Some applications of the Chebyshev polynomials to polynomials in general

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The Chebyshev polynomials: some basic facts

The *Chebyshev polynomials* T_n, named after the Russian mathematician P. L. Chebyshev (1821-1894), are defined by

$$\cos nt = T_n(\cos t). \tag{1}$$

For example, since $\cos 2t = 2\cos^2 t - 1$, we have $T_2(x) = 2x^2 - 1$. Also, $T_0(x) = 1$ and $T_1(x) = x$.

The Chebyshev polynomials have numerous applications, especially in approximation theory. A very comprehensive account can be seen in [1]. Here we will focus on applications of one particular kind.

The polynomials satisfy a simple recurrence relation. Since

$$\cos(n+1)t + \cos(n-1)t = 2\cos nt \cos t,$$

we have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{2}$$

(Strictly, the substitution $x = \cos t$ establishes this for $|x| \leqslant 1$, but since they are polynomials, it extends to all x). Using this, it is easy to write down successive expressions for $T_n(x)$, for example

$$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x,$$

$$T_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1.$$

It follows at once from (2), by induction, that $T_n(x)$ has degree n, with integer coefficients and leading term $2^{n-1}x^n$ for $n \geqslant 1$. We will use the notation $T_n(x) = \sum_{r=0}^{n} c_{n,r}x^r$. So, for example, $c_{n,n} = 2^{n-1}$.

A number of further facts follow easily from (1) and (2). We record some of them.

(i) Taking $t = 0$, we see that $T_n(1) = 1$ for all n. So $\sum_{r=0}^{n} c_{n,r} = 1$.

(ii) $|T_n(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$. Further, if $x_j = \cos\frac{j\pi}{n}$, then $T_n(x_j) = \cos j\pi = (-1)^j$. Note that $1 = x_0 > x_1 > \ldots > x_n = -1$. In this way, T_n reproduces the oscillations of $\cos nt$ in the interval $-1 \leqslant x \leqslant 1$. The importance of the polynomials derives largely from these two properties.

(iii) $T_n(0) = T_n\left(\cos\frac{\pi}{2}\right) = \cos\frac{n\pi}{2}$, so $T_{2n}(0) = (-1)^n$ and $T_{2n+1}(0) = 0$. Of course, $T_n(0) = c_{n,0}$.

(iv) If $\cos t = x$, then $\cos(t + \pi) = -x$. So

$$T_{2n}(-x) = \cos 2n(t + \pi) = \cos 2nt = T_{2n}(x).$$

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In other words, T_{2n} is an even function, so only even powers of x appear in its expression. Similarly, $T_{2n+1}(-x) = -T_{2n+1}(x)$, so T_{2n+1} is an odd function: only odd powers of x appear. (This also follows easily from (2).)

(v) $T_n(\cosh t) = \cosh nt$. For if we define (temporarily) $\tilde{T}_n$ by this equation, then the sequence $\tilde{T}_n$ also satisfies (2) and starts with $\tilde{T}_0(x) = 1$ and $\tilde{T}_1(x) = x$. Alternatively, the statement follows at once if we accept that (1) extends to complex z, so that t can be replaced by it. Either way, this identity describes the values of $T_n(x)$ when $x > 1$: in particular, $T_n(x) > 1$ for such x. (In fact $T_n(x)$ grows very rapidly outside $[-1, 1]$: for example, $T_6(2) = 1351$.)

(vi) Differentiating both sides in (1), we have $-n \sin nt = -T_n'(\cos t) \sin t$, so $T_n'(\cos t) = \dfrac{n \sin nt}{\sin t}$. Note that $T_n'(0) = c_{n,1}$. Taking $t = \frac{\pi}{2}$, we see that $c_{n,1} = n \sin \frac{n\pi}{2}$, hence $c_{2n,1} = 0$ (which we also knew from (iv)) and $c_{2n+1,1} = (-1)^n(2n + 1)$.

Note: Write $T_n^*(x) = \frac{1}{2n-1}T_n(x)$. Then T_n^* is *monic*: the leading term is x^n. Some writers have defined the Chebyshev polynomials to be T_n^* rather than T_n. This, I think, is no longer usual, and our purposes are definitely better served by the definition given in (1).

General polynomials: comparison betweeen the coefficients and $M(f)$

For any function f that is bounded on $[-1, 1]$, write

$$M(f) = \sup\{|f(x)| : -1 \leqslant x \leqslant 1\}.$$

(In the context of approximation theory and Banach space theory, the customary notation is $\|f\|_\infty$.) For two functions f, g, we regard $M(f - g)$ as a measure of 'distance' between f and g. As we have seen, $M(T_n) = 1$ for all n.

For polynomials $f(x) = \sum_{r=0}^{\infty} a_r x^r$ (with a_r real), we will establish inequalities comparing the coefficients a_r with $M(f)$. Of course, the coefficients are given, while the value of $M(f)$ is generally far from transparent. A trivial start is: $|a_0| = |f(0)| \leqslant M(f)$.

Write $S(f) = \sum_{r=0}^{\infty} |a_r|$. We shall compare $M(f)$ with $S(f)$ as well as with individual coefficients. Clearly $M(f) \leqslant S(f)$, and equality holds if $a_r \geqslant 0$ for all r, since then $S(f) = \sum_{r=0}^{n} a_r = f(1)$. Equality also holds if $(-1)^r a_r \geqslant 0$ for all r: then $S(f) = f(-1)$. We will be looking for reverse inequalities of the form $S(f) \leqslant CM(f)$.

We now state our first result. In a very neat way, the Chebyshev polynomials deliver the following comparison between $|a_n|$ and $M(f)$.

Theorem 1: Let $f(x) = \sum_{r=0}^{n} a_r x^r$, a general real polynomial of degree n. Then $|a_n| \leqslant 2^{n-1}M(f)$.

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Proof: As in (ii), let $x_j = \cos\frac{j\pi}{n}$ for $0 \leqslant j \leqslant n$, so that $T_n(x_j) = (-1)^j$. Let $\lambda = \frac{2^{n-1}}{a_n}$, so that the leading term of $\lambda f(x)$ is $2^{n-1}x^n$, and the degree of $T_n - \lambda f$ is at most $n - 1$. We have to show that $|\lambda| M(f) \geqslant 1$. Suppose that $|\lambda| M(f) < 1$, so that $|\lambda f(x_j)| < 1$ for each j. Then $T_n(x_j) - \lambda f(x_j)$ is strictly positive for even j and strictly negative for odd j. By the intermediate value theorem, $T_n - \lambda f$ has a zero in each open interval (x_j, x_{j-1}), hence at least n distinct zeros. But this is impossible, since its degree is at most $n - 1$.

Though it says the same thing, perhaps a slightly different message is conveyed by writing this inequality as $M(f) \geqslant \frac{|a_n|}{2^{n-1}}$ (since normally a_n is known and $M(f)$ unknown). In particular, if f is any monic polynomial of degree n, then $M(f) \geqslant \frac{1}{2^{n-1}}$ (and equality occurs for $f = T_n^*$). This fact can be reformulated again as follows:

Corollary: Let n denote the set of all polynomials of degree not greater than n, and let $u_n(x) = x_n$. Then $M(u_n - f) \geqslant \frac{1}{2^{n-1}}$ for all f in Π_{n-1}. Equality occurs for $f = u_n - T_n^*$.

In other words, using the measure of distance $M(f - g)$, $u_n - T_n^*$ is the closest approximation to u_n among polynomials of degree not greater than $n - 1$.

In one or other of these formulations, Theorem 1 can be seen in most books on approximation theory, e.g. [2, p. 61], [1, p. 67–68]. Before leaving it, we show that the same reasoning also delivers the following rather remarkable bound for values of a polynomial outside $[-1, 1]$.

Theorem 2: Let $f(x)$ be any real polynomial of degree n. Then $|f(x)| \leqslant M(f)|T_n(x)|$ for all x with $|x| \geqslant 1$, hence $|f(\cosh t)| \leqslant M(f)\cosh nt$ for all t.

Proof: Choose a point x^* with $|x^*| \geqslant 1$. Assume that $f(x^*) \neq 0$ (or the statement is trivial), and let $\lambda = \dfrac{T_n(x^*)}{f(x^*)}$, so that $T_n(x^*) - \lambda f(x^*) = 0$. We have to show that $|\lambda| M(f) \geqslant 1$. If $|\lambda| M(f) < 1$, then, as in Theorem 1, the polynomial $T_n - \lambda f$ has at least n distinct zeros in $(-1, 1)$. So, with x^*, it has at least $n + 1$ zeros, which is impossible since its degree is at most n.

Theorem 1 compared $M(f)$ with the leading coefficient a_n. We now prove a result comparing it with the other coefficients a_r. This is somewhat less well known. It appears in [1] as Remark 2, p. 112, but only as a consequence of a long chain of other theorems. Here we show how it can be established by a simple adaptation of the proof of Theorem 1. The key is the following variation on the fact that polynomials of degree n have at most n zeros. We say that a polynomial has *length* n if it has n non-zero terms. So, for example, T_{2n} and T_{2n+1} have length $n + 1$.

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Lemma 1: A polynomial of length n has at most $n - 1$ positive zeros.

The proof, using induction and Rolle's theorem, is almost the same as the proof that polynomials of degree n have at most n zeros. However, it is seemingly much less well known, so we include it here. Note that it refers to *positive* zeros.

Proof: A polynomial of length 1 is of the form ax^k for some k, with $a \neq 0$, so is non-zero for all positive x. Now assume the statement true for a certain n, and let $f(x)$ have length $n + 1$, so it is of the form $\sum_{r=1}^{n+1} a_r x^{p_r}$, where $p_1 < p_2 < \ldots < p_{n+1}$ (actually, we do not even need the p_r to be integers). Let

$$g(x) = x^{-p_1} f(x) = a_1 + \sum_{r=2}^{n+1} a_r x^{p_r - p_1}.$$

Then $g(x)$ has the same positive zeros as $f(x)$ and

$$g'(x) = \sum_{r=2}^{n+1} (p_r - p_1) a_r x^{p_r - p_1 - 1},$$

so $g'(x)$ has length n. By hypothesis, $g'(x)$ has at most $n - 1$ positive zeros. By Rolle's theorem, between any two consecutive zeros of $g(x)$, there is a zero of $g'(x)$. Hence $g(x)$ has at most n positive zeros.

Lemma 1 is actually a special case of "Descartes' rule of signs". This, together with further generalisations, was discussed in [3].

We present our comparison first for even and odd polynomials separately.

Theorem 3: Let

$$g(x) = \sum_{r=0}^{n} a_{2r} x^{2r}, \qquad h(x) = \sum_{r=0}^{n} a_{2r+1} x^{2r+1}.$$

Then

$$|a_{2r}| \leqslant |c_{2n,2r}| M(g), \tag{3}$$

$$|a_{2r+1}| \leqslant |c_{2n+1,2r+1}| M(h). \tag{4}$$

Proof: Recall from (ii) that if $x_j = \cos\frac{j\pi}{2n}$, then $T_{2n}(x_j) = (-1)^j$ and $1 = x_0 > x_1 > \ldots > x_n = 0$. Choose r. Assume that $a_{2r} \neq 0$ (or the result is trivial), and let $\lambda a_{2r} = c_{2n,2r}$. In $T_{2n}(x) - \lambda g(x)$, the x^{2r} term cancels to zero, so the length of $T_{2n} - \lambda g$ is at most n. We have to show that $|\lambda| M(g) \geqslant 1$. As in Theorem 1, if $|\lambda| M(g) < 1$, then $T_{2n}(x_j) - \lambda g(x_j)$ is strictly positive for even j and strictly negative for odd j, so, by the intermediate value theorem, it has at least n zeros (all positive). This contradicts Lemma 1.

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For $h(x)$, the reasoning is the same, with $x_j = \cos\frac{j\pi}{2n+1}$, so that $x_n > 0$.

Now consider any polynomial f of odd degree $2n+1$. Separate the even and odd powers: $f(x) = g(x) + h(x)$, where $g(x)$ and $h(x)$ are as above. Then the coefficients a_{2r} and a_{2r+1} satisfy (3) and (4). Also, $g(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$ and $h(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x)$, so $M(g)$ and $M(h)$ are not greater than $M(f)$, and can (if we wish) be replaced by $M(f)$ in both statements.

If, instead, the degree of f is $2n$, then $h(x)$ becomes $\sum_{r=0}^{n-1} a_{2r+1}x^{2r+1}$, so $|a_{2r+1}| \leqslant |c_{2n-1,2r+1}|M(h)$.

Of course, to complete this story, we need to know $c_{n,r}$. Theorem 1 is the case deriving from the fact that $c_{n,n} = 2^{n-1}$. By (vi), we know that $|c_{2n+1,1}| = 2n+1$, so we can state:

Corollary: For any polynomial $f(x) = \sum_{r=0}^{n} a_r x^r$, we have $|a_1| \leqslant nM(f)$. (More exactly, if n equals $2k-1$ or $2k$, then $|a_1| \leqslant (2k-1)M(h)$, with $h(x)$ as above.)

Unfortunately, the general expression for $c_{n,r}$ is not very pleasant. It can be expressed as follows: if $r = n - 2s$ (where $s \leqslant \frac{1}{2}n$), then

$$c_{n,r} = (-1)^s \frac{n}{n-s}\binom{n-s}{s}2^{n-2s-1}. \tag{5}$$

One way to prove (5) is given in [1, p. 36-37]. We will sketch an alternative proof later. With these values inserted, we have a purely numerical statement of Theorem 3.

More pleasantly, we can simply exploit known expressions for $T_n(x)$, as in the following example.

Example 1: Let $f(x)$ be a general quartic, expressed as $g(x) + h(x)$, where $g(x) = a_0 + a_2x^2 + a_4x^4$ and $h(x) = a_1x + a_3x^3$. From the expressions for $T_3(x)$ and $T_4(x)$, we have at once

$$|a_1| \leqslant 3M(h), \qquad |a_3| \leqslant 4M(h), \qquad |a_2| \leqslant 8M(g), \qquad |a_4| \leqslant 8M(g).$$

An alternative method: At this point, we digress briefly to discuss an alternative, purely algebraic, approach which may have occurred to the reader. A polynomial of degree n is determined by the values at $n+1$ points, for example the points x_j seen in (ii). So the coefficients a_r can be expressed as combinations of these values, found, in principle, by a system of linear equations. Once known, this will imply inequalities of the type we want.

We illustrate this by one of the simplest non-trivial cases: consider again $h(x) = a_1x + a_3x^3$. Using the values at 1 and $\frac{1}{2}$, we have $h(1) = a_1 + a_3$ and $h(\frac{1}{2}) = \frac{1}{2}a_1 + \frac{1}{8}a_3$, hence

$$a_1 = \tfrac{8}{3}h\left(\tfrac{1}{2}\right) - \tfrac{1}{3}h(1), \qquad a_3 = \tfrac{4}{3}h\left(\tfrac{1}{2}\right) - \tfrac{8}{3}h\left(\tfrac{1}{2}\right).$$

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This implies $|a_1| \leqslant \left(\frac{8}{3} + \frac{1}{3}\right)M(h) = 3M(h)$ and $|a_3| \leqslant \left(\frac{4}{3} + \frac{8}{3}\right)M(h) = 4M(h)$, as seen in Example 1.

This might appear easy, but the general case is by no means trivial. In fact the somewhat lengthy route to Theorem 3 in [1] is essentially a realisation of this strategy.

Comparison between $M(f)$ and $S(f)$

Recall that $S(f)$ means $\sum_{r=0}^{n} |a_r|$, where $f(x) = \sum_{r=0}^{n} a_r x^r$, and that $M(f) \leqslant S(f)$. By Theorem 3, if f is an even or odd polynomial, then $S(f) \leqslant S(T_n)M(f)$, and the two statements combine to imply that $S(f) \leqslant \left[S(T_n) + S(T_{n-1})\right]M(f)$ for all polynomials. We now show that this actually holds without the term $S(T_{n-1})$.

Again write $f(x) = g(x) + h(x)$, where $g(x)$ is even and $h(x)$ is odd. For any real numbers a, b, note that $|a| + |b|$ equals $|a + b|$ if a and b have the same sign and $|a - b|$ if they have opposite signs. Hence $|g(x)| + |h(x)|$ equals either $|f(x)|$ or $|f(-x)|$, so is not greater than $M(f)$ when $|x| < 1$. Hence also $|g(x) \pm xh(x)|$ and $|xg(x) \pm h(x)|$ are not greater than $M(f)$ when $|x| \leqslant 1$.

Theorem 4: For all real polynomials of degree n, we have $S(f) \leqslant S(T_n)M(f)$.

Proof: The details again force us to consider even and odd degree separately. First, let $f(x) = \sum_{r=0}^{2n} a_r x^r$, expressed as $g(x) + h(x)$. We show that for each $r \geqslant 1$,

$$|a_{2r}| + |a_{2r-1}| \leqslant |c_{2n,2r}|M(f). \tag{6}$$

Together with $|a_0| \leqslant M(f)$ and $|c_{2n,0}| = 1$, this implies that $S(f) \leqslant S(T_{2n})M(f)$. Now $|a_{2r}| + |a_{2r-1}|$ is either $|a_{2r} + a_{2r-1}|$ or $|a_{2r} - a_{2r-1}|$. Let

$$p_1 = g(x) + xh(x) = a_0 + \sum_{r=1}^{n} \left(a_{2r} + a_{2r-1}\right)x^{2r}.$$

As noted above, $M(p_1) \leqslant M(f)$. By Theorem 3, $|a_{2r} + a_{2r-1}| \leqslant |c_{2n,2r}|M(p_1)$. Similarly, using $p_2(x) = g(x) - xh(x)$, we see that $|a_{2r} - a_{2r-1}| \leqslant |c_{2n,2r}|M(p_2)$. This establishes (6).

Now suppose that $f(x) = \sum_{r=0}^{2n+1} a_r x^r$. In the same way, considering $h(x) \pm xg(x)$, we show that

$$|a_{2r+1}| + |a_{2r}| \leqslant |c_{2n+1,\,2r+1}|M(f),$$

so that $S(f) \leqslant S(T_{2n+1})M(f)$.

The good news is that there is a much more pleasant expression for $S(T_n)$ than for the individual coefficients $c_{n,r}$. We show how to derive it.

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Lemma 2: We have

$$S(T_{n+1}) = 2S(T_n) + S(T_{n-1}). \tag{7}$$

Proof: Using (2), we show by induction that $T_n(x)$ is of the form $\sum_{s \leqslant \frac{n}{2}} (-1)^s d_{n,s} x^{n-2s}$, where $d_{n,s} \geqslant 0$ and

$$d_{n+1,s} = 2d_{n,s} + d_{n-1,s-1} \tag{8}$$

for $s \leqslant \frac{1}{2}(n+1)$. Given this, we have $S(T_n) = \sum_{s \leqslant \frac{n}{2}} d_{n,s}$, so (7) follows. Assume the statement correct for T_{n-1} and T_n. Then the coefficient of x^{n+1-2s} in $2xT_n(x)$ is $2(-1)^s d_{n,s}$ (if $s = \frac{1}{2}(n+1)$, then $n+1-2s = 0$ and this still holds with $d_{n,s}$ counted as 0). Also, $n+1-2s = (n-1)-2(s-1)$, so the coefficient of x^{n+1-2s} in $T_{n-1}(x)$ is $(-1)^{s-1} d_{n-1,s-1}$. So by (2), the coefficient of x^{n+1-2s} in $T_{n+1}(x)$ is $(-1)^s[2d_{n,s} + d_{n-1,s-1}]$, as stated in (8). The essential point is that the contributions from $T_n(x)$ and $-T_{n-1}(x)$ have the same sign, so they reinforce instead of cancelling.

With a bit of effort, this reasoning can be developed further to establish the expression for $d_{n,s}$ stated earlier in (5), but for present purposes we only need (7).

The standard procedure for solving recurrence relations (alias difference equations) now delivers the following evaluation.

Theorem 5: We have

$$S(T_n) = \tfrac{1}{2}(1+\sqrt{2})^n + \tfrac{1}{2}(1-\sqrt{2})^n.$$

Proof: Write $S(T_n) = S_n$. Then $S_{n+1} = 2S_n + S_{n-1}$. This recurrence relation is satisfied by $S_n = x^n$ if $x^2 = 2x+1$: the solutions are $x_1 = 1+\sqrt{2}$ and $x_2 = 1-\sqrt{2}$, so $c_1 x_1^n + c_2 x_2^n$ satisfies the recurrence for any c_1, c_2. If we take $c_1 = c_2 = \frac{1}{2}$, this matches the starting values $S_0 = S_1 = 1$, so S_n is as stated.

So these values can be inserted to complete the statement of Theorem 4. Also, $(1-\sqrt{2})^n \to 0$ as $n \to \infty$, so $S(T_n)$ is closely approximated by $\frac{1}{2}(1+\sqrt{2})^n$ for large n. In fact this approximation is already quite good for $n = 4$: it gives 16.985 to 3 d.p., while $S(T_4) = 17$.

Concluding remark

The situation is totally different for *complex* polynomials. The appropriate definition of $M(f)$ is now $\sup\{|f(z)| : z \text{ complex}, |z| \leqslant 1\}$. By Parseval's identity, one finds that $M(f) \geqslant \sqrt{\sum_{r=0}^{n} |a_r|^2}$ (so certainly $M(f) \geqslant |a_r|$ for all r) and $S(f) \leqslant \sqrt{n+1}\,M(f)$. Further, this estimation is asymptotically optimal. This topic is discussed in a companion Article [4].

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