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Finite time stability and comparison principles*

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1. *Introduction.* Motivated by discussion on practical stability in LaSalle and Lefschetz (3), Weiss and Infante (5), have discussed various notions of stability over finite time interval of a given differential system. This theory of stability differs from the usual stability theory mainly by the preassigned limits to which any given solution of the differential system must adhere. Sufficient conditions for these notions of stability are given in (5) in terms of certain Lyapunov-like functions satisfying some appropriate differential inequalities. The purpose of this article is to introduce some complementary notions of finite time stability and weaken the conditions on the differential inequalities involving Lyapunov-like functions by the use of comparison principles.

In sections 2 and 3, we introduce the notations and various notions of finite time stability. Two comparison principles are given in section 4 in a form which will be used later. Section 5 contains a list of theorems on various notions of finite time stability in terms of some Lyapunov-like functions and differential inequalities. Concluding remarks and an example to illustrate our results are presented in the last section.

2. *Notations and preliminaries.* Let J denote the interval $[t_0, t_0+T)$ in the non-negative real line R_+, with $T > 0$, and R^n be the real Euclidean n-space with norm $\| \ \|$. Consider the differential system

$$\dot{x} = f(t,x), \quad \left(\cdot \equiv \frac{d}{dt}\right), \tag{2·1}$$

where $f: J \times R^n \to R^n$ is a smooth function so that existence of solutions of (2·1) together with their continuous dependence on initial conditions are ensured. We do not assume uniqueness of solutions, nor do we assume the existence of the null solution.

Let $B(a) = \{x: x \in R^n, \|x\| < a\}$ and $\bar{B}(a) = \{x: x \in R^n, \|x\| \leqslant a\}$. A function

$$V = V(t,x): J \times R^n \to R$$

is said to be of class $\mathscr{V}$ if it is continuous in both t and x and satisfies for each $t \in J$ local Lipschitz condition in x. A function $\omega = \omega(t,r): J \times R \to R$ is said to be of class Ω if it is smooth enough to ensure the existence of the maximal solution of the differential equation

$$\dot{r} = \omega(t,r) \tag{2·2}$$

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over J. A function $\omega \in \Omega$ is said to be of class Ω_M if in addition $\omega(t,r)$ is monotonic increasing in r for each $t \in J$. We also need the following notations in the sequel:

$$V_m^{\alpha}(t) = \min_{\|x\|=\alpha} V(t,x),$$

$$V_M^{\alpha}(t) = \max_{\|x\|=\alpha} V(t,x),$$

$$V_{\delta}^{\alpha}(t) = \max_{x \in \overline{B}(\alpha)-B(\delta)} V(t,x) \quad (\delta < \alpha),$$

and

$$V^*(t,x) = \limsup_{h \to 0} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}, \tag{2·3}$$

where f is the given function in (2·1).

3. *Definitions of stability over J.*

DEFINITION 1. *A system* (2·1) *is stable with respect to* $(\alpha, \gamma, t_0, T, \| \|)$, $\alpha \leqslant \gamma$, *if for every trajectory* $x(t)$ *of* (2·1), $\|x(t_0)\| < \alpha$ *implies* $\|x(t)\| < \gamma$ *for all* $t \in J$.

DEFINITION 2. *A system* (2·1) *is quasi-contractively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$, $\alpha > \beta$, *if for every trajectory* $x(t)$ *of* (2·1), *the condition* $\|x(t_0)\| < \alpha$ *implies that there exists* $t_1 \in J$ *such that* $\|x(t)\| < \beta$ *for* $t \in (t_1, t_0+T)$.

DEFINITION 3. *A system* (2·1) *is contractively stable with respect to* $(\alpha, \beta, \gamma, t_0, T, \| \|)$, $\beta < \alpha \leqslant \gamma$ *if it is stable with respect to* $(\alpha, \gamma, t_0, T, \| \|)$ *and it is quasi-contractively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$.

DEFINITION 4. *A system* (2·1) *is quasi-expansively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$, $\alpha \leqslant \beta$, *if for every trajectory* $x(t)$ *of* (2·1), *the condition* $\|x(t_0)\| < \alpha$ *implies that there exists* $t_1 \in J$ *such that* $\|x(t_0)\| < \beta$ *for* $t \in (t_1, t_0+T)$.

DEFINITION 5. *A system* (2·1) *is expansively stable with respect to* $(\alpha, \beta, \gamma, t_0, T, \| \|)$, $\alpha \leqslant \beta < \gamma$, *if it is stable with respect to* $(\alpha, \gamma, t_0, T, \| \|)$ *and it is quasi-expansively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$.

Remark (3·1). Definitions 1, 2, 3 are due to Weiss and Infante (5). However, from the practical point of view, Definitions 4, 5 seem equally plausible and important.

Remark (3·2). The significance of these definitions, as remarked in (5), lies in the fact that the parameters $\alpha, \beta, \gamma, t_0, T, \| \|$ are usually pre-assigned in a particular problem. We observe here that if Definition 2 holds for every β, $0 < \beta < \alpha$, then all trajectories of (2·1) with the condition $\|x(t_0)\| < \alpha$ will reach origin for some $t \in J$. The system (2·1) with this property is defined to be strongly stable over J by Roxin (4).

Remark (3·3). Suppose that the system (2·1) is $(\alpha, \gamma, t_0, T, \| \|)$-stable, then it is trivially stable with respect to $(\alpha, \delta, t_0, T, \| \|)$ for every $\delta > \gamma$. Denote by $S(\alpha) = \{\gamma : (2\cdot1)$ is $(\alpha, \gamma, t_0, T, \| \|)$-stable$\}$. If $a = \inf_{\gamma \in S(\alpha)} \gamma \in S(\alpha)$, then system (2·1) may be said to be optimally stable with respect to $(\alpha, a, t_0, T, \| \|)$. Similarly, optimal contractive stability may be defined.

4. *Comparison principles.* The principle tool used in establishing our results is the following form of comparison principle due to Lakshmikantham (2):

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LEMMA 1. *Let there exist a function* $V \in \mathscr{V}$ *such that the function* $V^*(t,x)$ *defined by* (2·3) *satisfies the condition*

$$V^*(t,x) \leqslant \omega(t, V(t,x))$$

for all $(t,x) \in J \times R^n$, *where* $\omega \in \Omega$. *Let* $x(t,x_0)$ *denote any trajectory of* (2·1) *with* $x(t_0,x_0) = x_0$, *and* $r(t,r_0)$ *denote the maximal solution of* (2·2) *with* $r(t_0,r_0) = r_0$, *then*

$$V(t,x(t,x_0)) \leqslant r(t, V_M^{\|x_0\|}(t_0))$$

for all $t \in J$.

In order to present results which will include results of Weiss and Infante(5) as special cases, we need the following variant of Lemma 1.

LEMMA 2. *Let there exist a function* $V \in \mathscr{V}$ *such that the function* $V^*(t,x)$ *defined by* (2·3) *satisfies the condition*

$$V^*(t,x) < \omega(t, V(t,x)) \tag{4·1}$$

for all $(t,x) \in J \times R^n$, *where* $\omega \in \Omega_M$. *Let* $x(t,x_0)$ *denote any trajectory of system* (2·1) *with* $x(t_0,x_0) = x_0$, *and* $r(t,r_0)$ *denote the maximal solution of* (2·2) *with* $r(t_0,r_0) = r_0$. *Then,*

$$V(t,x(t,x_0)) < r(t, V_M^{\|x_0\|}(t_0)) \tag{4·2}$$

for all $t \in J, t > t_0$.

Proof. Let $x(t) = x(t,x_0)$. We observe

$$\begin{aligned}
&(1/h)\,[V(t+h,x(t+h)) - V(t,x(t)]\\
&\quad = (1/h)\,[V(t+h,x(t+h)) - V(t+h,x(t)+hf(t,x(t))]\\
&\qquad + V(t+h,x(t)+hf(t,x(t))) - V(t,x(t))]\\
&\quad \leqslant K_t \left\| \frac{x(t+h)-x(t)}{h} - f(t,x(t)) \right\| + \frac{V(t+h,x(t)+hf(t,x(t))) - V(t,x(t))}{h},
\end{aligned}$$

where K_t is the Lipschitz constant corresponding to t. Taking lim sup on both sides as $h \to 0$, by (2·1), (4·1) we have

$$\limsup_{h \to 0} \frac{V(t+h,x(t+h)) - V(t,x(t))}{h} \leqslant V^*(t,x(t)) < \omega(t, V(t,x(t)). \tag{4·3}$$

Note that $V(t_0,x(t_0)) \leqslant V_M^{\|x_0\|}(t_0)$. Let $r(t)$ be the maximal solution of (2·2) with $r(t_0) = V_M^{\|x_0\|}(t_0)$, then by Lemma 1 we have $V(t,x(t)) \leqslant r(t)$ for all $t \in J$. Since the lim sup on the left of (4·3) is just the upper Dini derivative of $V(t,x(t))$ with respect to t, it follows that $V(t,x(t)) < r(t)$ in a small neighbourhood of t_0. We now claim that (4·2) indeed holds throughout J. Assume that contrary, there exists $t_1 \in J$ such that $V(t_1,x(t_1)) = r(t_1)$, and $t_1 > t_0$. Consider $m(t) = r(t) - V(t,x(t))$ in the interval $I = (t_0,t_1)$. Clearly $m(t)$ is a continuous non-negative function on I and satisfies $m(t_0) = m(t_1) = 0$. Hence there exists $t_2 \in I$ at which $m^+(t_2) \leqslant 0$, where m^+ is the upper Dini derivative of $m(t)$. Observe that

$$\begin{aligned}
0 &\geqslant r^+(t_2) - V^+(t_2,x(t_2))\\
&> \dot{r}(t_2) - \omega(t_2,x(t_2))\\
&\geqslant \dot{r}(t_2) - \omega(t_2,r(t_2)) = 0
\end{aligned}$$

which is the desired contradiction.

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5. THEOREM 1. *A system* (2·1) *is stable with respect to* $(\alpha, \gamma, t_0, T, \| \|)$, $\alpha < \gamma$, *if there exist functions* $V \in \mathscr{V}$ *and* $\omega \in \Omega_M$ *such that*

(i) *for all* $t \in J$ *and* $x \in \bar{B}(\gamma) - B(\alpha)$, (4·1) *holds, and*

(ii) *the maximal solution* $r(t)$ *of* (2·2) *with initial condition* $r(t_1) = V_M^\alpha(t_1)$ *at any* $t_1 \in J$ *satisfies the condition*

$$r(t, V_M^\alpha(t_1)) \leqslant V_m^\gamma(t) \tag{5·1}$$

for all $t \in J$, $t > t_1$.

Proof. Let $x(t)$ denote any arbitrary trajectory of (2·1) with $\|x(t_0)\| < \alpha$. Assume that there exists $t_2 \in J$ such that $\|x(t_2)\| = \gamma$, and we may also assume that t_2 is the first such time in the interval J. Then there must exist $t_1 \in J$, $t_0 < t_1 < t_2$, such that $\|x(t_1)\| = \alpha$ and $\gamma \geqslant \|x(t)\| \geqslant \alpha$, for $t_1 \leqslant t \leqslant t_2$. Hence by hypothesis (i) and Lemma 2, we have

$$V(t, x(t)) < r(t, V_M^\alpha(t_1)) \tag{5·2}$$

for all $t \in (t_1, t_2]$. On the other hand, it follows from (5·2) and (5·1) that

$$V_m^\gamma(t_2) \leqslant V(t_2, x(t_2)) < r(t_2, V_M^\alpha(t_1)) \leqslant V_m^\gamma(t_2),$$

which contradicts the existence of $t_2 \in J$ with $\|x(t_2)\| = \gamma$. We then conclude that $\|x(t)\| < \gamma$ for all $t \in J$. Since this argument is clearly independent of the particular trajectory chosen, it holds for all trajectories $x(t)$ with $\|x(t_0)\| < \alpha$, and the proof is complete.

THEOREM 2. *A system* (2·1) *is stable with respect to* $(\alpha, \alpha, t_0, T, \| \|)$ *if there exist functions* $V \in \mathscr{V}$, $\omega \in \Omega_M$ *and a positive number* $\delta < \alpha$ *such that*

(i) *for all* $t \in J$, $x \in \bar{B}(\alpha) - B(\delta)$, (4·1) *holds, and*

(ii) *the maximal solution* $r(t)$ *of* (2·2) *with initial condition* $r(t_1) = V_\delta^\alpha(t_1)$ *at any* $t_1 \in J$ *satisfies the condition*

$$r(t, V_\delta^\alpha(t_1)) \leqslant V_m^\alpha(t) \tag{5·3}$$

for all $t \in J$, $t > t_1$.

Proof. Assume that there is a trajectory $x(t)$ of (2·1) with $\|x(t_0)\| < \alpha$ such that $\|x(t_2)\| = \alpha$ for some $t_2 \in J$. We may also assume that $t_2 \in J$ is the first such t in J. Then there is a $t_1 < t_2$, $t_1 \geqslant t_0$ such that $\delta \leqslant \|x(t)\| < \alpha$ for $t \in [t_1, t_2)$. Hence in the interval $[t_1, t_2]$ the hypothesis (i) holds and again from Lemma 2 we have

$$V(t, x(t)) < r(t, V_\delta^\alpha(t_1)) \tag{5·4}$$

for all $t \in [t_1, t_2]$. On the other hand, we have at $t = t_2$

$$V_m^\alpha(t_2) \leqslant V(t_2, x(t_2)) < r(t_2, V_\delta^\alpha(t_1)) \leqslant V_m^\alpha(t_2)$$

on account of (5·3), and (5·4), which is the desired contradition.

THEOREM 3. *A system* (2·1) *is quasi-expansively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$ *if there exist functions* $V \in \mathscr{V}$ *and* $\omega \in \Omega_M$ *such that*

(i) *for all* $t \in J$, $x \notin B(\beta)$, (4·1) *holds,*

(ii) *the maximal solution* $r(t)$ (2·2) *with initial condition* $r(t_1) = V_M^\beta(t_1)$ *at any* $t_1 \in J$ *satisfies the condition*

$$r(t_0 + T, V_m^\beta(t_1)) \leqslant V_m^\beta(t_0 + T), \tag{5·5}$$

(iii) *for all* $x \notin B(\beta)$,

$$V(t_0 + T, x) \geqslant V_m^\beta(t_0 + T). \tag{5·6}$$

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Proof. To prove the theorem it is sufficient to show that for every trajectory $x(t)$ of (2·1) the condition $\|x(t_0)\| < \alpha$ implies that $\|x(t_0+T)\| < \beta$. Assume therefore that there exists a trajectory $x(t)$ of (2·1), with $\|x(t_0)\| < \alpha$ such that $\|x(t_1)\| = \beta$ and $\|x(t)\| \geqslant \beta$ for all $t \in [t_1, t_0+T)$, $t_1 \in J$. Thus the hypothesis (i) holds in this interval and from Lemma 2, we have the inequality

$$V(t, x(t)) < r(t, V_M^\beta(t_1)) \tag{5·7}$$

for all $t \in (t_1, t_0+T]$. However, from (5·5), (5·6), and (5·7) we have

$$\begin{aligned} V_m^\beta(t_0+T) &\leqslant V(t_0+T, x(t_0+T)) < r(t_0+T, V_M^\beta(t_1)) \\ &\leqslant V_m^\beta(t_0+T), \end{aligned}$$

which is a contradiction, and hence the theorem follows.

THEOREM 4. *A system* (2·1) *is quasi-contractively stable with respect to* $(\alpha, \beta, t_0, T, \|\ \|)$ *if there exist functions* $V \in \mathscr{V}$ *and* $\omega \in \Omega_M$ *such that*

(i) *for all* $t \in J$ *and* $x \notin B(\beta)$, (4·1) *holds,*

(ii) *the maximal solution* $r(t)$ *of* (2·2) *with appropriate initial conditions satisfies*

(*a*) $r(t_0+T, V_M^\beta(t_1)) \leqslant V_m^\beta(t_0+T)$ *for all* $t_1 \in J$,

(*b*) $r(t_0+T, V_\beta^\alpha(t_0)) \leqslant V_m^\beta(t_0+T)$,

and

(iii) *for all* $x \notin B(\beta)$

$$V(t_0+T, x) \geqslant V_m^\beta(t_0+T).$$

Proof. The proof may be divided into two cases. First, assume that there exists a trajectory of (2·1), with $\|x(t_0)\| < \beta$ such that $\|x(t_0+T)\| \geqslant \beta$, then there is a $t_1 \in J$ such that $\|x(t_1)\| = \beta$ and $\|x(t)\| \geqslant \beta$ for all $t \in [t_1, t_0+T]$. In this case hypotheses (i), (ii) (*a*), (ii) imply as a consequence of Theorem 3 that $\|x(t_0+T)\| < \beta$. On the other hand, there may exist trajectory starting in $\beta(\alpha) - \bar{B}(\beta)$ but never reach $B(\beta)$ at any time. Let $x(t)$ be such a trajectory, with $x(t_0) \in B(\alpha) - \bar{B}(\beta)$, and $\|x(t_0+T)\| \geqslant \beta$. Observe in this case, at $t = t_0+T$

$$\begin{aligned} V_m^\beta(t_0+T) &\leqslant V(T_0+T, x(t_0+T)) < r(t_0+T, V_M^{\|x_0\|}(t_0)) \\ &\leqslant r(t_0+T, V_\beta^\alpha(t_0)) \leqslant V_m^\beta(t_0+T), \end{aligned}$$

using (i), (ii) (*b*), (iii), and Lemma 2, which is impossible. This completes the proof.

THEOREM 5. *A system* (2·1) *is expansively stable with respect to* $(\alpha, \beta, \gamma, t_0, T, \|\ \|)$, $\alpha \leqslant \beta < \gamma$, *if there exist functions* $V \in \mathscr{V}$ *and* $\omega \in \Omega_M$ *such that*

(i) *for all* $t \in J$ *and* $x \in \bar{B}(\gamma) - B(\alpha)$, (4·1) *holds,*

(ii) *the maximal solution* $r(t)$ *of Equation* (2·2) *with appropriate initial conditions satisfies*

(*a*) $r(t, V_M^\alpha(t_1)) \leqslant V_m^\gamma(t)$ *for all* $t_1 \in J$, *and* $t > t_1$,

(*b*) $r(t_0+T, V_M^\beta(t_1)) \leqslant V_m^\beta(t_0+T)$ *for all* $t_1 \in J$,

(iii) *for all* $x \in B(\gamma) - B(\beta)$

$$V(t_0+T, x) \geqslant V_m^\beta(t_0+T).$$

Proof. Hypothesis (i), (ii) (*a*) ensure that the system (2·1) is stable with respect to $(\alpha, \gamma, t_0, T, \|\ \|)$ by Theorem 1. The fact that for every trajectory $x(t)$, $\|x(t_0)\| < \alpha$ implies

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$\|x(t)\| < \gamma$ for all $t \in J$, together with (i), (ii) (b), and (iii) show that the system (2·1) is quasi-expansively stable with respect to $(\alpha, \beta, t_0, T, \| \|)$ by Theorem 3.

THEOREM 6. *A system* (2·1) *is contractively stable with respect to* $(\alpha, \beta, \gamma, t_0, T, \| \|)$, $\beta < \alpha < \gamma$, *if there exist functions* $V \in \mathscr{V}$ *and* $\omega \in \Omega_M$ *such that*

(i) *for all* $t \in J$, $x \in \bar{B}(\gamma) - B(\beta)$, *inequality* (4·1) *holds,*

(ii) *the maximal solution* $r(t)$ *of Equation* (2·2) *with appropriate initial conditions satisfies*

(a) $r(t, V_M^{\alpha}(t_1)) \leqslant V_M^{\gamma}(t)$ *for all* $t_1 \in J$ *and* $t > t_1$,

(b) $r(t_0+T, V_M^{\beta}(t_1)) \leqslant V_m^{\beta}(t_0+T)$ *for all* $t_1 \in J$,

(c) $r(t_0+T, V_{\beta}^{\alpha}(t_0)) \leqslant V_m^{\beta}(t_0+T)$, *and*

(iii) *for all* $x \in \bar{B}(\gamma) - B(\beta)$,

$$V(t_0+T, x) \geqslant V_m^{\beta}(t_0+T).$$

Proof. Theorem 6 follows from Theorems 1 and 4 in the same way as Theorem 5 follows from Theorems 1 and 3.

Patterned after the results of (5), the following theorems may be proved in the same way. We leave the details to the reader.

THEOREM 7. *A system* (2·1) *is quasi-contractively stable with respect to* $(\alpha, \beta, t_0, T, \| \|)$ *if there exist functions* $V \in \mathscr{V}$, $\omega \in \Omega_M$ *and a positive constant* $\delta < \beta$ *such that*

(i) *for all* $t \in J$ *and* $x \notin B(\beta)$, (4·1) *holds,*

(ii) *the maximal solution* $r(t)$ *of* (2·2) *with appropriate initial conditions satisfies*

(a) $r(t_0+T, V_M^{\alpha}(t_1)) \leqslant V_m^{\beta}(t_0+T)$, *for all* $t_1 \in J$;

(b) $r(t_0+T, V_{\delta}^{\alpha}(t_0)) \leqslant V_m^{\beta}(t_0+T)$;

and

(iii) *for all* $x \notin B(\beta)$, $V(t_0+T, x) \geqslant V_m^{\beta}(t_0+T)$.

THEOREM 8. *A system* (2·1) *is contractively stable with respect to* $(\alpha, \beta, \gamma, t_0, T, \| \|)$, $\beta < \alpha < \gamma$, *if there exist functions* $V \in \mathscr{V}$, $\omega, \psi \in \Omega_M$ *and a positive constant* $\delta < \beta$ *such that*

(i) *for all* $x \in \bar{B}(\gamma) - B(\alpha)$, (4·1) *holds,*

(ii) *for all* $x \in B(\gamma) - B(\delta)$, *we have*

$$V^*(t, x) < \psi(t, V(t, x)),$$

(iii) *the maximal solution* $r(t)$ *of* (2·2) *satisfies*

$$r(t, V_M^{\alpha}(t_1)) \leqslant V_m^{\gamma}(t) \text{ for all } t_1 \in J \text{ and } t > t_1,$$

(iv) *the maximal solution* $p(t)$ *of* $\dot{p}(t) = \psi(t, p)$

satisfies

(a) $p(t_0+T, V_{\delta}^{\alpha}(t_0)) \leqslant V_m^{\beta}(t_0+T)$, *and*

(b) $p(t_0+T, V_M^{\delta}(t_1)) \leqslant V_m^{\beta}(t_0+T)$ *for all* $t_1 \in J$, *and*

(v) *for all* $x \in B(\gamma) - B(\beta)$, $V(t_0+T, x) \geqslant V_m^{\beta}(t_0+T)$.

In the same way, theorems on quasi-expansive stability and expansive stability may be formulated similar to Theorems 7 and 8.

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6. *Remarks and example.*

Remark (6·1). Let $V(t,x) \in C^1 \times C^0$, and $\omega(t,r) = \phi(t)$, where $\phi(t)$ is integrable over any finite subinterval in J. Clearly $V \in \mathscr{V}$ and $\omega \in \Omega_M$, and the maximal solution of Equation (2·2) with initial condition $r(t_1) = V^a_M(t_1)$ at $t_1 \in J$ is

$$r(t, V^a_M(t_1)) = V^a_M(t_1) + \int_{t_1}^{t} \phi(s)\, ds. \tag{6·1}$$

Thus the condition (5·1) becomes

$$\int_{t_1}^{t} \phi(s)\, ds \leqslant V^\gamma_m(t) - V^a_M(t_1) \tag{6·2}$$

for all $t, t_1 \in J$, $t > t_1$, and Theorem 1 here reduces to Theorem (1·1) of (5). Similarly, Theorems 2, 7, 8 will reduce to corresponding Theorems (1·2), 3, 4 of (5) by straight forward computations.

Remark (6·2). Results such as Theorem 2 of (5) on instability may be formulated analogously.

Remark (6·3). Extensions of our results to sets may be carried over in the same manner as that of Hallam and Komkov (1) who generalize part of the results in (5) to sets.

Remark (6·4). The assumption that $\omega \in \Omega_M$ can be relaxed to $\omega \in \Omega$ in each of the above theorems provided that the strict inequality in (4·1) and other inequalities are suitably interchanged. We state the following variant of Theorem 1, others may be formulated similarly.

THEOREM 1′. *A system* (2·1) *is stable with respect to* $(\alpha, \gamma, t_0, T, \| \|)$, $\alpha < \gamma$, *if there exist functions* $V \in \mathscr{V}$, $\omega \in \Omega$ *such that*

(i) *for all* $t \in J$, $x \in \bar{B}(\gamma) - B(\alpha)$,

$$V^*(t,x) \leqslant \omega(t, V(t,x)), \tag{6·3}$$

and

(i) *the maximal solution of* (2·2) *satisfies the condition*

$$r(t, V^a_M(t_1)) < V^\gamma_m(t) \tag{6·4}$$

for all, $t_1 \in J$, $t > t_1$.

Remark (6·5). To illustrate the versatility of our results, we shall give for specific functions $\omega(t,r)$ the corresponding equations to (6·1) and (6·2).

We shall consider $\omega(t,r) = \phi(t) r^p$ for all $p \geqslant 0$ and $\phi(t) \geqslant 0$. In this case the maximal solution of (2·2) in (6·1) becomes

$$r(t, V^a_M(t_1)) = \begin{cases} V^a_M(t_1) \exp \int_{t_1}^{t_2} \phi(s)\, ds & (p = 1), \\ \left\{ (V^a_M(t_1))^{1-p} + (1-p) \int_{t_1}^{t_2} \phi(s)\, ds \right\}^{1/(1-p)} & (p \neq 1), \end{cases} \tag{6·5}$$

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where we also assume that the radical in (6·5) is meaningful when $p > 1$. Correspondingly, (6·2) may be formulated as

$$\int_{t_1}^{t_2} \phi(s)\,ds \leqslant \begin{cases} \log|V_m^{\gamma}(t_2)| - \log|V_M^{\alpha}(t_1)| & (p = 1), \\ \dfrac{|V_m^{\beta}(t_2)|^{1-p} - |V_M^{\alpha}(t_1)|^{1-p}}{(1-p)} & (p \neq 1). \end{cases} \tag{6·6}$$

If we do not wish to assume $\phi(t) \geqslant 0$, we may use (6·3) and (6·4) in Theorem 1 to obtain a corresponding statement where strict inequality instead of inequality occurs in (6·6).

Remark (6·6). In contrast to the usual results on stability, no requirement of definiteness on either $V(t,x)$ or $\dot{V}(t,x)$ is made.

We finally close our discussion with an example. Consider the following differential system:

$$\left.\begin{aligned} \dot{x}_1 &= x_1 \cos t + \beta x_2, \\ \dot{x}_2 &= -\beta x_1 + x_2 \cos t, \end{aligned}\right\} \tag{6·7}$$

where $x = (x_1, x_2) \in R^2$, and β real. In this case, we have $\|x\| = \sqrt{(x_1^2 + x_2^2)}$.

Let $\alpha = 1, \gamma = 6$, and we shall consider the stability of the system (6·7) with respect to $(1, 6, 0, \frac{1}{2}\pi, \|\ \|)$. Take the usual energy function $V = x_1^2 + x_2^2$. It is easily verified that

$$\dot{V} = 2\cos t V < 2(\cos t + \mu)\, V$$

for any $\mu > 0$. Here $\omega(t,r) = 2(\cos t + \mu) r$ which clearly belongs to Ω_M, for all $t \in J$. Since $V_M^1(t_1) = 1$ for any $t_1 \in J$, the maximal solution of (2·2) is

$$r(t, V_M^1(t_1)) = \exp[2(\sin t - \sin t_1) + \mu(t - t_1)].$$

Condition (5·1) of Theorem 1 for $(1, 6, 0, \frac{1}{2}\pi, \|\ \|)$ stability becomes

$$\exp[2(\sin t - \sin t_1) + \mu(t - t_1)] \leqslant V_M^6(t_2) = 36,$$

which is readily satisfied if μ is sufficiently small, say $\leqslant 1$. On the other hand, if we take the same Lyapunov function V and apply Theorem (1·1) of (5), we are unable to deduce the same stability property. Observe that

$$\dot{V} = 2\cos t(x_1^2 + x_2^2) < 72\cos t + \mu.$$

for any $\mu > 0$ whenever $x \in \bar{B}(6) - B(1)$, and we must take $\phi(t) = 72\cos t + \mu$ in the above-mentioned result. However, condition (6·2) now becomes

$$72(\sin t_2 - \sin t_1) + \mu(t_2 - t_1) \leqslant 36 - 1 = 35,$$

which cannot be satisfied if $t_1 = 0$ and $t_2 \geqslant \frac{1}{6}\pi$.

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