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Kuzmin's theorem revisited

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Abstract. Iosifescu pointed out that earlier attempts to generalize Kuzmin's theorem to fibred systems have a gap. In this paper a new proof is given. The convergence rate is worse, but the result applies to multi-dimensional continued fractions and complex continued fractions.

1. *Introduction*

One of the milestones in the history of metric number theory was Kuzmin's proof of Gauss' conjecture [**8**]. A historical account of metric number theory of continued fractions can be found in Rieger [**13**]. A critical exposition of Kuzmin's approach is given in Rieger [**14**]. A common feature of Kuzmin's proof and later refinements can be seen in the use of peculiar properties of the density of the invariant measure, namely

$$h(x) = \frac{1}{\log 2} \cdot \frac{1}{1+x}.$$

Iosifescu [**5**] took a completely new route to Kuzmin's theorem. Fluch [**3**] following Wirsing [**19**] gave numerous interesting numerical calculations.

In my early work on the Jacobi–Perron algorithm I could prove the existence of an absolutely continuous invariant measure, but I was unable to derive the density explicitly (and the problem is still open). Therefore I tried to give a proof of a Kuzmin theorem for multi-dimensional continued fractions without using the shape of an existing density. In Schweiger [**15**] a first attempt was given but the proof contains a substantial error (see [**15**, §2]). Schweiger [**16**] claimed to circumvent the difficulty. In both papers a very good convergence rate of order $\sigma(n)$ was asserted (note that for continued fractions this means an order $((3-\sqrt{5})/2)^n$ which should be compared with [**6**]). Strange enough, it was not before 1997 that a serious flaw in the second 'proof' was detected, by Berechet and Iosifescu. This error was not detected by the referee, and unfortunately, it was copied into subsequent papers (e.g. Schweiger and Waterman [**18**], Nakada [**12**]).

Here I make a new attempt for a correct proof. I give the proof for a class of fibred systems which includes the Jacobi–Perron algorithm and complex continued fractions [**12**].

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2. *A suitable class of fibred systems*

Let B be a compact and connected subset of $\mathbb{R}^d$ and λ denote the d-dimensional Lebesgue measure. For convenience we assume $\lambda(B) = 1$. Furthermore, let $T : B \to B$ be a map. The pair (B, T) is called a *fibred system* (see [**17**]) if:

(a) there is a finite or countable set I (the set of 'digits');

(b) the sets $B(i) \subseteq B, i \in I$, form a partition of B;

(c) the restriction of T to $B(i), i \in I$, is an injective map. Its inverse is denoted $V(i) : TB(i) \to B(i)$;

(d) T is measurable and non-singular (i.e. $\lambda(E) = 0$ implies $\lambda(T^{-1}E) = 0$).

We define cylinders recursively:

$$B(i_1, \ldots, i_s) := B(i_1) \cap T^{-1}B(i_2, \ldots, i_s).$$

Let $V(i_1, \ldots, i_s)$ denote the inverse map of T^s restricted to $B(i_1, \ldots, i_s)$. A cylinder is called *full* if $T^s B(i_1, \ldots, i_s) = B$. We further assume the following.

(A) Let $\sigma(s) := \sup_{(k_1,\ldots,k_s)} \operatorname{diam} B(k_1, \ldots, k_s)$ then $\lim_{s\to\infty} \sigma(s) = 0$.

(B) There are finitely many subsets $U_1, \ldots, U_N$ of B such that for any non-empty cylinder $B(i_1, \ldots, i_s)$ there is an index $j = j(i_1, \ldots, i_s)$ with the property $T^s B(i_1, \ldots, i_s) = U_j$. The sets $U_j, 1 \le j \le N$, generate a finite partition $\mathcal{F}$ of B. Furthermore, we define $L := \min_{1\le j\le N} \lambda(U_j)$.

(C) Let the functions $\omega(i_1, \ldots, i_s)$ be defined by the equation

$$\int_E \omega(i_1, \ldots, i_s)\, d\lambda = \int_{T^{-s}E\cap B(i_1,\ldots,i_s)} d\lambda.$$

Then there is a constant $C \ge 1$ such that

$$\operatorname*{ess\,sup}_{x\in U_j} \omega(i_1, \ldots, i_s; x) \le C \operatorname*{ess\,inf}_{x\in U_j} \omega(i_1, \ldots, i_s; y)$$

('Rényi's condition').

LEMMA 1.

$$\operatorname*{ess\,inf}_{x\in U_j} \omega(k_1, \ldots, k_s; x) \le \frac{\lambda(B(k_1, \ldots, k_s))}{\lambda(U_j)} \le \operatorname*{ess\,sup}_{x\in U_j} \omega(k_1, \ldots, k_s; x).$$

Proof. This follows from $\lambda(B(k_1, \ldots, k_s)) = \int_{U_j} \omega(k_1, \ldots, k_s)\, d\lambda$. □

(D) Any set U_j contains a full cylinder.

It is well known that conditions (A)–(D) together imply that T is ergodic (with respect to λ) and admits an invariant probability measure μ such there is a version h of the density $d\mu/d\lambda$ which satisfies

$$\frac{1}{Q} \le h \le Q$$

for a constant $Q \ge 1$ (see, for example, [**17**]).

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Remark. Nakada [**12**] assumes that there is an $s \geq 1$ such that any set U_j contains a full cylinder of rank s. However, conditions (B) and (D) together imply this seemingly stronger condition. We proceed by induction. The case $N = 1$ is clear. Let $r \geq 1$ be a number such that there are full cylinders $B(a_1^{(j)}, \ldots, a_r^{(j)}) \subseteq U_j,\ 1 \leq j \leq N-1$. There is also a full cylinder $B(b_1, \ldots, b_m) \subseteq U_N$. Then the cylinders $B(a_1^{(j)}, \ldots, a_r^{(j)}, b_1, \ldots, b_m)$ and $B(b_1, \ldots, b_m, a_1,^{(j)}, \ldots, a_r^{(j)})$ are full cylinders of rank $s = r + m$. Clearly,

$$B(A_1^{(j)}, \ldots, a_r^{(j)}, b_1, \ldots, b_m) \subseteq U_j, \quad 1 \leq j \leq N-1$$
$$B(b_1, \ldots, b_m, a_1^{(j)}, \ldots, a_r^{(j)}) \subseteq U_N.$$

Let $\|\cdot\|$ denote the usual norm in the Euclidean space $\mathbb{R}^d$.

(E) There exist versions of $\omega(k_1, \ldots, k_s)$ which are Lipschitz continuous, i.e. there is a constant $R_1 \geq 0$ such that

$$|\omega(k_1, \ldots, k_s; x) - \omega(k_1, \ldots, k_s; y)| \leq R_1 \lambda(B(k_1, \ldots, k_s))\|x - y\|,$$

for $x, y \in U_j$.

(F) There is a constant $R_2 \geq 0$ such that

$$\|V(k_1, \ldots, k_s)x - V(k_1, \ldots, k_s)y\| \leq R_2\|x - y\|,$$

for $x, y \in U_j$.

Next we introduce the transfer operator A (also called the Frobenius–Perron operator). For $f \in L_1(\lambda)$, the equation

$$\int_{T^{-1}E} f\, d\lambda = \int_E (Af)\, d\lambda$$

defines a new function $Af \in L_1(\lambda)$. Then $Af = \sum_{k \in I}(f \circ V(k))\omega(k)$. The transfer operator A is defined for functions $f \in L_1(\lambda)$, but can be extended to non-negative measurable functions. A measurable function $f \geq 0$ satisfies the equation $Af = f$ iff it is a scalar multiple of the invariant density.

The transfer operator has the following properties (see [**9, 17**]):

(i) A is linear;

(ii) if $f \geq 0$ then $Af \geq 0$;

(iii) $\int_B f\, d\lambda = \int_B (Af)\, d\lambda$.

Let $\mathcal{L}$ be the class of real-valued functions defined on B which satisfy the conditions:

(α) there are constant $m_0 = m_0(f)$ and $M_0 = M_0(f)$ such that

$$0 < m_0 \leq f \leq M_0;$$

(β) there is a constant $N_0 = N_0(f)$ such that

$$|f(x) - f(y)| \leq N_0\|x - y\|$$

for all $x, y \in F$ where F is a cell of the partition $\mathcal{F}$.

LEMMA 2. *There are constants g and G, $0 < g \leq G$ such that*

$$g \cdot h \leq A^s f \leq G \cdot h$$

for all $s \geq 0$.

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Proof. Since $0 < Q^{-1} \le h \le Q$ and $0 < m_0 \le f \le Q$ there are constants g and G such that

$$g \cdot h \le f \le G \cdot h.$$

Applying the transfer operator to this inequality we obtain the result. □

For the remainder we introduce the following notation: $\sum_{(s)}$ is the summation over all blocks $(k_1, \dots, k_s)$ such that $B(k_1, \dots, k_s) \neq \emptyset$; $\sum_{(s;x)}$ is the summation over all blocks $(k_1, \dots, k_s)$ such that $x \in T^s B(k_1, \dots, k_s)$ (note that $\sum_{(s;x)} = \sum_{(s;y)}$ if x and y belong to the same cell $F \in \mathcal{F}$); and $\sum'_{(s)}$ is the summation over all blocks $(k_1, \dots, k_s)$ such that $T^s B(k_1, \dots, k_s) = B$.

LEMMA 3. *If x and y belong to the same cell $F \in \mathcal{F}$ then $V(a_1, \dots, a_m)x$ and $V(a_1, \dots, a_m)y$ belong to the same cell $G \in \mathcal{F}$, say.*

Proof. Let $V(a_1, \dots, a_m)x \in T^s B(k_1, \dots, k_s)$. Then $x \in T^{s+m} B(k_1, \dots, k_s, a_1, \dots, a_m)$. Therefore $y \in T^{s+m} B(k_1, \dots, k_s, a_1, \dots, a_m)$ which implies $V(a_1, \dots, a_m)y \in T^s B(k_1, \dots, k_s)$. By symmetry, we see that $V(a_1, \dots, a_m)x \in T^s B(k_1, \dots, k_s)$ iff $V(a_1, \dots, a_m)y \in T^s B(k_1, \dots, k_s)$. □

LEMMA 4. *Let $f \in \mathcal{L}$. There is a constant N_1 such that*

$$|(A^s f)(x) - (A^s f)(y)| \le N_1 \|x - y\|$$

for all $s \ge 0$ and $x, y \in F$.

Proof.

$$\begin{aligned}
&|(A^s f)(x) - (A^s f)(y)| \\
&\quad = \Bigg| \sum_{(s;x)} f(V(k_1, \dots, k_s)x)\omega(k_1, \dots, k_s; x) \\
&\qquad - \sum_{(s;y)} f(V(k_1, \dots, k_s)y)\omega(k_1, \dots, k_s; y) \Bigg| \\
&\quad \le \sum_{(s;x)} \omega(k_1, \dots, k_s; x) |f(V(k_1, \dots, k_s)x) - f(V(k_1, \dots, k_s)y)| \\
&\qquad + \sum_{(s;x)} f(V(k_1, \dots, k_s)y) |\omega(k_1, \dots, k_s; x) - \omega(k_1, \dots, k_s; y)| \\
&\quad \le (CL^{-1} N_0 R_2 + M_0 R_1) \|x - y\|.
\end{aligned}$$

Hence $N_1 := CL^{-1} N_0 R_2 + M_0 R_1$ is a suitable constant. □

The last condition is the following.

(G) Let $\mathcal{L}_m := \{(a_1, \dots, a_m) : B(a_1, \dots, a_m)$ is not contained in a cell $F \in \mathcal{F}\}$ and define $\gamma(m) := \sum_{\mathcal{L}_m} \lambda(B(a_1, \dots, a_m))$, then $\lim_{m \to \infty} \gamma(m) = 0$. Note that, if (B, T) satisfies a Markov condition, i.e. there is an $n = n(T) \ge 1$ such that $T^n B(i_1, \dots, i_n)$ is a union of cylinders of rank n, then clearly $\gamma(m) = 0$ if $m \ge n$.

THEOREM 1. *There is a version h of the invariant density $d\mu/d\lambda$ which is Lipschitz continuous on any cell $F \in \mathcal{F}$.*

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Proof. The sequence $(A^s f),\ s \in \mathbb{N}$, is uniformly bounded and satisfies a uniform Lipschitz condition. Therefore the same is true for the sequence $(t^{-1}\sum_{s=0}^{t-1} A^s f)$, $t \in \mathbb{N}$. Since B is compact the theorem of Arzelà–Ascoli implies the existence of a limit point h which satisfies $Ah = h$.

From Theorem 1 it follows that there is a version h of $d\mu/d\lambda$ which also satisfies

$$|h(x) - h(y)| \le N_1\|x - y\|;\ x, y \in F \in \mathcal{F}. \qquad \square$$

3. *Kuzmin's theorem*

THEOREM 2. *There is a constant α, $0 < \alpha < 1$, such that for any $f \in \mathcal{L}$ the convergence result*

$$A^s f = \left(\int_B f\, d\lambda\right)h + O(\alpha^{\sqrt{s}} + \sigma(\sqrt{s}) + \gamma(\sqrt{s}))$$

holds.

By Lemmas 2 and 4 we may assume that we have chosen $m_1 \le M_1$ such that for all $s \ge 0$:

($\alpha^\star$) $0 < m_1 \le A^s f \le M_1$;

($\beta^\star$) $|(A^s f)(x) - (A^s f)(y)| \le N_1\|x - y\|$ for all $x, y \in F$.

LEMMA 5. *Let $P_m := \bigcup_{T^m B(k_1,\dots,k_m)=B} B(k_1, \dots, k_m)$. Then there is a constant $q > 0$ such that*

$$\lambda(P_n) \ge q \quad \textit{for all } n \ge 1.$$

Proof. Let $B(a_1, \dots, a_k) \ne \emptyset$ and $T^k B(a_1, \dots, a_k) = U_j$; say. We first assume that s is the value mentioned in the remark to condition (D). Then $B(a_1, \dots, a_k, b_1, \dots, b_s)$ is a full cylinder iff $B(b_1, \dots, b_s)$ is a full cylinder which is contained in U_j.

Let $\sum^{(j)}$ denote the summation over all of the blocks $(b_1, \dots, b_s)$. Then

$$\sum_{(b_1,\dots,b_s)}^{(j)} \lambda(B(a_1, \dots, a_k, b_1, \dots, b_s)) \ge C^{-1}\lambda(B(a_1, \dots, a_k)) \sum_{(b_1,\dots,b_s)}^{(j)} \lambda(B(b_1, \dots, b_s)).$$

Put $q_1 := C^{-1} \min_{0 \le j \le N} \sum^{(j)} \lambda(B(b_1, \dots, b_s))$.

Then for $m \ge s + 1$

$$\begin{aligned}
&\sum_{(b_1,\dots,b_m)} \lambda(B(a_1, \dots, a_k, b_1, \dots, b_m)) \\
&\quad = \sum_{(b_1,\dots,b_{m-s})} \sum_{(b_{m-s+1},\dots,b_m)} \lambda(B(a_1, \dots, a_k, b_1, \dots, b_{m-s})) \\
&\qquad \times \frac{\lambda(B(a_1, \dots, a_k, b_1, \dots, b_m))}{\lambda(B(a_1, \dots, a_k, b_1, \dots, b_{m-s}))} \\
&\quad \ge q_1 \sum_{(b_1,\dots,b_{m-s})} \lambda(B(a_1, \dots, a_k, b_1, \dots, b_{m-s})) = q_1\lambda(B(a_1, \dots, a_k)).
\end{aligned}$$

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Here we used that

$$\sum_{(b_{m-s+1},\dots,b_m)}^{(j)} \lambda(B(a_1,\dots,a_k,b_1,\dots,b_{m-s},b_{m-s+1},\dots,b_m)$$
$$\geq q_1\lambda(B(a_1,\dots,a_k,b_1,\dots,b_{m-s}))$$

since again $T^{k+m}B(a_1,\dots,a_k,b_1,\dots,b_m) = U_j$ for some j.

Then

$$\sum_{(a_1,\dots,a_m)}{}' \lambda(B(a_1,\dots,a_m)) \geq q_1$$

provided $m \geq s+1$. However, there is a constant q_2 such that for $1 \leq m \leq s$ we find

$$\sum_{(a_1,\dots,a_m)}{}' \lambda(B(a_1,\dots,a_m)) \geq q_2.$$

Therefore $q = \min(q_1, q_2)$ is a suitable value. □

LEMMA 6. *Let $\varphi \geq 0$ satisfy $\varphi \leq M_2$ and the Lipschitz condition $|\varphi(u) - \varphi(v)| \leq N_2\|u-v\|$ for $u, v \in F$. Then*

$$A^s\varphi \geq \frac{1}{C}\Big(\int_{P_s} \varphi\, d\lambda - N_2\sigma(s) - M_2\gamma(s)\Big).$$

Proof. For suitable points $u^{(s)} \in B(k_1,\dots,k_s)$ we find

$$\int_B \varphi\, d\lambda = \sum_{(s)} \varphi(u^{(s)})\lambda(B(k_1,\dots,k_s)).$$

Then we calculate

$$\begin{aligned}(A^s\varphi)(x) &= \sum_{(s;x)} \varphi(V(k_1,\dots,k_s)x)\omega(k_1,\dots,k_s;x)\\ &\geq \frac{1}{C}\sum_{(s)}{}' \varphi(V(k_1,\dots,k_s)x)\lambda(B(k_1,\dots,k_s))\\ &\geq \frac{1}{C}\sum_{(s)}{}' (\varphi(V(k_1,\dots,k_s)x) - \varphi(u^{(s)})\lambda(B(k_1,\dots,k_s))\\ &\quad + \frac{1}{C}\sum_{(s)}{}' \varphi(u^{(s)})\lambda(B(k_1,\dots,k_s))\\ &\geq \frac{1}{C}\int_{P_s} \varphi\, d\lambda - \frac{N_2}{C}\sigma(s) - \frac{M_2}{C}\gamma(s).\end{aligned}$$ □

We apply Lemma 6 first to the sequence $\varphi_j(x) = (A^{js}f)(x) - g_j(s)h(x)$ where $g_1(s) := g \leq g_j(s) \leq \int_B f\, d\lambda$, $1 \leq j \leq s$, and $\varphi_j(x) \geq 0$ for $s \geq s(f)$. The conditions of Lemma 6 are satisfied with $M_2 = M_1 + Q$ and $N_2 = \big(1 + \int_B f\, d\lambda\big)N_1$. Then

$$\begin{aligned}A^s\varphi_j &\geq \frac{1}{C}\Big(\int_{P_s} \varphi_j\, d\lambda - N_2\sigma(s) - M_2\gamma(s)\Big)\\ &\geq h\Big(\frac{1}{CQ}\int_{P_s} \varphi_j\, d\lambda - \frac{QN_2}{C}\sigma(s) - \frac{QM_2}{C}\gamma(s)\Big).\end{aligned}$$

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Therefore,

$$(A^{(j+1)s}f)(x) \geq g_{j+1}(s)h(x)$$

where

$$\begin{aligned} g_{j+1}(s) &= g_j(s)\alpha(s) + \beta(s,j) \\ \alpha(s) &= 1 - \frac{1}{CQ}\int_{P_s} h\,d\lambda \\ \beta(s,j) &= \frac{1}{CQ}\int_{P_s}(A^{js}f)\,d\lambda - \frac{QN_2}{C}\sigma(s) - \frac{QM_2}{C}\gamma(s). \end{aligned}$$

Since $\int_{P_s}(A^{js}f)\,d\lambda \geq qm_1$ we choose $s \geq s(f)$ such that $\beta(s,j) \geq 0$. Furthermore, we observe that

$$\alpha(s) \leq \alpha := 1 - \frac{q}{CQ^2} < 1.$$

Next we apply Lemma 6 to the sequence

$$\Phi_j(x) = G_j(s)h(x) - (A^{js}f)(x)$$

where $\int_B f\,d\lambda \leq G_j(s) \leq G =: G_1(s)$, $1 \leq j \leq s$, and $\Phi_j(x) \geq 0$ for $s \geq s(f)$. Again, the conditions of Lemma 6 are satisfied with $M_3 = M_1 + GQ$ and $N_3 = (G+1)N_1$. Then

$$\begin{aligned} A^s\Phi_j &\geq \frac{1}{C}\left(\int_{P_s}\Phi_j\,d\lambda - N_3\sigma(s) - M_3\gamma(s)\right) \\ &\geq h\left(\frac{1}{CQ}\int_{P_s}\Phi_j\,d\lambda - \frac{QN_3}{C}\sigma(s) - \frac{QM_3}{C}\gamma(s)\right). \end{aligned}$$

Therefore

$$(A^{(j+1)s}f)(x) \leq G_{j+1}(s)h(x)$$

where

$$\begin{aligned} G_{j+1}(s) &= G_j(s)\alpha(s) + \delta(s,j) \\ \alpha(s) &= 1 - \frac{1}{CQ}\int_{P_s} h\,d\lambda \\ \delta(s,j) &= \frac{1}{CQ}\int_{P_s}(A^{js}f)\,d\lambda + \frac{QN_3}{C}\sigma(s) + \frac{QM_3}{C}\gamma(s). \end{aligned}$$

Now

$$G_{j+1}(s) - g_{j+1}(s) = \alpha(s)(G_j(s) - g_j(s)) + \frac{Q(N_3+N_2)}{C}\sigma(s) + \frac{Q(M_3+M_2)}{C}\gamma(s).$$

Therefore

$$G_{j+1}(s) - g_{j+1}(s) \leq \alpha^j(G_1 - g_1) + K(\sigma(s) + \gamma(s))$$

for a suitable constant K.

If we introduce the 'diagonal sequence'

$$\gamma_s = g_s(s), \quad \Gamma_s = G_s(s)$$

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then we find

$$\gamma_s h \le A^{s^2} f \le \Gamma_s h$$
$$\Gamma_s - \gamma_s = O(\alpha^s + \sigma(s) + \gamma(s)).$$

Since $\gamma_s \le \int_B f\,d\lambda \le \Gamma_s$ we see that

$$\lim_{s\to\infty} \gamma_s = \lim_{s\to\infty} \Gamma_s = \int_B f\,d\lambda.$$

Therefore

$$A^{s^2} f - \left(\int_B f\,d\lambda\right) h = O(\alpha^s + \gamma(s) + \sigma(s)).$$

If we replace s^2 by s and observe property (ii) of the transfer operator A we obtain

$$A^s f = \left(\int_B f\,d\lambda\right) h + O(\alpha^{\sqrt{s}} + \gamma(\sqrt{s}) + \sigma(\sqrt{s})).$$

Clearly $\gamma(\sqrt{s})$ (respectively $\sigma(\sqrt{s})$) has to be interpreted as $\gamma([\sqrt{s}])$ (respectively $\sigma([\sqrt{s}])$) where $[\Theta]$ denotes the integral part of Θ. □

4. *Final remarks*

For several multi-dimensional maps the exponential rate of decay has been proved by different methods (see, for example, Gordin [**4**], Manu-Iosifescu [**10**], Mayer [**11**], Broise [**2**]; for systems satisfying the 'Gibbs property' see Aaronson [**1**]). Therefore I am convinced that an estimate of order $0(\sigma(n)+\alpha^n+\gamma(n))$ should be true. Since several multi-dimensional continued fractions satisfy a Markov condition and $\sigma(n) = 0(\gamma^n)$, $0 < \gamma < 1$, exponential decay is to be expected. A possible approach could be to show that the transfer operator is quasi-compact (see Keller [**7**]).

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