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TWO NEW TRUNCATED IDENTITIES AND PARTITION THEOREMS

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Abstract

We establish two new truncated identities for theta series. Their partition-theoretic interpretations are also given.

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1. Introduction

Euler's pentagonal number theorem plays an important role in the theory of partitions. It gives

$$(q;q)_\infty = \sum_{j=0}^{\infty} (-1)^j q^{j(3j+1)/2}(1-q^{2j+1}). \tag{1.1}$$

In (1.1) and for the rest of this article, we use the notation

$$(x_1, x_2, \ldots, x_k; q)_m := \prod_{n=0}^{m-1} (1-x_1q^n)(1-x_2q^n)\cdots(1-x_kq^n),$$

$$(x_1, x_2, \ldots, x_k; q)_\infty := \prod_{n=0}^{\infty} (1-x_1q^n)(1-x_2q^n)\cdots(1-x_kq^n),$$

$$\begin{bmatrix} L \\ K \end{bmatrix}_q := \begin{cases} 0 & \text{if } K<0 \text{ or } K>L, \\ \dfrac{(q;q)_L}{(q;q)_K(q;q)_{L-K}} & \text{otherwise,} \end{cases}$$

and we require $|q|<1$ for absolute convergence.

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Andrews and Merca [2] proved the truncated pentagonal number theorem

$$\frac{1}{(q)_\infty}\sum_{j=0}^{k-1}(-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{(k+1)n+k(k-1)/2}}{(q;q)_n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{1.2}$$

Using (1.2), they proved a new partition theorem. Recall that a partition of an integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n.

THEOREM 1.1 [2, Theorem 1.1]. *For $n > 0, k \geq 1$, let*

$$M_k(n) := (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j(p(n-j(3j+1)/2) - p(n-j(3j+5)/2-1)).$$

Then $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $> k$ than there are $< k$.

The first result of this paper is the following new partition theorem, which is an analogue of Theorem 1.1.

THEOREM 1.2. *For $n, k \geq 0$, let*

$$N_k(n) := (-1)^k\Big\{p(n) + \sum_{j=1}^{k}(-1)^j(p(n-j(3j-1)/2) + p(n-j(3j+1)/2))\Big\}.$$

Then $N_k(n)$ is the number of partitions of n with every integer $\leq k$ appearing as a part, and in which the first part larger than k appears at least $k+1$ times.

For example, $N_2(16) = 6$ and the partitions in question are $4+4+4+2+1+1$, $4+3+3+3+2+1, 3+3+3+3+2+1+1, 3+3+3+2+2+2+1$, $3+3+3+2+2+1+1+1, 3+3+3+2+1+1+1+1+1$.

To prove Theorem 1.2, we need a new truncated pentagonal number theorem. Note that the pentagonal number theorem is also given by

$$(q;q)_\infty = 1+\sum_{j=1}^{\infty}(-1)^j q^{j(3j-1)/2}(1+q^j). \tag{1.3}$$

Then we have the truncated version of (1.3).

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THEOREM 1.3. *For $k \geq 0$,*

$$\frac{1}{(q;q)_\infty}\Big\{1+\sum_{j=1}^{k}(-1)^j q^{j(3j-1)/2}(1+q^j)\Big\} = 1+(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(j+k+1)(k+1)}}{(q^{j+k+1};q)_\infty}. \tag{1.4}$$

We remark that letting $k \to \infty$ in (1.4) gives (1.3). Andrews and Merca's work inspired a wide study on truncated identities and their partition-theoretic interpretations (see, for example, [4, 6–10, 12–17]). In particular, Guo and Zeng [5] obtained truncated forms of two identities of Gauss (see [1, page 23]): that is,

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2}, \tag{1.5}$$

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{j=0}^{\infty}(-q)^{j(j+1)/2}. \tag{1.6}$$

They found that

$$\frac{(-q)_\infty}{(q)_\infty}\Big(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\Big) = 1+(-1)^k\sum_{n=k+1}^{\infty}\frac{q^{(k+1)n}(-1;q)_{n-k}(-q;q)_k}{(q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q, \tag{1.7}$$

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=1}^{k}(-1)^j q^{2j^2-j}(1+q^{2j}) = 1+(-1)^{k-1}\sum_{n=k}^{\infty}\frac{q^{2(k+1)n-k}(-q;q^2)_k(-q;q^2)_{n-k}}{(q^2;q^2)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_{q^2}. \tag{1.8}$$

Partition-theoretic interpretations of the two truncated sums in (1.7) and (1.8) are obtained in [3]. Rewrite (1.5) and (1.6) as

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{j=0}^{\infty}(-1)^j q^{j^2}(1-q^{2j+1}), \tag{1.9}$$

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = 1+\sum_{j=1}^{\infty}(-1)^j q^{2j^2-j}(1+q^{2j}). \tag{1.10}$$

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Wang and Yee [13] obtained the following truncated version of (1.9): for $k \geq 0$,

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{j=0}^{k}(-1)^j q^{j^2}(1-q^{2j+1}) = 1+(-1)^k q^{-k-1}\sum_{n=k+1}^{\infty}\frac{q^{(k+2)n}(-1;q)_{n-k-1}(-q;q)_{k+1}}{(q;q)_n}\begin{bmatrix} n-1 \\ m \end{bmatrix}_q.$$

We give a truncated version of (1.10).

THEOREM 1.4. *For $k \geq 0$,*

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{1+\sum_{j=1}^{k}(-1)^j q^{2j^2-j}(1+q^{2j})\right\} = 1+(-1)^k\frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty}. \tag{1.11}$$

Letting $k \to \infty$ in (1.11), we obtain (1.10). Denote the number of partitions of n without repeated odd parts by $pod(n)$. Then we give a partition-theoretic interpretation of the truncated sum in (1.11).

COROLLARY 1.5. *For $n > 0, k \geq 1$, let*

$$NP_k(n) := (-1)^k\left\{pod(n)+\sum_{j=1}^{k}(-1)^j(pod(n-j(2j-1))+pod(n-j(2j+1)))\right\}.$$

Then $NP_k(n)$ is the number of partitions of n in which $2k+1$ appears $k+1$ or $k+2$ times and all other odd parts appear at most once, or $2k+2$ does not appear, and the first part larger than $2k+2$ is odd and appears exactly $k+1$ times, and all other odd parts appear at most once.

For example, $NP_1(13) = 9$ and the partitions in question are $7+3+3, 6+3+3+1, 5+3+3+2, 4+3+3+3, 4+3+3+2+1, 3+3+3+2+2, 3+3+2+2+2+1, 5+5+3, 5+5+2+1$. We remark that Corollary 1.5 does not hold when $k = 0$. For example, $NP_0(4) = 3$, but there are only two partitions in question, namely, $3+1, 2+1+1$.

The proofs of truncated identities (1.4) and (1.11) are inspired by [3]. The Rogers–Fine identity (see (2.2)) plays a key role. The paper is organised as follows. In Section 2, we prove Theorems 1.2 and 1.3. Theorem 1.4 and Corollary 1.5 are obtained in Section 3.

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2. Ordinary partitions

LEMMA 2.1. *For* $|\alpha| < 1$,

$$\sum_{n=0}^{\infty}(\alpha;q)_n\alpha^n = \sum_{n=0}^{\infty}(-1)^n\alpha^{3n}q^{n(3n-1)/2}(1+\alpha q^n). \tag{2.1}$$

PROOF. Recall the Rogers–Fine identity [11, page 15] (see also [3, Equation (1.10)])

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n\tau^n}{(\beta;q)_n} = \sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}. \tag{2.2}$$

Setting $\tau = \alpha$ and letting $\beta \to 0$ in (2.2) yields (2.1). □

PROOF OF THEOREM 1.3. Equation (1.3) implies that

$$(q;q)_\infty - \Big\{1 + \sum_{j=1}^{k}(-1)^j q^{j(3j-1)/2}(1+q^j)\Big\} = \sum_{j=k+1}^{\infty}(-1)^j q^{j(3j-1)/2}(1+q^j)$$
$$= (-1)^{k+1}q^{(k+1)(3k+2)/2}\sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2+(3k+3)n}(1+q^{n+k+1}).$$

Replacing α by q^{k+1} in (2.1) gives

$$\sum_{j=0}^{\infty}(q^{k+1};q)_j q^{(k+1)j} = \sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2+3n(k+1)}(1+q^{n+k+1}).$$

It follows that

$$(q;q)_\infty - \Big\{1 + \sum_{j=1}^{k}(-1)^j q^{j(3j-1)/2}(1+q^j)\Big\} = (-1)^{k+1}q^{(k+1)(3k+2)2}\sum_{j=0}^{\infty}(q^{k+1};q)_j q^{(k+1)j}$$
$$= (-1)^{k+1}q^{(k+1)k/2}\sum_{j=0}^{\infty}(q^{k+1};q)_j q^{(k+1)(j+k+1)}.$$

Thus,

$$1 - \frac{1}{(q;q)_\infty}\Big\{1 + \sum_{j=1}^{k}(-1)^j q^{j(3j-1)/2}(1+q^j)\Big\} = (-1)^{k+1}\frac{q^{k(k+1)/2}}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(j+k+1)(k+1)}}{(q^{j+k+1};q)_\infty}.$$

This proves Theorem 1.3. □

PROOF OF THEOREM 1.2. Theorem 1.3 gives

$$\sum_{n=1}^{\infty}N_k(n)q^n = \frac{q^{k(k+1)/2}}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(j+k+1)(k+1)}}{(q^{j+k+1};q)_\infty}.$$

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The term

$$\frac{q^{k(k+1)/2}}{(q;q)_k}$$

is the generating function for partitions with parts $\leq k$ and each integer $\leq k$ appearing as a part. In addition, $q^{(j+k+1)(k+1)}$ produces the $k+1$ appearances of the first part, $j+k+1$, larger than k, and $1/(q^{j+k+1};q)_\infty$ generates partitions with parts $\geq j+k+1$. Thus, $N_k(n)$ enumerates the partitions of n in question. □

3. Partitions without repeated odd parts

LEMMA 3.1. *For* $|\alpha|<1$,

$$\sum_{n=0}^{\infty}\frac{(\alpha;q^2)_n\alpha^n}{(\alpha q;q^2)_n}=\sum_{n=0}^{\infty}\alpha^{2n}q^{2n^2-n}(1+\alpha q^{2n}). \tag{3.1}$$

PROOF. Replacing q by q^2 and setting $\beta=\alpha q,\tau=\alpha$ in (2.2) yields (3.1). □

PROOF OF THEOREM 1.4. By (1.10),

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}-\left\{1+\sum_{j=1}^{k}(-1)^jq^{2j^2-j}(1+q^{2j})\right\}=\sum_{j=k+1}^{\infty}(-1)^jq^{2j^2-j}(1+q^{2j})$$
$$=(-1)^{k+1}q^{(2k+1)(k+1)}\sum_{n=0}^{\infty}(-1)^nq^{2n^2+(4k+3)n}(1+q^{2n+2k+2}).$$

Replacing q by $-q$ and setting $\alpha=q^{2k+2}$ in (3.1) gives

$$\sum_{j=0}^{\infty}\frac{(q^{2k+2};q^2)_jq^{(2k+2)j}}{(-q^{2k+3};q^2)_j}=\sum_{n=0}^{\infty}(-1)^nq^{2n^2+(4k+3)n}(1+q^{2n+2k+2}).$$

Then

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}-\left\{1+\sum_{j=1}^{k}(-1)^jq^{2j^2-j}(1+q^{2j})\right\}=(-1)^{k+1}\sum_{j=0}^{\infty}\frac{(q^{2k+2};q^2)_jq^{(2j+2k+1)(k+1)}}{(-q^{2k+3};q^2)_j}.$$

It follows that

$$1-\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{1+\sum_{j=1}^{k}(-1)^jq^{2j^2-j}(1+q^{2j})\right\}$$
$$=(-1)^{k+1}\frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty}.$$

This proves Theorem 1.4. □

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PROOF OF COROLLARY 1.5. By Theorem 1.4,

$$\sum_{n=1}^{\infty} NP_k(n)q^n = \frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty}$$
$$= \frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k} \sum_{j=1}^{\infty} \frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty} + \frac{q^{(2k+1)(k+1)}(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

(i) The term

$$\frac{q^{(2k+1)(k+1)}(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

is the generating function for partitions in which $2k+1$ appears $k+1$ or $k+2$ times and all other odd parts appear at most once.

(ii) In

$$\frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k} \sum_{j=1}^{\infty} \frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty},$$

the expression preceding the summation is the generating function for partitions with parts $\leq 2k+1$ and odd parts distinct, the expression inside the sum is the generating function for partitions in which the first part, $2j+2k+1$, appears exactly $k+1$ times, and all other odd parts appear at most once. Thus,

$$\frac{(-q;q^2)_{k+1}}{(q^2;q^2)_k} \sum_{j=1}^{\infty} \frac{q^{(k+1)(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2j+2k+2};q^2)_\infty}$$

is the generating function of partitions in which $2k+2$ does not appear, and the first part larger than $2k+2$ is odd and appears exactly $k+1$ times, and all other odd parts appear at most once.

Moreover, the sets of partitions of n described in (i) and (ii) are disjoint when $k \geq 1$. Thus, $NP_k(n)$ enumerates the partitions of n in question. □

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