation. Every extension of minimal systems can be lifted to an open extension by almost one-to-one modifications. To be precise, for

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every extension $\pi : (X, T) \to (Y, T)$ between minimal systems, there exists a canonically defined commutative diagram of extensions (called the *shadow diagram*)

$$\begin{array}{ccc} X^* & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y^* & \xrightarrow{\tau} & Y \end{array}$$

with the following properties.

(1) σ and τ are almost one-to-one extensions.

(2) π^* is an open extension, i.e., for any open set $U \subset X^*$, $\pi^*(U)$ is an open set of Y^*.

(3) X^* is the unique minimal set in $R_{\pi\tau} = \{(x, y) \in X \times Y^* : \pi(x) = \tau(y)\}$ and σ and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^*, respectively.

We refer to [**1, 8, 30, 31**] for the details of this construction.

In [**8**], it was shown that a metric minimal system (X, T) with the property that n-proximal tuples are dense in X^n for every $n \geq 2$ is weakly mixing. This was extended by van der Woude [**33**] as follows (see also [**9**]).

THEOREM 4.6. *Let $\pi : (X, T) \to (Y, T)$ be a factor map between the metric minimal systems (X, T) and (Y, T). Suppose that π is an open proximal extension, then π is a weakly mixing extension, i.e., $(R_\pi, T \times T)$ is transitive.*

Now we give the proof of Theorem 4.5.

Proof of Theorem 4.5. Let (X, T) be a minimal system without $\mathrm{Ind}_{\mathrm{fip}}$-pairs. We denote by (Y, T) the quotient system of (X, T) determined by the equivalence relation $\mathbf{RP}^{[\infty]}(X, T)$ and let $\pi : (X, T) \to (Y, T)$ be the canonical projection map.

We first prove that π is a proximal extension. Remark that if $(x, y) \in R_\pi = \mathbf{RP}^{[\infty]}$ is a $T \times T$ minimal point, according to Corollary 4.4, we have that (x, y) is an $\mathrm{Ind}_{\mathrm{fip}}$-pair and thus we must have $x = y$. Now consider any $(x, y) \in R_\pi$ and $u \in E(X, T)$ a minimal idempotent. Since (ux, uy) is a $T \times T$ minimal point, we have from the previous observation that $ux = uy$, which implies (x, y) is a proximal pair.

As the shadow diagram shows, there exists a canonically defined commutative diagram of extensions

$$\begin{array}{ccc} X^* & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y^* & \xrightarrow{\tau} & Y \end{array}$$

verifying properties (1)–(3) above.

Since π is proximal and σ is almost one-to-one, we have that $\pi \circ \sigma$ is proximal. For any $(x, x') \in R_{\pi^*}$, $\pi \circ \sigma(x) = \pi \circ \sigma(x')$, so (x, x') is a proximal pair, which implies π^* is proximal too. By Theorem 4.6, π^* is a weakly mixing extension, and hence there exists $(x_1, x_2) \in R_{\pi^*}$ such that $R_{\pi^*} = \overline{\mathrm{orb}((x_1, x_2), T \times T)}$. Let π_1 be the projection of R_{π^*} to the first coordinate. It is easy to get that for any open sets $U, V \subset X^*$, $\pi_1(R_{\pi^*} \cap (U \times V)) = U \cap (\pi^*)^{-1}(\pi^*(V))$ is an open set. So we get that π_1 is an open map too.

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Since (x_1, x_2) is a proximal pair, we have that (x_1, x_2) is an $\mathrm{Ind}_{\mathrm{fip}}$-pair by Corollary 4.4. Hence $(\sigma(x_1), \sigma(x_2))$ is an $\mathrm{Ind}_{\mathrm{fip}}$-pair too. Then we must have $\sigma(x_1) = \sigma(x_2)$, and thus $R_{\pi^*} \subset R_\sigma$.

Since τ is almost one-to-one, we can choose $y \in Y$ such that $\tau^{-1}(y)$ contains only one point. Suppose $x_1, x_2 \in \pi^{-1}(y)$, then there exist $x_1^*, x_2^* \in X^*$ such that $\sigma(x_1^*) = x_1$ and $\sigma(x_2^*) = x_2$. As $\tau \circ \pi^*(x_1^*) = \pi \circ \sigma(x_1^*) = y = \pi \circ \sigma(x_2^*) = \tau \circ \pi^*(x_2^*)$, we have $\pi^*(x_1^*), \pi^*(x_2^*) \in \tau^{-1}(y)$ and so $\pi^*(x_1^*) = \pi^*(x_2^*)$, i.e., $(x_1^*, x_2^*) \in R_{\pi^*} \subset R_\sigma$. Hence $x_1 = x_2$, which implies that $\pi^{-1}(y)$ contains only one point, too. We conclude that π is almost one-to-one. □

PROPOSITION 4.7. *Let (X, T) be a minimal distal system. Then there are no non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs if and only if (X, T) is an ∞-step nilsystem.*

Proof. It is a direct consequence of Theorem 4.5. □

4.3. *The assumption of semi-openness in Lemma 4.2 cannot be removed.* In this subsection, we give an example to show that the condition of semi-openness in Lemma 4.2 cannot be removed. First, we recall some notions.

Let (X, T) be a TDS. For any open cover $\mathcal{U}$, let $N(\mathcal{U})$ denote the smallest possible cardinality among finite subcovers of $\mathcal{U}$. Given an increasing sequence $\mathcal{A} = \{t_1, t_2, \ldots\}$ in $\mathbb{Z}_+$, the sequence entropy of (X, T) or just T with respect to $\mathcal{A}$ and the cover $\mathcal{U}$ is

$$h_{\mathcal{A}}(T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}\right)$$

and the sequence entropy of T with respect to $\mathcal{A}$ is $h_{\mathcal{A}}(T) = \sup_{\mathcal{U}} h_{\mathcal{A}}(T, \mathcal{U})$, where the supremum is taken over all finite open covers $\mathcal{U}$ of X. The system (X, T) is a *null system* if for any sequence $\mathcal{A} \subset \mathbb{Z}_+$, $h_{\mathcal{A}}(T) = 0$.

Similar to $\mathrm{Ind}_{\mathrm{fip}}$-pair, we can define IN-pairs. Let (X, T) be a TDS and $(x_1, x_2) \in X \times X$. Then, (x_1, x_2) is a *IN-pair* if for any neighborhoods U_1, U_2 of x_1, x_2, respectively, $\mathrm{Ind}(U_1, U_2)$ contains arbitrarily long finite independence sets. In [**24**], it is shown that (X, T) is a null system if and only if it contains no non-trivial IN-pairs. It is obvious that a null system contains no non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs as $\mathrm{Ind}_{\mathrm{fip}}$-pairs are IN-pairs.

The following example is classical.

Example 4.8. Sturmian system.

Let α be an irrational number in the interval $(0, 1)$ and R_α be the irrational rotation on the (complex) unit circle $\mathbb{T}$ generated by $e^{2\pi i\alpha}$. Set

$$A_0 = \{e^{2\pi i\theta} : 0 \le \theta < (1-\alpha)\} \quad \text{and} \quad A_1 = \{e^{2\pi i\theta} : (1-\alpha) \le \theta < 1\}.$$

Consider $z \in \mathbb{T}$ and define $x \in \{0, 1\}^{\mathbb{Z}}$ as follows: for all $n \in \mathbb{Z}$, $x_n = i$ if and only if $R_\alpha^n(z) \in A_i$. Let $X \subset \{0, 1\}^{\mathbb{Z}}$ be the orbit closure of x under the shift map σ on $\{0, 1\}^{\mathbb{Z}}$, i.e., for any $y \in \{0, 1\}^{\mathbb{Z}}$, $(\sigma(y))_n = y_{n+1}$. This system is called a Sturmian system. It is well known that (X, σ) is a minimal almost one-to-one extension of $(\mathbb{T}, R_\alpha)$. Moreover, it is an asymptotic extension. Also, it is not hard to prove that it is a null system.

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Let $\pi : X \to \mathbb{T}$ be the former extension and consider $(x_1, x_2) \in R_\pi \setminus \Delta_X$. Then (x_1, x_2) is an asymptotic pair and thus $(x_1, x_2) \in \mathbf{RP}^{[d]}$ for any integer $d \geq 1$. In particular, $(x_2, x_1, \ldots, x_1) \in \mathbf{Q}^{[d+1]}$ for any integer $d \geq 1$. Now, if $\{x_1, x_2\}^{[d+1]} \subset \mathbf{Q}^{[d+1]}$ for any integer $d \geq 1$, then, by Lemma 4.1, (x_1, x_2) is a non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pair, and so an IN-pair, which contradicts the fact that (X, T) is a null system. Therefore, there exists an integer $d \geq 1$ such that $\{x_1, x_2\}^{[d]} \not\subseteq \mathbf{Q}^{[d]}$, which implies that the condition $\pi_1 : \overline{\mathrm{orb}((x, y), T \times T)} \to X$ is semi-open in Lemma 4.2 cannot be removed.

5. *Minimal distal systems which are not ∞-step nilsystems*

In the previous section, we showed that a minimal distal system is an ∞-step nilsystem if and only if it has no non-trivial $\mathrm{Ind}_{\mathrm{fip}}$-pairs. In this section, we will give examples of minimal distal systems which are not ∞-step nilsystems. We remark that if (X, T) is minimal distal and $\pi : (X, T) \longrightarrow (X_{eq}, T)$ is the maximal equicontinuous factor of (X, T), then each pair in $R_\pi \setminus \Delta_X$ is an untame pair (see [**17, 20, 27**]). In fact, the result in the previous section shows that if $\pi_\infty : (X, T) \longrightarrow (Z_\infty, T)$ is the factor map from X to its maximal ∞-step nilfactor, then each pair in $R_{\pi_\infty} \setminus \Delta_X$ is an $\mathrm{Ind}_{\mathrm{fip}}$-pair. We do not know how to combine the results.

5.1. *The existence.* To show the existence of minimal distal systems which are not ∞-step nilsystems, we use some abstract results from [**26**]. We freely use the notations therein.

PROPOSITION 5.1. [**26**, Theorem 4.4] *Every ergodic measurable distal system $(X, \mathcal{B}, \mu, T)$ can be represented as a minimal topologically distal system equipped with a Borel measure of full support.*

LEMMA 5.2. [**26**, Claim 5.5] *Suppose $(Y, \mathcal{D}, \nu, T)$ is an isometric extension of the ergodic rotation $(\mathbb{T}, \mathcal{B}, \lambda, R_\alpha)$ by a finite group that is not a Kronecker system. Then $(Y, \mathcal{D}, \nu, T)$ does not have a uniquely ergodic distal model.*

The following result produces as many examples as we want.

THEOREM 5.3. *Suppose $(Y, \mathcal{D}, \nu, T)$ is an isometric extension of the ergodic rotation $(\mathbb{T}, \mathcal{B}, \lambda, R_\alpha)$ by a finite group, that is, not a Kronecker system. Then any minimal distal topological model of $(Y, \mathcal{D}, \nu, T)$ is not an ∞-step nilsystem.*

Proof. By Proposition 5.1, $(Y, \mathcal{D}, \nu, T)$ has a minimal topologically distal system (X, T) model equipped with a Borel measure of full support. By Lemma 5.2, (X, T) cannot be uniquely ergodic. It is clear that a minimal ∞-step nilsystem is uniquely ergodic, so (X, T) is not an ∞-step nilsystem. □

5.2. *An explicit example.* A way to produce an explicit example is to use the following Furstenberg result. It appeared first in [**6**]. We recall that a topological dynamical system is *strictly ergodic* if it is minimal and uniquely ergodic.

THEOREM 5.4. [**6**, Theorem 3.1] *Let (Ω_0, T_0) be a strictly ergodic system and μ_0 its unique ergodic measure. Let $\Omega = \Omega_0 \times \mathbb{T}$ and let $T : \Omega \to \Omega$ be defined*

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by $T(\omega_0, s) = (T_0(\omega_0), g(\omega_0)s)$, *where* $g : \Omega_0 \to \mathbb{T}$ *is a continuous function. Then, if the equation*

$$g^k(\omega_0) = R(T_0(\omega_0))/R(\omega_0) \tag{5.1}$$

has a solution $R : \Omega_0 \to \mathbb{T}$ *which is measurable but not equal almost everywhere to a continuous function, then* $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} f \circ T^n(\omega)$ *cannot exist for all continuous functions* f *and all* $\omega \in \Omega$.

Now we recall some elements of the example from [**6**] satisfying the criterion of the previous theorem. The first step (which we omit here) is the construction of a sequence of integers $(n_k)_{k\in\mathbb{Z}}$ and an irrational number α such that

$$h(\theta) = \sum_{k\neq 0} \frac{1}{|k|}(e^{2\pi i n_k \alpha} - 1)e^{2\pi i n_k \theta}$$

and $g(e^{2\pi i\theta}) = e^{2\pi i\lambda h(\theta)}$, where λ is as yet undetermined, are C^∞ functions of $[0, 1)$ and $\mathbb{T}$, respectively. Clearly, $h(\theta) = H(\theta + \alpha) - H(\theta)$, where

$$H(\theta) = \sum_{k\neq 0} \frac{1}{|k|} e^{2\pi i n_k \theta}.$$

Thus $H(\cdot)$ is in $L^2(0, 1)$ and in particular defines a measurable function. However, $H(\cdot)$ cannot correspond to a continuous function since $\sum_{k\neq 0}(1/|k|) = \infty$ and hence the series is not Cesàro summable at $\theta = 0$ [**35**]. Therefore, for some λ, $e^{2\pi i\lambda H(\theta)}$ cannot be a continuous function either. Considering $R(e^{2\pi i\theta}) = e^{2\pi i\lambda H(\theta)}$, we get

$$R(e^{2\pi i\alpha}s)/R(s) = g(s)$$

with $R : \mathbb{T} \to \mathbb{T}$ measurable but not continuous.

By Theorem 5.4, the transformation T of $\mathbb{T} \times \mathbb{T}$ given by

$$T(s_1, s_2) = (e^{2\pi i\alpha}s_1, g(s_1)s_2)$$

will not possess all its ergodic averages, i.e., there are a continuous function f and $\omega \in \mathbb{T} \times \mathbb{T}$ such that $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} f \circ T^n(\omega)$ does not exist.

Let $\Omega = \overline{\mathrm{orb}(\omega, T)} \subset \mathbb{T} \times \mathbb{T}$. It is easy to get that $(\mathbb{T} \times \mathbb{T}, T)$ is distal, and so (Ω, T) is minimal distal. If (Ω, T) is an ∞-step nilsystem, then it is an inverse limit of minimal nilsystems, and of course (Ω, T) is strictly ergodic. Let $h = f|_\Omega$ and h be apparently continuous on Ω, so $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} g \circ T^n(\omega)$ exists, contradicting the choice of f and ω. Therefore, (Ω, T) is minimal distal but not an ∞-step nilsystem.

6. *Discussion of unique ergodicity*

In this section, we aim to investigate the question whether a minimal system without $\mathrm{Ind}_{\mathrm{fip}}$-pairs is uniquely ergodic. First, we observe that when (X, T) is minimal, then $(X_\infty = X/\mathbf{RP}^{[\infty]}, T)$ is uniquely ergodic since it is an inverse limit of uniquely ergodic systems.

Assume that (X, T) is a minimal system and let μ be an ergodic measure for (X, T). Then $(X, \mathcal{B}, \mu, T)$ is an ergodic measure-preserving system, where $\mathcal{B}$ is the σ-algebra

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of Borel sets of X (we omit $\mathcal{B}$ in what follows). In [**15**], to prove the convergence in $L^2(X, \mathcal{B}, \mu)$ of some non-conventional ergodic averages, the authors introduced measures $\mu^{[d]} \in \mathcal{M}(X^{[d]}, T^{[d]})$ for any integer $d \geq 1$ and used them to produce the maximal measure theoretical factor of order d of (X, μ, T) (in the measurable context this means that the system is an inverse limit, with measurable factor maps, of d-step nilsystems), denoted by $(Z_d, \mathcal{Z}_d, \mu_d, T)$.

In the topological setting, $(X_d = X/\mathbf{RP}^{[d]}, T)$ is the maximal factor of order d of (X, T) and is uniquely ergodic. In [**16**], it was observed that (X_d, T) is also a system of order d in the measurable sense for its unique invariant measure. This implies that X_d is a factor of Z_d in the measurable sense.

Let $\mu = \int_{Z_d} \mu_z \, d\mu_d(z)$ be the disintegration of μ over μ_d. Pairs in the support of the measure

$$\lambda_d = \int_{Z_d} \mu_z \times \mu_z \, d\mu_d(z)$$

are called $\mathcal{F}_d^\mu$-pairs, where $\mathcal{F}_d^\mu = \mathrm{supp}(\lambda_d)$. To study $\mathcal{F}_d^\mu$, we will need the following lemma from [**15**].

LEMMA 6.1. [**15**, Propostion 4.7, Theorem 13.1] *Let $d \geq 1$ be an integer and $V_d = \{0, 1\}^{[d]}$.*

(1) *For f_ϵ, $\epsilon \in V_d$, bounded measurable functions on X,*

$$\int_{X^{[d]}} \bigotimes_{\epsilon \in V_d} f_\epsilon \, d\mu^{[d]} = \int_{Z_{d-1}^{[d]}} \bigotimes_{\epsilon \in V_d} \mathbb{E}(f_\epsilon | \mathcal{Z}_{d-1}) \, d\mu_{d-1}^{[d]},$$

where $(Z_{d-1}^{[d]}, \mu_{d-1}^{[d]})$ is the joining of 2^d copies of (Z_{d-1}, μ_{d-1}).

(2) *For f_ϵ, $\epsilon \in V_d$, bounded measurable functions on X, the average*

$$\prod_{i=1}^{d} \frac{1}{N_i - M_i} \sum_{\mathbf{n} \in [M_1, N_1) \times \cdots \times [M_d, N_d)} \int_X \prod_{\epsilon \in V_d} f_\epsilon \circ T^{\mathbf{n} \cdot \epsilon} \, d\mu$$

converges to

$$\int_{X^{[d]}} \prod_{\epsilon \in V_d} f_\epsilon(x_\epsilon) \, d\mu^{[d]}(\mathbf{x})$$

as $N_i - M_i \to \infty$ for all $1 \leq i \leq d$.

We get the following theorem about $\mathcal{F}_d^\mu$.

THEOREM 6.2. *Let (X, T) be a minimal system and μ an ergodic measure on X.*

(1) *Let $d \geq 1$ be an integer; then $\mathcal{F}_d^\mu \subset \mathbf{RP}^{[d]}$.*

(2) $\bigcap_{d \in \mathbb{N}} \mathcal{F}_d^\mu \subset \mathrm{Ind}_{\mathrm{fip}}(X, T)$.

Proof. Let $(x_0, x_1) \in \mathcal{F}_d^\mu$. Then, for any neighborhood $U_0 \times U_1$ of (x_0, x_1),

$$\lambda_d(U_0 \times U_1) = \int_{Z_d} \mathbb{E}(1_{U_0} | \mathcal{Z}_d) \mathbb{E}(1_{U_1} | \mathcal{Z}_d) \, d\mu_d > 0.$$

Let $\{U_\epsilon : \epsilon \in V_{d+1}\} \subset \{U_0, U_1\}$ be open sets. We claim that

$$\int_{X^{[d+1]}} \prod_{\epsilon \in V_{d+1}} 1_{U_\epsilon}(x_\epsilon) d\mu^{[d+1]}(\mathbf{x}) > 0.$$

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Proof of the claim. Since $\int_{Z_d} \mathbb{E}(1_{U_0}|\mathcal{Z}_d)\mathbb{E}(1_{U_1}|\mathcal{Z}_d)d\mu_d > 0$, there exists $B \in \mathcal{Z}_d$ with $\mu_d(B) > 0$ such that for any $z \in B$, $\mathbb{E}(1_{U_0}|\mathcal{Z}_d)(z)\mathbb{E}(1_{U_1}|\mathcal{Z}_d)(z) > 0$. Also, since $U_\epsilon = U_0$ or $U_\epsilon = U_1$ for $\epsilon \in V_{d+1}$, we have for any $\mathbf{z} \in B^{[d+1]}$ that $\prod_{\epsilon\in V_{d+1}} \mathbb{E}(1_{U_\epsilon}|\mathcal{Z}_d)(z_\epsilon) > 0$. Now, by the previous lemma, we obtain

$$\begin{aligned}\int_{X^{[d+1]}} \prod_{\epsilon\in V_{d+1}} 1_{U_\epsilon}(x_\epsilon)d\mu^{[d+1]}(\mathbf{x}) &= \int_{Z_d^{[d+1]}} \prod_{\epsilon\in V_{d+1}} \mathbb{E}(1_{U_\epsilon}|\mathcal{Z}_d)(z_\epsilon)d\mu_d^{[d+1]}(\mathbf{z}) \\ &\geq \int_{B^{[d+1]}} \prod_{\epsilon\in V_{d+1}} \mathbb{E}(1_{U_\epsilon}|\mathcal{Z}_d)(z_\epsilon)d\mu_d^{[d+1]}(\mathbf{z}) > 0.\end{aligned}$$

This completes the proof of the claim since it was shown in [**15**] that $\mu_d^{[d+1]}(B^{[d+1]}) \geq \mu_d(B)^{2^{d+1}} > 0$.

Again, by Lemma 6.1,

$$\frac{1}{N^{d+1}} \sum_{0\leq n_1,\ldots,n_{d+1}<N} \int_X \prod_{\epsilon\in V_{d+1}} 1_{U_\epsilon} \circ T^{n_1\epsilon_1+\cdots+n_{d+1}\epsilon_{d+1}}\, d\mu$$

converges to

$$\int_{X^{[d+1]}} \prod_{\epsilon\in V_{d+1}} 1_{U_\epsilon}(x_\epsilon)d\mu^{[d+1]}(\mathbf{x}) > 0$$

as $N \to \infty$. Hence there exists a Borel set B with $\mu(B) > 0$ and $n_1, \ldots, n_{d+1} \in \mathbb{Z}_+$ such that, for each $x \in B$,

$$\prod_{\epsilon\in V_{d+1}} 1_{U_\epsilon} \circ T^{n_1\epsilon_1+\cdots+n_{d+1}\epsilon_{d+1}}(x) > 0,$$

that is, for each $x \in B$, $T^{n_1\epsilon_1+\cdots+n_{d+1}\epsilon_{d+1}}(x) \in U_\epsilon$ or

$$\bigcap_{\epsilon\in V_{d+1}} T^{-n_1\epsilon_1-\cdots-n_{d+1}\epsilon_{d+1}} U_\epsilon \neq \emptyset.$$

First, we prove statement (1). Let $(x_0, x_1) \in \mathcal{F}_d^\mu$. For $\delta > 0$, let $U_0 \times U_1$ be a neighborhood of (x_0, x_1) where the diameters of U_0 and U_1 are less than δ. Set $U_{(0,\ldots,0)} = U_0$, $U_{(0,\ldots,0,1)} = U_1$ and $U_\epsilon = U_0$ for any other $\epsilon \in V_{d+1}$. By the previous discussion, there exist $x \in X$ and $n_1, \ldots, n_{d+1} \in \mathbb{Z}_+$ such that $T^{n_1\epsilon_1+\cdots+n_{d+1}\epsilon_{d+1}}(x) \in U_\epsilon$, for any $\epsilon \in V_{d+1}$. Let $y_0 = x = T^0 x \in U_{(0,\ldots,0)} = U_0$, $y_1 = T^{n_{d+1}}x \in U_{(0,\ldots,0,1)} = U_1$. Then, for any $\epsilon \in V_d\backslash\{(0,\ldots,0)\}$, we have $T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}(y_0) \in U_{\epsilon 0} = U_0$ and $T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}(y_1) \in U_{\epsilon 1} = U_0$. Since the diameters of U_0 and U_1 are less than δ, we get that $d(x_0, y_0) < \delta$, $d(x_1, y_1) < \delta$, and for any $\epsilon \in V_d\backslash\{(0,\ldots,0)\}$, $d(T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}(y_0), T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}(y_1)) < \delta$. Therefore, $\mathcal{F}_d^\mu \subset \mathbf{RP}^{[d]}$.

To show (2), by Lemma 4.1, it remains to prove that if $(x_0, x_1) \in \bigcap_{d\in\mathbb{N}} \mathcal{F}_d^\mu$, then $\{x_0, x_1\}^{[d]} \subset \mathbf{Q}^{[d]}$ for any integer $d \geq 1$. Let $d \geq 1$ be an integer and take $\mathbf{x} = (x_\epsilon)_{\epsilon\in V_d} \in \{x_0, x_1\}^{[d]}$. Given a neighborhood $\mathbf{V}$ of $\mathbf{x}$, there exists a neighborhood $U_0 \times U_1$ of (x_0, x_1) such that if we set $U_\epsilon = U_i$ depending on $x_\epsilon = x_0$ or $x_\epsilon = x_1$, then $\bigotimes_{\epsilon\in V_d} U_\epsilon \subset \mathbf{V}$. From the conclusion of part (1), there exist $x \in X$ and $n_1, \ldots, n_d \in \mathbb{Z}_+$ such that $T^{n_1\epsilon_1+\cdots+n_d\epsilon_d}(x) \in U_\epsilon$ for any $\epsilon \in V_d$. Let $\mathbf{n} = (n_1, \ldots, n_d)$. We have $(T^{\mathbf{n}\cdot\epsilon}x)_{\epsilon\in V_d} \in \mathbf{V}$, and thus $\mathbf{x} \in \mathbf{Q}^{[d]}$. The proof is complete. □

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Remark 6.3. We have shown that $\mathcal{F}_d^\mu \subset \mathbf{RP}^{[d]}$. However, the converse is not true in general. For example, let (Z, S) be a non-trivial d-step nilsystem and ν an ergodic measure on Z. In [**32**], it was shown that there exists a weakly mixing minimal uniquely ergodic system (X, T) with the uniquely ergodic measure μ satisfying that for all $x, y \in X$ with $y \notin \{T^n x\}_{n\in\mathbb{Z}}$, the orbit $\{(T^n x, T^n y)\}_{n\in\mathbb{Z}}$ is dense in $X \times X$, and (Z, ν, S) and (X, μ, T) are isomorphic. Then (X, μ) coincides with $(Z_d(X), \mu_d)$, and so $\mathcal{F}_\mu^d = \Delta_X$. Since (X, T) is weakly mixing, we get that $\mathbf{RP}^{[d]} = X \times X$, which implies that $\mathcal{F}_\mu^d \subsetneq \mathbf{RP}^{[d]}$.

A direct application of the above theorem is the following.

THEOREM 6.4. *Let (X, T) be a minimal system with $\mathrm{Ind}_{\mathrm{fip}}(X, T) = \Delta_X$. Then for each ergodic measure μ, (X, μ, T) is measure theoretically isomorphic to an ∞-step nilsystem.*

Proof. Applying Theorem 6.2, we get that $\bigcap_{d\in\mathbb{N}} \mathcal{F}_d^\mu \subset \mathrm{Ind}_{\mathrm{fip}}(X, T) = \Delta_X$. The result follows, since it is easy to check that $\bigcap_{d\in\mathbb{N}} \mathcal{F}_d^\mu = \mathcal{F}_\infty^\mu$, where $\mathcal{F}_\infty^\mu$ is the support of $\lambda_\infty = \int_{Z_\infty} \mu_z \times \mu_z d\mu_\infty(z)$ with (Z_∞, μ_∞) the inverse limit of (Z_d, μ_d). □

To show the unique ergodicity of a minimal system with $\mathrm{Ind}_{\mathrm{fip}}(X, T) = \Delta_X$, the following question is crucial.

Question 6.5. Let (X, T) be an E-system (i.e., it is transitive and admits an invariant measure with full support), let x be a transitive point and p be a fixed point of (X, T). Is it true that $(p, x, \ldots, x) \in \mathbf{Q}^{[d]}$ for any integer $d \geq 1$?

If this question has a positive answer, then, by Lemma 4.2, we have $\{x, p\}^{[d]} \subset \mathbf{Q}^{[d]}$ for any integer $d \geq 1$ (since $\overline{\mathrm{orb}((x, p), T \times T)} = X \times \{p\}$), and thus (x, p) is an $\mathrm{Ind}_{\mathrm{fip}}$-pair by Lemma 4.1.

CONJECTURE 6.6. *Let (X, T) be a minimal system with $\mathrm{Ind}_{\mathrm{fip}}(X, T) = \Delta_X$. Then (X, T) is uniquely ergodic.*

If Question 6.5 has a positive answer, then, using the proof of [**20**, Theorem 4.4] and the lifting property of $\mathrm{Ind}_{\mathrm{fip}}$-pairs [**19**], we may conclude that the conjecture holds.

7. *Topological complexity of ∞-step nilsystems*

The big development in the study of non-conventional ergodic averages during the last decade has provided evidence, among other facts, of the crucial role of nilsystems when studying 'polynomial' phenomena in dynamical systems theory. The objective of this section is to prove that ∞-step nilsystems have polynomial topological complexity. It is well known that bounded complexity characterizes minimal rotations on compact Abelian groups (see [**4**]). Some basic symbolic examples (substitution systems, for example) show that polynomial complexity cannot characterize ∞-step nilsystems, so the characterization of polynomial complexity seems to be a deep problem far from being solved.

In order to study the quantitative distribution of polynomial orbits in nilmanifolds, Green and Tao introduced in [**13**] a metric induced by the Mal'cev basis on a nilmanifold and they studied its behavior under left multiplication. We obtain as an application a polynomial bound for the topological complexity of nilsystems and consequently of ∞-step nilsystems. We use freely the notations and results from [**13**] and [**2**].

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7.1. *Polynomial behavior of orbits.* In this subsection, we assume G is a connected and simply connected Lie group and $\Gamma \subset G$ is a cocompact subgroup. We will denote by G_i the ith subgroup of the associated lower central series. Under these assumptions, Mal'cev proved the following theorem.

THEOREM 7.1. (Mal'cev basis) *Let G be an m-dimensional nilpotent Lie group and $\Gamma \subset G$ a cocompact subgroup. There exists a basis $\mathcal{X} = \{X_1, \ldots, X_m\}$ of the associated Lie algebra $\mathfrak{g}$ such that:*

(1) *for each $j \in \{0, \ldots, m-1\}$, the subspace $\mathfrak{h}_j = \mathrm{Span}(X_{j+1}, \ldots, X_m)$ is an ideal in $\mathfrak{g}$ and $\exp(\mathfrak{h}_j)$ is a normal subgroup in G;*

(2) *$G_i = \exp(\mathfrak{h}_{m-m_i})$, where $m_i = \dim(G_i)$;*

(3) *each $g \in G$ can be written uniquely as $\exp(t_1 X_1) \cdots \exp(t_m X_m)$, where $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$; and*

(4) *$\Gamma = \{\exp(n_1 X_1) \cdots \exp(n_m X_m) : n = (n_1, \ldots, n_m) \in \mathbb{Z}^m\}$.*

We say that $\mathcal{X} = \{X_1, \ldots, X_m\}$ is a Mal'cev basis for G/Γ adapted to the lower central series $(G_i)_{i \geq 0}$.

Let $g = \exp(t_1 X_1) \cdots \exp(t_m X_m) \in G$ and denote $\psi(g) = (t_1, \ldots, t_m) \in \mathbb{R}^m$. Let $|\psi(g)| = \|\psi(g)\|_\infty$. Fix a Mal'cev basis $\mathcal{X}$. In [**13**], the authors introduced the following metric on G and G/Γ.

Definition 7.2. Let $x, y \in G$ and define

$$d(x, y) = \inf\Bigg\{\sum_{i=0}^{n-1} \min(|\psi(x_{i-1}x_i^{-1})|, |\psi(x_i x_{i-1}^{-1})|) : n \in \mathbb{N}, \\ x_0, \ldots, x_n \in G, x_0 = x, x_n = y\Bigg\},$$

which is right invariant, i.e., $d(x, y) = d(xg, yg)$ for all $g \in G$ and $d(x, y) \leq |\psi(xy^{-1})|$.

This metric induces a metric on G/Γ that we also call $d(\cdot, \cdot)$ by setting

$$\begin{aligned} d(x\Gamma, y\Gamma) &= \inf\{d(x', y') : x' \in x\Gamma, y' \in y\Gamma\} \\ &= \inf\{d(x\gamma_1, y\gamma_2) : \gamma_1, \gamma_2 \in \Gamma\} = \inf\{d(x, y\gamma) : \gamma \in \Gamma\}, \end{aligned}$$

where the last equality follows from the right-invariance of the metric.

In the following, we use some results obtained in [**13**] and we rephrase some others in a convenient way. The Mal'cev basis $\mathcal{X}$ is fixed.

LEMMA 7.3. (Multiplication and inversion) *Let $x, y \in G$, $t = \psi(x)$, $u = \psi(y)$. Then:*

(1) *$\psi(xy) = (t_1 + u_1, t_2 + u_2 + P_1(t_1, u_1), \ldots, t_m + u_m + P_{m-1}(t_1, \ldots, t_{m-1}, u_1, \ldots, u_{m-1}))$, where for each $i \in \{1, \ldots, m-1\}$, P_i is a real polynomial; and*

(2) *$\psi(x^{-1}) = (-t_1, -t_2 + Q_1(t_1), \ldots, -t_m + Q_{m-1}(t_1, \ldots, t_{m-1}))$, where for each $i \in \{1, \ldots, m\}$, Q_i is a real polynomial.*

We get easily that $\psi(xy^{-1}) = (R_1(t, u), R_2(t, u), \ldots, R_m(t, u))$, where R_i is a real polynomial for each $i \in \{1, \ldots, m\}$. In order to simplify notations in what follows, we will write P to refer to any generic real polynomial with positive coefficients, not necessarily the same. This will be clear from the context.

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LEMMA 7.4. (Coordinates polynomial bound) *Let $x, y \in G$. Then,*

$$d(x, y) \leq P(\max(|\psi(x)|, |\psi(y)|))|\psi(x) - \psi(y)|.$$

Proof. Let $\psi(x) = t$ and $\psi(y) = u$. From the definition of the distance and Lemma 7.3, we have

$$d(x, y) \leq |\psi(xy^{-1})| = |(R_1(t, u), R_2(t, u), \ldots, R_m(t, u))|.$$

Write $R_i(t, u) = \sum_{\vec{\alpha},\vec{\beta}} C^{(i)}_{\vec{\alpha},\vec{\beta}} t^{\vec{\alpha}} u^{\vec{\beta}}$. Since $R_i(t, t) = 0$, one deduces

$$R_i(t, u) = R_i(t, u) - R_i(t, t) = \sum_{\vec{\alpha},\vec{\beta}} C^{(i)}_{\vec{\alpha},\vec{\beta}} t^{\vec{\alpha}} (u^{\vec{\beta}} - t^{\vec{\beta}}).$$

Expanding $(u^{\vec{\beta}} - t^{\vec{\beta}}) = \sum_{i=1}^{m} (u_i - t_i) W_{\vec{\beta},i}(t, u)$, where $W_{\vec{\beta},i}$ are polynomials, we get

$$\begin{aligned} d(x, y) &\leq |\psi(x) - \psi(y)| \sum_{\vec{\alpha},\vec{\beta}} \sum_{i=1}^{m} |C^{(i)}_{\vec{\alpha},\vec{\beta}}||t^{\vec{\alpha}}||W_{\vec{\beta},i}(t, u)| \\ &\leq |\psi(x) - \psi(y)| P(\max(|\psi(x)|, |\psi(y)|)). \end{aligned}$$

□

LEMMA 7.5. *Let $g, x, y \in G$; then*

$$d(gx, gy) \leq P(|\psi(g)|, |\psi(x)|, |\psi(y)|)\, d(x, y).$$

Proof. Let $g, z \in G$. From Lemma 7.3, we see that $\psi(gzg^{-1})$ is a polynomial function of $\psi(z)$ and $\psi(g)$ that vanishes when $\psi(z) = 0$. This polynomial function can be written as $\psi(z) P(\psi(g), \psi(z))$. Then

$$|\psi(gzg^{-1})| \leq |\psi(z)| P(|\psi(z)|, |\psi(g)|). \tag{7.1}$$

By Lemma 7.4, $d(x, y) \leq P(|\psi(x)|, |\psi(y)|)$, and then for computing the distance between x and y we can restrict to paths $x = x_0, \ldots, x_n = y$ satisfying $k(x_i, x_{i+1}) \leq P(|\psi(x)|, |\psi(y)|)$, where $k(x_i, x_{i+1}) = \min(|\psi(x_i x_{i+1}^{-1})|, |\psi(x_{i+1} x_i^{-1})|)$. Let us observe (using Lemma 7.3) that this property implies that

$$\max(|\psi(x_i x_{i+1}^{-1})|, |\psi(x_{i+1} x_i^{-1})|) \leq P(|\psi(x)|, |\psi(y)|).$$

Consider such a path $x = x_0, \ldots, x_n = y$, and the path $gx = gx_0, gx_1, \ldots, gx_n = gy$. From (7.1),

$$\begin{aligned} k(gx_i, gx_{i+1}) &\leq |\psi(gx_i x_{i+1}^{-1} g^{-1})| \\ &\leq |\psi(x_i x_{i+1}^{-1})| P(|\psi(x_i x_{i+1}^{-1})|, |\psi(g)|) \\ &\leq |\psi(x_i x_{i+1}^{-1})| P(|\psi(x)|, |\psi(y)|, |\psi(g)|). \end{aligned}$$

In the same way,

$$|\psi(gx_{i+1} x_i^{-1} g^{-1})| \leq |\psi(x_{i+1} x_i^{-1})| P(|\psi(x)|, |\psi(y)|, |\psi(g)|)$$

and then

$$k(gx_i, gx_{i+1}) \leq P(|\psi(x)|, |\psi(y)|, |\psi(g)|) |k(x_i, x_{i+1})|.$$

We conclude that

$$\begin{aligned} d(gx, gy) &\leq k(gx_0, gx_1) + \cdots + k(gx_{n-1}, gx_n) \\ &\leq P(|\psi(x)|, |\psi(y)|, |\psi(g)|)(k(x_0, x_1) + \cdots + k(x_{n-1}, x_n)) \end{aligned}$$

and the lemma follows, taking the infimum. □

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LEMMA 7.6. *Let $g \in G$ and $n \in \mathbb{N}$; then $|\psi(g^n)| \leq P(n)$, where P is a polynomial with coefficients depending on $|\psi(g)|$.*

Proof. By the multiplication formula, we observe that $(\psi(g^n))_1 = n\psi(g)_1$, i.e., the first coordinate is controlled polynomially. Suppose now that the ith coordinate is controlled polynomially, then the same happens with the $(i+1)$th coordinate. In fact, we see inductively that

$$\begin{aligned}\psi(g^{n+1})_{i+1} &= \psi(g^n g)_{i+1} \\ &= \psi(g^n)_{i+1} + \psi(g)_{i+1} + P(\psi(g^n)_1, \ldots, \psi(g^n)_i, \psi(g)_1, \ldots, \psi(g)_i)\end{aligned}$$

and then $|\psi(g^{n+1})_{i+1}| - |\psi(g^n)_{i+1}| \leq P(n)$, and we conclude that $|\psi(g^{n+1})_{i+1}| \leq (n+1)P(n+1)$. (Here all polynomials P are not necessarily the same.) □

LEMMA 7.7. (Factorization) *Each $g \in G$ can be written in a unique way as $g = \{g\}[g]$ with $\psi(\{g\}) \in [0, 1)^m$ and $[g] \in G$.*

Therefore, when writing $x\Gamma$, we can assume x is such that $|\psi(x)| \leq 1$.

LEMMA 7.8. *Let $x, y \in G$. Then*

$$d(x\Gamma, y\Gamma) = \inf_{\gamma\in\Gamma} d(x, y\gamma) = \inf_{\gamma\in\Gamma, |\psi(\gamma)|\leq C} d(x, y\gamma),$$

where C is a constant depending only on $|\psi(x)|$ and $|\psi(y)|$.

Combining the last two lemmas, we see that

$$d(x\Gamma, y\Gamma) = \inf_{\gamma\in\Gamma, |\psi(\gamma)|\leq C} d(x, y\gamma),$$

where C is a constant.

We obtain

$$d(g^n x\Gamma, g^n y\Gamma) \leq \inf_{\gamma\in\Gamma, |\psi(\gamma)|\leq C} d(g^n x, g^n y\gamma).$$

Using Lemmas 7.5, 7.6 and 7.8, we get the following.

COROLLARY 7.9. *Let $x, y, g \in G$ with $|\psi(x)| \leq 1$ and $|\psi(y)| \leq 1$. Then,*

$$d(g^n x\Gamma, g^n y\Gamma) \leq \inf_{\gamma\in\Gamma, |\psi(\gamma)|\leq C} P(|\psi(g^n)|)\, d(x, y\gamma) \leq P(n)\, d(x\Gamma, y\Gamma). \tag{7.2}$$

7.2. *Complexity computation.* Let (X, T) be a TDS. We say that a subset $F \subseteq X$ is (n, ϵ)-shadowing if for all $x \in X$, there exists $y \in F$ such that $d(T^i x, T^i y) \leq \epsilon$ for all $i \in \{0, \ldots, n\}$. Write $r(n, \epsilon) = \min\{|F| : F \subseteq X, F \text{ is } (n, \epsilon)\text{-shadowing}\}$. Given an open cover $\mathcal{U} = \{U_1, \ldots, U_k\}$, the topological complexity function of $\mathcal{U}$ is the sequence on n : $c(\mathcal{U}, n) = \min\{M \geq 1 : \exists V_1, \ldots, V_M \in \bigvee_{i=0}^n T^{-i}\mathcal{U}, \text{ such that } \bigcup_{i=1}^M V_i = X\}$. One has that

$$c(\mathcal{U}, n) \leq r\left(n, \frac{\delta}{2}\right),$$

where $\delta > 0$ is the Lebesgue number of $\mathcal{U}$.

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To compute the topological complexity of a nilsystem in the general case (i.e., when G is not necessarily a connected and simply connected Lie group), we will use an argument given by Leibman in [**25**]. For this, we require an extra definition and one lemma from [**25**].

Let $X = G/\Gamma$ be a nilmanifold and $T : X \to X$ the transformation given by $x \to gx$ with a fixed $g \in G$.

Definition 7.10. We say that a closed subset $Y \subset X$ is a submanifold of X if $Y = Hx$, where H is a closed subgroup of G and $x \in X$.

LEMMA 7.11. *There exists a connected, simply connected nilpotent Lie group $\widehat{G}$ and $\widehat{\Gamma} \subseteq \widehat{G}$ a cocompact subgroup such that X with the action of G is isomorphic to a submanifold $\widetilde{X}$ of $\widehat{X} = \widehat{G}/\widehat{\Gamma}$ representing the action of G in $\widehat{G}$.*

This is the main result of the section.

THEOREM 7.12. *Let $X = G/\Gamma$ be a nilmanifold and $T : X \to X$ the transformation given by $x \to gx$ with a fixed $g \in G$. Let $\mathcal{U}$ be an open cover of X. Then, for all $n \in \mathbb{N}$,*

$$c(\mathcal{U}, n) \leq P(n),$$

where P is a polynomial.

Proof. First, assume that G is a connected, simply connected nilpotent Lie group. For $\epsilon > 0$, let $N(\epsilon)$ be the smallest number of balls of ratio ϵ needed to cover X. The upper Minkowski dimension or box dimension (see [**28**]) is defined by

$$\limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}.$$

This dimension coincides with the usual dimension of the manifold X and hence there exists a constant K such that

$$N(\epsilon) \leq K \left(\frac{1}{\epsilon}\right)^{\dim(X)+1}.$$

Using the bound in (7.2), we observe that if $x, y \in X$ and $d(x, y) \leq \delta/2P(n)$, then $d(T^i x, T^i y) \leq (P(i)/P(n))(\delta/2) \leq \delta/2$ if $i \leq n$ since P has positive coefficients. Let δ be the Lebesgue number of $\mathcal{U}$. We get

$$c(\mathcal{U}, n) \leq N\left(\frac{\delta}{2P(n)}\right) \leq K \frac{(2P(n))^{\dim(X)+1}}{\delta^{\dim(X)+1}}$$

and the polynomial bound of the complexity is obtained. □

Now consider the general case. Denote by $\pi : X \to \widetilde{X}$ the isomorphism given by Lemma 7.11. We see that (X, T) is conjugate to $(\widetilde{X}, \widetilde{T})$, where $\widetilde{T} : \widehat{X} \to \widehat{X}$ is defined by $\widetilde{T}(\widehat{x}) = \pi(g)\widehat{x}$. Hence $(\widetilde{X}, \widetilde{T})$ is a subsystem of $(\widehat{X}, \widetilde{T})$.

If $\mathcal{U} = (U_1, \ldots, U_m)$ is an open cover of X, $\pi(\mathcal{U}) = (\pi(U_1), \ldots, \pi(U_m))$ is an open cover of $\widetilde{X}$ and $c(\mathcal{U}, n) = c(\pi(\mathcal{U}), n) \leq c(\widehat{\mathcal{U}}, n)$, where $\widehat{\pi}(U) = (V_1, \ldots, V_m, \widetilde{X}^c)$ is an open cover of $\widehat{X}$ with $\pi(U_i) = \widetilde{X} \cap V_i$ for all $i \in \{1, \ldots, m\}$. Thus, we get a polynomial bound of the complexity in the general case.

Finally, we consider the complexity of an ∞-step nilsystem. For this, we need the following easy lemma.

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LEMMA 7.13. *Suppose X is an inverse limit of the systems $(X_i, T)_{i\in\mathbb{N}}$, where (X_i, T) has a polynomial complexity for each $i \in \mathbb{N}$. Then X has polynomial complexity.*

Proof. We will show that the product system has polynomial complexity and therefore the inverse limit has the same property. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\delta = \epsilon - 2^{-N} > 0$. Then $r_X(\epsilon, n) \le \prod_{i\le N} r_{X_i}(\delta, n)$ and, by assumption, the right side is polynomially bounded. □

We conclude the following.

THEOREM 7.14. *If X is an ∞-step nilsystem, then it has a polynomial complexity.*

Proof. By the above discussion, a d-step nilsystem has polynomial complexity. By Theorem [**16**], the factors (X_d, T) defined by the relation $\mathbf{RP}^{[d]}$ are inverse limits of d-step nilsystems and therefore they have polynomial complexity. Again, using the inverse limit argument, we conclude the polynomial bound for the complexity of (X, T). □

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A. *Appendix. The proof of Theorem 3.8*

In this appendix, we give the proof of Theorem 3.8. First, we discuss Theorem 3.8(2). The idea to prove this fact was inspired by personal communications with Kra [**23**]; here we give details of the proof.

LEMMA A.1. [**14**] *Let (X, μ, T) be an ergodic d-step nilsystem with $X = G/\Gamma$, μ be its Haar probability measure and T be the translation by the element $t \in G$. Moreover, assume that the group G can be spanned by the connected component of the identity and the element t (it is always possible to reduce to this case, see* [**3**]*). Let $d \ge 1$ be an integer. If Z_k is the maximal factor of order d of (X, μ, T) with $k \le d$, then Z_k has the form $G/(G_{k+1}\Gamma)$ endowed with the translation by the projection of t on G/G_{k+1}, where $G_1 = G$, $G_2 = [G, G_1]$, $G_3 = [G, G_2]$, ..., $G_{d+1} = \{e\}$.*

Now we prove Theorem 3.8(2).

Proof of Theorem 3.8(2). Let $n > d$ be any integer; we will show $Z_n = Z_d$. In [**15**], it was shown that Z_n is an inverse limit of n-step nilsystems $(Z_{n,i})_{i\in\mathbb{N}}$. For any $Z_{n,i}$, assume it has the form G/Γ, where the group G is spanned by the connected component of the identity and the translation element t.

Let G^o be the identity component of G. As shown in [**3**], $G_2 = [G, G] = [G^o, G]$ is connected, and, inductively, for any integer $k \ge 2$, G_k is connected. By Lemma A.1, the d-step maximal nilfactor and $(d+1)$-step maximal nilfactor of G/Γ are $G/(G_{d+1}\Gamma)$

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and $G/(G_{d+2}\Gamma)$, respectively. We have that $G/(G_{d+2}\Gamma)$ is also a $(d+1)$-step nilfactor of X, so it is a factor of $Z_{d+1} = Z_d$, which implies that $G/(G_{d+2}\Gamma)$ is also a d-step nilsystem. Now, by the maximality of $G/(G_{d+1}\Gamma)$, we have that $G/(G_{d+2}\Gamma)$ and $G/(G_{d+1}\Gamma)$ coincide. Then $G_{d+1}\Gamma = G_{d+2}\Gamma$, and, of course, the nilpotent Lie groups G_{d+1} and G_{d+2} have the same dimension, since Γ is discrete.

For any positive integer k, let $\mathfrak{g}_k$ be the associated Lie algebra of G_k. Then $\mathfrak{g}_{d+1}$ and $\mathfrak{g}_{d+2}$ have the same dimension, which implies that $\mathfrak{g}_{d+1}$ and $\mathfrak{g}_{d+2}$ coincide since $\mathfrak{g}_{d+2}$ is a subalgebra of $\mathfrak{g}_{d+1}$. Since G_{d+1} and G_{d+2} are connected, $G_{d+1} = \exp(\mathfrak{g}_{d+1}) = \exp(\mathfrak{g}_{d+2}) = G_{d+2}$, and then we have that

$$G_{d+3} = [G, G_{d+2}] = [G, G_{d+1}] = G_{d+2} = G_{d+1}.$$

Inductively, we have $G_{k+1} = G_{d+1}$ for all $d \le k \le n$, which implies that $G_{d+1} = \{e\}$, G is d-step nilpotent and $Z_{n,i} = G/\Gamma$ is a d-step nilsystem. So the inverse limit Z_n is a d-step nilsystem. By the maximality of Z_d, we conclude $Z_n = Z_d$. The proof is complete. □

To show the next lemma, we need some results from [**16**].

PROPOSITION A.2. *Let (X, T) be a minimal system. If (X, T) is an inverse limit of some d-step nilsystems, then each measurable factor is a topological factor.*

Proof. It is a combination of [**16**, Proposition 5.2] and [**16**, Lemma 6.1]. □

LEMMA A.3. *Let (X, T) be a minimal system of order n; then the maximal measurable and topological factors of order d coincide, where $d \le n$.*

Proof. Let μ be the unique invariant probability measure of X, and Z_d the maximal measurable factor of order d of (X, μ, T). It is clear that Z_d is a topological factor of order d of (X, μ, T) by Proposition A.2. Endow the maximal topological factor X_d of order d of X with its unique invariant probability measure. Clearly, it is a measurable factor of order d of (X, μ, T) and so is a measurable factor of Z_d. By Proposition A.2 again, X_d is a topological factor of Z_d, and so $Z_d = X_d$ is the maximal topological factor of order d. □

Now we can finish the proof of Theorem 3.8.

Proof of Theorem 3.8. (1) For any $k \ge 1$, recall $X_k = X/\mathbf{RP}^{[k]}$ and let μ_k be its unique invariant probability measure. Let $n > d$ be an integer. By the Lemma A.3, and since X_n is a minimal system of order n, then the maximal measurable and topological factors of order k of X_n coincide and $X_n/\mathbf{RP}^{[k]}(X_n) = X_k = Z_k$, $k \le n$. As $(Z_d, \mu_d, T) = (Z_{d+1}, \mu_{d+1}, T)$, by Theorem 3.8(2), we have for any $d \le k \le n$, $(Z_k, \mu_k, T) = (Z_d, \mu_d, T)$. Therefore, (Z_n, μ_n, T) and (Z_d, μ_d, T) coincide in the measurable sense, and by Proposition A.2, they coincide in the topological sense too, i.e., $X_n = Z_n = Z_d = X_d$, which implies that $\mathbf{RP}^{[n]}(X) = \mathbf{RP}^{[d]}(X)$.

(3) If Z_n is measure theoretically isomorphic with $X_n = X/\mathbf{RP}^{[n]}(X)$ for some $n \in \mathbb{N}$, by the Lemma A.3, for any positive integer $k \le n$, the measurable and topological maximal factors of order k coincide, which implies Z_k is measure theoretically isomorphic with $X_k = X/\mathbf{RP}^{[k]}(X)$. The proof of the theorem is complete. □

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