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Ergodic boundary representations

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Abstract. We prove a von Neumann-type ergodic theorem for averages of unitary operators arising from the Furstenberg–Poisson boundary representation (the quasi-regular representation) of any lattice in a non-compact connected semisimple Lie group with finite center.

1. *Introduction*

Let Γ be an infinite group and let π be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. What can be learned from the asymptotic behavior of weighted averages of the type

$$\sum_{\gamma\in\Gamma} a_\gamma \pi(\gamma)$$

with $a_\gamma \in \mathbb{C}$ carefully chosen? This question has been intensively studied for representations associated with measurable actions. The weakest form of the von Neumann ergodic theorem (convergence in the weak operator topology) is one of the founding results of this line of thought. Recall that a sequence A_n in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ converges to $A \in \mathcal{B}(\mathcal{H})$ with respect to the weak operator topology (WOT) if and only if for any $v, w \in \mathcal{H}$

$$\lim_{n\to\infty} \langle A_n v, w\rangle = \langle Av, w\rangle.$$

1.1. *Unitary representations and measurable transformations.*

Suppose Γ acts by measure preserving transformations on a probability space (B, ν). Let π be the associated canonical unitary representation on the Hilbert space $L^2(B, \nu)$. That is:

$$\pi(\gamma)\varphi(b) = \varphi(\gamma^{-1}b) \quad \text{for all } b \in B \text{ for all } \varphi \in L^2(B, \nu) \text{ for all } \gamma \in \Gamma.$$

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Let $\mathbb{1}_B$ be the characteristic function of the whole space B and let $P_{\mathbb{1}_B} \in \mathcal{B}(L^2(B, \nu))$ denote the orthogonal projection onto the complex line generated by $\mathbb{1}_B$. *The existence of a sequence* $\mu_n \in L^1(\Gamma)$ *such that*

$$\lim_{n\to\infty} \pi(\mu_n) = \lim_{n\to\infty} \sum_{\gamma} \mu_n(\gamma)\pi(\gamma) = P_{\mathbb{1}_B}$$

in the WOT implies the ergodicity of the action.

If Γ *is amenable, the converse implication also holds with* μ_n *the uniform probability measures defined by any Følner sequence* F_n*: if the action is ergodic, then in the WOT*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{\gamma\in F_n} \pi(\gamma) = P_{\mathbb{1}_B}.$$

This is a straightforward consequence of the L^2-mean ergodic theorem for amenable groups. More generally, *for any locally compact second countable amenable group* G*, a measure preserving action on a probability space* (B, ν) *is ergodic if and only if, for any Følner sequence* F_n*,*

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(F_n)} \int_{F_n} \pi(g)\, dg = P_{\mathbb{1}_B}$$

in the WOT, where dg *denotes a Haar measure on* G *and* $\mathrm{vol}(F_n)$ *denotes the corresponding Haar volume of* F_n, see [**33**, Ch. IV] and references therein. See also [**26**, §5.1, p. 894] for a short proof of the L^2-mean ergodic theorem which is a variant of Riesz's proof of von Neumann's ergodic theorem. In fact *the ergodicity of the action implies almost everywhere (a.e.) point-wise convergence of Birkhoff sums provided the Følner sequence is carefully chosen* [**25**].

The paper [**2**] about the equidistribution of the orbit of a point on the sphere under the action of a free group generated by two rotations is an early result about ergodic properties of measure preserving actions of non-amenable groups. Let us mention results for measure preserving actions of families of non-amenable groups. *The ergodicity of the action implies the convergence in the WOT of* $\pi(\mu_n)$ *to* $P_{\mathbb{1}_B}$ in the following cases: the group Γ *is Gromov-hyperbolic and* μ_n *is the family of Cesaro means with respect to concentric balls or spherical shells defined by any word metric associated with a finite symmetric generating set of* Γ (see [**21**] for the special case of the free groups), the group Γ *is a lattice in a connected semisimple Lie group with finite center and* μ_n *is the uniform measure on the intersection of* Γ *with a bi-*K*-invariant lift in* G *of a ball of radius* n *in the symmetric space* G/K. See [**1**, **6**, **19**, **26**] and references therein for many other and stronger convergence results (L^p-convergence, point-wise a.e. convergence, equidistribution, rates of convergence, etc.).

When the acting group Γ is non-amenable there may be no invariant measures but only quasi-invariant ones. In this case the associated quasi-regular representation is made unitary with the help of the Radon–Nikodym cocycle:

$$\pi(\gamma)\varphi(b) = \varphi(\gamma^{-1}b)\sqrt{\frac{d\gamma_*\nu}{d\nu}(b)} \quad \text{for all } b \in B \text{ for all } \varphi \in L^2(B, \nu).$$

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Due to the possible rapid decay of the Radon–Nikodym cocycle as γ goes to infinity, if $E_n \subset \Gamma$ and $|E_n| \to \infty$, *the averages*

$$\frac{1}{|E_n|} \sum_{\gamma \in E_n} \pi(\gamma)$$

may converge in the WOT to the zero operator. (For example, *this is the case if* Γ *is any uniform lattice in* $\mathrm{SL}(2, \mathbb{R})$*, respectively in a non-compact connected semisimple Lie group* G *with finite center, and* π *is the quasi-regular representation of* Γ *on the circle at infinity of* $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$*, respectively on the Furstenberg–Poisson boundary of* G*, and* E_n *is the ball in* Γ *of radius* n *defined by a word metric associated with any finite symmetric set of* Γ.) Therefore it makes sense to normalize each unitary operator $\pi(\gamma)$ by the mean (the Harish–Chandra function associated with π evaluated on γ)

$$\Xi(\gamma) = \langle \pi(\gamma)\mathbb{1}_B, \mathbb{1}_B \rangle = \int_B \sqrt{\frac{d\gamma_* \nu}{d\nu}(b)}\, d\nu(b)$$

and rather to consider the averages

$$\frac{1}{|E_n|} \sum_{\gamma \in E_n} \frac{\pi(\gamma)}{\langle \pi(\gamma)\mathbb{1}_B, \mathbb{1}_B \rangle}.$$

In this paper, we define various types of subsets E_n by geometric conditions with the help of *length functions*.

Definition 1.1. Let G be a locally compact group and let $e \in G$ denote its identity element. A length function L on G is a Borel map $L : G \to [0, \infty[$ such that:

(1) $L(e) = 0$;

(2) for all $g, h \in G$, $L(gh) \leq L(g) + L(h)$;

(3) for all $g \in G$, $L(g) = L(g^{-1})$.

We only consider length functions such that the sets

$$L^{-1}([0, r]) = \{g \in G : L(g) \leq r\}$$

are relatively compact for any $r \geq 0$.

1.2. *Bader–Muchnik's theorem on negatively curved manifolds and generalizations.* Let M be a closed Riemannian manifold with strictly negative sectional curvature and let (X, d_X) be the universal cover of M endowed with the unique Riemannian metric locally isometric to M. Fix a base point $x_0 \in X$. For any $x \in X$ with $x \neq x_0$ there is a unique geodesic ray $c_x : [0, \infty[\to X$ such that $c_x(0) = x_0$ and $c_x(d_X(x_0, x)) = x$. Let $B = \partial X$ be the visual boundary of X endowed with the Patterson–Sullivan measure ν corresponding to $x_0 \in X$. Up to normalization, ν is the Hausdorff measure defined by the Bourdon metric d_{x_0} on B (see [**5**]) and Γ acts by conformal transformations on B with Radon–Nikodym cocycle

$$\frac{d\gamma_* \nu(b)}{d\nu(b)} = e^{-\delta B_b(\gamma x_0, x_0)};$$

here δ is the Hausdorff dimension of (B, d_{x_0}) and $B_b(\gamma x_0, x_0)$ is the Busemann function on X defined by the point $b \in B$ (see [**28**]). As $\Gamma = \pi_1(M)$ acts (from the left) freely on X,

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each $\gamma \in \Gamma\backslash\{e\}$ defines a unique geodesic ray $c_{\gamma x_0}$ emanating from x_0 and passing through γx_0. Let $c_{\gamma x_0}(\infty) \in B$ be the point it defines in the visual boundary B and consider the boundary map

$$\mathbf{b} : \Gamma\backslash\{e\} \to B$$

defined by $\mathbf{b}(\gamma) = c_{\gamma x_0}(\infty)$. The map $\mathbf{b}$ allows one to associate to any function f defined on the boundary B and any finite subset E of $\Gamma\backslash\{e\}$ the weighted average

$$\frac{1}{|E|} \sum_{\gamma \in E} f(\mathbf{b}(\gamma)) \frac{\pi(\gamma)}{\langle \pi(\gamma)\mathbb{1}_B, \mathbb{1}_B \rangle}.$$

If $f \in L^\infty(B, \nu)$ we denote $m(f) \in \mathcal{B}(L^2(B, \nu))$ the multiplication operator defined by f, that is: $m(f)\phi = f\phi$ for all $\phi \in L^2(B, \nu)$. In a remarkable paper [**3**], Bader and Muchnik prove that *if f is continuous then in the WOT*

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{\gamma \in B_n} f(\mathbf{b}(\gamma)) \frac{\pi(\gamma)}{\langle \pi(\gamma)\mathbb{1}_B, \mathbb{1}_B \rangle} = m(f) P_{\mathbb{1}_B},$$

where B_n is the ball of radius n in Γ defined by the length function

$$L(\gamma) = d_X(x_0, \gamma x_0)$$

with center $e \in \Gamma$ removed (the statement in [**3**] involves annuli rather than balls but this is equivalent; see Proposition 2.11 below).

The starting point of the present work was the guess—hinted by the authors in their paper: 'we will resist the temptation of stating things in a greater generality than needed'—that Bader–Muchnik's theorem and its consequences hold in a much more general setting than the one of closed Riemannian manifolds with strictly negative curvature. One of the authors of this paper (Boyer) generalized Bader–Muchnik's theorem to discrete isometry groups of proper CAT(-1) spaces, having non-arithmetic length spectrum, finite Bowen–Margulis–Sullivan measure and δ-Ahlfors regular Patterson–Sullivan conformal density of dimension δ (see [**7**]). This covers convex-cocompact groups with non-arithmetic spectrum as well as finite volume locally symmetric spaces of rank one, but does not cover complete finite volume Riemannian manifolds with pinched negative curvature (see the interesting examples from [**14**]). Recall that the length spectrum is non-arithmetic if by definition the subgroup of the real line generated by the lengths of the closed geodesics of the quotient space is not cyclic, and that a compact locally CAT(-1) space has an arithmetic length spectrum if and only if it is a finite graph with edge lengths rationally dependent [**27**, Theorem 4]. In the case that the CAT(-1) space is the Cayley graph (with all edges of length 1) of the free group on n letters, all closed geodesics in the wedge of n circles of length 1 have integral length, hence the spectrum is obviously arithmetic and the general theory does not apply. Nevertheless a direct counting argument [**9**] allows one to prove Bader–Muchnik's theorem in this case as well. Following the same lines of ideas as Bader and Muchnik, L. Garncarek was able to prove that the boundary representation associated with a Patterson–Sullivan measure of a Gromov hyperbolic group is irreducible [**17**]. In the same vein, the irreducibility of some boundary representations associated with Gibbs measures has been established in [**10**].

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In the unpublished note 'A brief presentation of property RD' 2006, E. Breuillard speculates that a special case of Bader–Muchnik's theorem should also hold true for lattices in semisimple Lie groups of higher rank and that the proof should easily follow from the work of Gorodnik and Oh [**20**]. More precisely, Breuillard asks if the quasi-regular representation $\lambda_{G/P}$ of a connected non-compact semisimple Lie group G with finite center on its Furstenberg–Poisson boundary $(G/P, \nu)$ satisfies for any lattice $\Gamma \subset G$

$$\lim_{T\to\infty} \frac{1}{|S_r(T)|} \sum_{\gamma\in S_r(T)} \frac{\langle \lambda_{G/P}(\gamma)\phi, \psi\rangle}{\langle \lambda_{G/P}(\gamma)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle} = \int_B \phi(b)\, d\nu(b) \int_B \overline{\psi(b)}\, d\nu(b)$$

for all (positive) $\phi, \psi \in L^2(G/P, \nu)$, where

$$S_r(T) = \{\gamma \in \Gamma : T - r \le L(\gamma) \le T + r\}$$

is the intersection with Γ of the annulus in G around $e \in G$ of radius T and thickness r with respect to a length function L on G.

1.3. *Statement of the main result.* Let G be a connected semisimple Lie group with finite center and let $\mathfrak{g}$ be its Lie algebra. Let K be a maximal compact subgroup of G and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form B. Among the abelian sub-algebras of $\mathfrak{g}$ contained in the subspace $\mathfrak{p}$, let $\mathfrak{a}$ be a maximal one. We assume $\dim \mathfrak{a} > 0$, i.e. the real rank of G is strictly positive (or equivalently that G is not compact). Let Σ be the root system associated with $(\mathfrak{g}, \mathfrak{a})$. Let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \alpha(H)X \ \forall H \in \mathfrak{a}\}$$

be the root space of $\alpha \in \Sigma$. Let

$$\mathfrak{a}^{\mathrm{sing}} = \bigcup_{\alpha\in\Sigma} \ker(\alpha)$$

be the union of the hyperplanes of $\mathfrak{a}$ defined as the kernels of the roots of Σ. Let us choose a positive open Weyl chamber $\mathfrak{a}^+$ (i.e. a connected component of $\mathfrak{a}\backslash\mathfrak{a}^{\mathrm{sing}}$). Let $\Sigma^+ \subset \Sigma$ be the set of positive roots ($\alpha \in \Sigma$ is positive if and only if $\alpha(H) > 0$ for all $H \in \mathfrak{a}^+$), and $\mathfrak{n}$ the nilpotent Lie algebra defined as the direct sum of root spaces of positive roots:

$$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha.$$

Let $A = \exp(\mathfrak{a})$, $A^+ = \exp(\mathfrak{a}^+)$ and $N = \exp(\mathfrak{n})$. Let $G = KAN$ be the Iwasawa decomposition defined by $\mathfrak{a}^+$. Let $Z(A)$ be the centralizer of A in G and $M = Z(A) \cap K$. The group M normalizes N. Let $P = MAN$ be the minimal parabolic subgroup of G associated with $\mathfrak{a}^+$. Let ν be the unique Borel regular K-invariant probability measure on the Furstenberg–Poisson boundary G/P. Let

$$\lambda_{G/P} : G \to \mathcal{U}(L^2(G/P, \nu))$$

be the associated quasi-regular representation and let

$$\Xi(g) = \langle \lambda_{G/P}(g)\mathbb{1}_{G/P}, \mathbb{1}_{G/P}\rangle$$

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be the Harish–Chandra function (see §§2.1 and 3.4 below for definitions and references). The subset $KA^+K \subset G$ is open and dense, and the map

$$\mathbf{b} : KA^+K \to G/P$$

defined as

$$\mathbf{b}(k_1 a k_2) = k_1 P$$

is continuous (to check that $\mathbf{b}$ is well defined see [**24**, Theorem 5.20 and its proof] or [**22**, Theorem 1.1 and Corollary 1.2 of Ch. IX and their proofs]). Let d_G be the left-invariant Riemannian metric on G whose scalar product on the tangent space of G at the identity e

$$\langle Z, Y\rangle = -B(Z, \Theta(Y)) \quad \text{for all } Z, Y \in \mathfrak{g}$$

is defined by the Killing form B and the Cartan involution Θ associated with the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

It is convenient to consider also normalizations of the scalar product $\langle \cdot, \cdot \rangle$ and d_G. Once a normalization is chosen, let d_X be the unique Riemannian metric on the symmetric space $X = G/K$ such that the canonical projection $G \to G/K$ becomes a G-equivariant Riemannian submersion. Let $x_0 \in X$ be the image of the identity element of G under the canonical projection. Notice that if $H \in \mathfrak{a}$, then

$$d_X(\exp(H)x_0, x_0) = \|H\|.$$

Let $T > 0$. We define

$$\mathfrak{a}_T = \{H \in \mathfrak{a} : \|H\| < T\}, \quad \mathfrak{a}_T^+ = \mathfrak{a}^+ \cap \mathfrak{a}_T.$$

We denote the images in G under the exponential map as

$$A_T = \exp(\mathfrak{a}_T), \quad A_T^+ = \exp(\mathfrak{a}_T^+).$$

Eventually, we define

$$B_T = KA_T^+K.$$

If $\psi \in L^\infty(G/P, \nu)$, we denote $m(\psi) \in \mathcal{B}(L^2(G/P, \nu))$ the corresponding multiplication operator. Let $P_{\mathbb{1}_{G/P}} \in \mathcal{B}(L^2(G/P, \nu))$ be the orthogonal projection onto the complex line spanned by $\mathbb{1}_{G/P}$. Let Γ be a discrete subgroup of G,

$$\Gamma_T = B_T \cap \Gamma$$

and let $|\Gamma_T|$ denote the cardinality of this finite set. Let $f : G/P \to \mathbb{C}$ be a continuous function. In the case Γ_T is non-empty we may consider the bounded operator

$$M_{\Gamma_T}^f = \frac{1}{|\Gamma_T|} \sum_{\gamma \in \Gamma_T} f(\mathbf{b}(\gamma)) \frac{\lambda_{G/P}(\gamma)}{\Xi(\gamma)}.$$

THEOREM 1.2. (Ergodicity of the quasi-regular representation of a lattice in a semisimple Lie group) *Let G be a non-compact connected semisimple Lie group with finite center. Let P be a minimal parabolic subgroup of G and Γ a lattice in G. Let f be a continuous function on G/P. With the notation as above we have*

$$\lim_{T\to\infty} M_{\Gamma_T}^f = m(f) P_{\mathbb{1}_{G/P}}$$

in the weak operator topology of $\mathcal{B}(L^2(G/P, \nu))$. That is, for any $\varphi, \psi \in L^2(G/P, \nu)$,

$$\lim_{T\to\infty} \frac{1}{|\Gamma_T|} \sum_{\gamma \in \Gamma_T} f(\mathbf{b}(\gamma)) \frac{\langle \lambda_{G/P}(\gamma)\varphi, \psi\rangle}{\Xi(\gamma)} = \langle \varphi, \mathbb{1}_{G/P}\rangle \langle f, \psi\rangle.$$

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1.4. *Main results and ideas.* Among quasi-regular representations defined by measurable actions of a locally compact second countable group G on a second countable Borel space B preserving the class of a Borel probability measure ν, we single out in Definition 2.1 those representations which are *ergodic with respect to some family* (E_n, e_n) *where each* $E_n \subset G$ *is relatively compact Borel and* $e_n : E_n \to B$ *is Borel, and with respect to some function* $f \in L^\infty(B, \nu)$.

The definitions are coined so that a quasi-regular representation which is ergodic with respect to a family (E_n, e_n) and sufficiently many functions $f \in L^\infty(B, \nu)$ has to be irreducible; see Proposition 2.8. The existence of such a family implies the equidistribution of the sets $e_n(E_n)$ in (B, ν); see Remark 2.7. The ergodicity of the quasi-regular representation associated with an action is strictly stronger than the ergodicity of the action; see the example after Remark 2.7.

We show that the ergodicity of a quasi-regular representation does not depend too much on the chosen family $E_n \subset G$: one is free to work with balls, annuli, coronas, cones and subcones; see §2.3.

Theorem 2.2 gives sufficient conditions for the quasi-regular representation of a unimodular locally compact second countable group G, endowed with a length function and acting on a second countable Borel metric probability space (B, d, ν), to be ergodic with respect to a symmetric family $E_n \subset G$, Borel maps $e_n : E_n \to B$ and functions $f \in L^\infty(B, \nu)$ belonging to the closure of the subspace generated by characteristic functions of Borel subsets $U \subset B$ such that $\nu(\partial U) = 0$.

As explained in the previous section, if G is a connected non-compact semisimple Lie group G with finite center, with maximal compact subgroup K, Cartan decomposition $K\overline{A^+}K$, minimal parabolic P and $M = Z(A) \cap K$, we consider its Furstenberg–Poisson boundary $B = G/P = K/M$ and the boundary map

$$\mathbf{b} : KA^+K \to K/M$$

defined a.e. on G as $\mathbf{b}(k_1 a k_2) = k_1 M$. If the real rank of G equals one, the Furstenberg–Poisson boundary is identified with the visual boundary of the symmetric space G/K, the boundary map $\mathbf{b}$ is defined on $G\backslash K$ and coincides with the map $\mathbf{b}(\gamma) = c_{\gamma x_0}(\infty)$ explained above. If $\Gamma \subset G$ is a lattice, we apply Theorem 2.2 to the intersections of Γ with some cones made bi-K-invariant in G and deduce (using Theorem 3.1) the ergodicity of the quasi-regular representation of a lattice in a semisimple Lie group (Theorem 1.2); the boundary maps $\mathbf{b}_T$ are the restrictions of $\mathbf{b}(k_1 a k_2) = k_1 M$ to the T-truncation of (the image by the exponential map of) a Weyl chamber made bi-K-invariant. The irreducibility of the restriction of the quasi-regular representation $\lambda_{G/P}$ to any lattice follows from a general result in [**13**]; according to Proposition 2.8 this is also a straightforward corollary of Theorem 1.2.

Applying Theorem 1.2 or Bader–Muchnik's theorem to the case

$$\mathrm{rank}_{\mathbb{R}} G = 1$$

brings essentially the same information, although the original statement of Bader and Muchnik applies only to uniform lattices.

Specializing Theorem 1.2 by choosing twice the vector

$$\mathbb{1}_{G/P} \in L^2(G/P)$$

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and applying the definition of the WOT one recovers the equidistribution of the radial K-component of the Cartan decomposition KA^+K of lattice points in the Furstenberg–Poisson boundary: *for every continuous function f on G/P,*

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{\gamma\in B_n} f(\mathbf{b}(\gamma)) = \int_{G/P} f(b)\, d\nu(b).$$

The analogous statement with the Iwasawa decomposition $G = KAN$ instead of the Cartan decomposition and the boundary map $\mathbf{b}(kan) = kM$ instead of $\mathbf{b}(k_1ak_2) = k_1M$ follows from [**18**, Theorem 1]. We refer the reader to [**19**, Theorem 7.2] for the equidistribution of both K-components of the Cartan decomposition KA^+K of lattice points with an explicit control on the rate of convergence for Lipschitz functions.

Specializing Theorem 1.2 by choosing $f = \mathbb{1}_{G/P} \in L^\infty(G/P)$ and applying the results of §2.3 confirms Breuillard's guess mentioned above.

1.5. *Key points from the proofs, questions and speculations.* The proof of Theorem 2.2 splits into two parts. The first part consists in 'the identification of the limit'—by which we mean the convergence

$$\lim_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_W\rangle = \nu(U\cap W)\nu(V)$$

(see the formula (1) below and §2 below for notation)—which is established for sufficiently many characteristic functions. The second part amounts to proving the uniform bound

$$\sup_n \|M^{\mathbb{1}_B}_{E_n}\|_{\mathrm{op}} < \infty$$

for the family of operators $M^{\mathbb{1}_B}_{E_n} \in \mathcal{B}(L^2(B,\nu))$. Assuming both parts proved, the main conclusion of Theorem 2.2, that is the ergodicity of the representation π is easily deduced from density arguments and the following lemma, whose proof is straightforward.

LEMMA 1.3. (From L^∞ to L^2) *Let $\mathcal{H}$ be a Hilbert space and let V be a dense subspace. Let A_n be a uniformly bounded sequence of $\mathcal{B}(\mathcal{H})$. If for all $v, w \in V$,*

$$\lim_{n\to\infty} \langle A_n v, w\rangle = 0,$$

then

$$\lim_{n\to\infty} A_n = 0$$

in the weak operator topology.

This twofold approach is a core idea in [**3**]; it works in strictly negative curvature. We show that it also works in higher rank. Theorem 2.2 may be viewed as an attempt to formalize this twofold approach in a general frame, flexible enough to cover different geometric situations. Although semisimple Lie groups over non-Archimedean local fields and their lattices and S-arithmetic groups are not touched in this paper, we believe that a similar approach applies to those groups as well. Also it would be interesting to obtain effective forms of convergence with a good control on the error term. (Reference [**19**] looks helpful for both purposes.) We now briefly indicate how the 'identification of the limit', respectively the 'uniform boundedness', is obtained in illustrative cases.

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1.5.1. *Identification of the limit.* As explained in §3.4 below, in the case of symmetric spaces the first hypothesis of Theorem 2.2, namely the inequality

$$\frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_{\{b\in B: d(b,e_n(g))\geq r\}}\rangle}{\Xi(g)} \leq h_r(L(g)),$$

follows from Fatou-type theorems (see [**29**, **31**, **32**]) for the normalized square root of the Poisson kernel

$$(\mathcal{P}_0\varphi)(g) = \frac{\langle \pi(g)\mathbb{1}_B, \overline{\varphi}\rangle}{\Xi(g)}.$$

An important condition in generalized Fatou theorems is the convergence type (weak and/or restricted tangential, admissible). We handle this technical issue by first proving the ergodicity of π with respect to subcones of a Weyl chamber in Theorem 3.1. The restrictions imposed by the subcones may then be removed thanks to the results of §2.3. As shown in [**7**, Lemma 5.1, Proposition 7.4], the above inequality is true for discrete isometry groups of proper CAT(-1) spaces, with δ-Ahlfors regular Patterson–Sullivan conformal density of dimension δ, and although the proof relies on geometric properties of negatively curved spaces it could be deduced from a Fatou-type theorem for the normalized square-root of the Poisson kernel [**7**, Proposition 7.4]. It would be interesting to develop the boundary theory with Fatou-type theorems for the normalized square-root of the Poisson kernels for more CAT(0) spaces and groups.

The second hypothesis of Theorem 2.2, namely the inequality

$$\limsup_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} \mathbb{1}_U(e_n(g^{-1}))\mathbb{1}_V(e_n(g))\, dg \leq \nu(U)\nu(V),$$

is satisfied thanks to counting results. For closed negatively curved manifolds it follows from a result which goes back to G. Margulis's thesis [**3**, Theorem C.1]. For CAT(-1) spaces and groups with finite Bowen–Margulis–Sullivan measure and non-arithmetic spectrum it follows (see [**7**]) from a result of Roblin [**28**, Theorem 4.1.1]. For free groups, a direct counting gives the result [**9**] and it would be interesting to understand how the normalizing constant in [**28**, Theorem 4.1.1] varies with $\alpha > 0$ in the case of the tree covering the wedge of two circles of lengths 1 and α. For locally symmetric spaces of finite volume it follows—as explained below in §3.5—from the wave-front lemma and counting results from [**20**].

1.5.2. *Uniform boundedness.* Bounding operator norms of averages of unitary operators may be a difficult problem; this is well illustrated by Valette's conjecture about property RD for uniform lattices in higher rank semisimple Lie groups (see [**11**]). In [**3**], Bader and Muchnik apply the Riesz–Thorin interpolation theorem and reduce their L^2 uniform bound problem to one about L^∞ norms. Following their idea we explain in §2.4 below how the problem boils down to bounding cocyles averages of the type

$$M_E^{\mathbb{1}_B}\mathbb{1}_B(b) = \frac{1}{\mathrm{vol}(E)} \int_E \frac{c(g,b)^{1/2}}{\Xi(g)}\, dg.$$

In CAT(-1) spaces, the concept of sampling from [**3**] and geometric inequalities [**10**, §4.4 "Uniform boundedness"] lead to the wanted uniform upper bound for the L^∞ norms. (An

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alternative proof of the uniform bound on the operator norms consists in applying property RD to the quasi-regular representation. This works for Gromov hyperbolic groups; see [**10**, §4.4 "Uniform boundedness"].) For lattices in semisimple Lie groups the upper bound is proved in two steps. The first step (Proposition 3.5) is an explicit computation of the cocycle averages: they are all equal to one (this was first observed in [**9**, Lemma 2.6]). The second step (Proposition 3.6), which goes back to [**8**], is a discretization of the cocycles averages; it works because the cocycles and the Harish–Chandra functions are stable in the sense of §2.5.

2. *Ergodic properties of quasi-regular representations*

2.1. *Quasi-regular representations.* Let B be a second countable topological space and let ν be a Borel probability measure on B. Let G be a locally compact second countable group. Let dg denote a left Haar measure on G. We assume that G acts continuously on B and that the action preserves sets of ν-measure zero. For all $\varphi \in L^2(B, \nu)$ and all $g \in G$ we have

$$\int_G \varphi(gb)c(g, b)\, d\nu(b) = \int_G \varphi(b)\, d\nu(b),$$

where

$$c(g, b) = \frac{dg_*^{-1}\nu(b)}{d\nu(b)}$$

is the Radon–Nikodym derivative at the point b of the transformation defined by g^{-1}. The formula

$$(\pi(g)\varphi)(b) = \varphi(g^{-1}b)c(g^{-1}, b)^{1/2}$$

defines a unitary representation

$$\pi : G \to \mathcal{U}(L^2(B, \nu))$$

called the *quasi-regular representation* associated with the action of G on (B, ν). (See [**4**, Proposition A.6.1 and Lemma A.6.2] for the strong continuity of π.) We will always assume that for any relatively compact open set $U \subset G$,

$$\sup_{g\in U} \left\| \frac{dg_*\nu}{d\nu} \right\|_{L^\infty(B,\nu)} < \infty,$$

equivalently, for any relatively compact open set $U \subset G$,

$$\sup_{g\in U} \|\pi(g)\mathbb{1}_B\|_{L^\infty(B,\nu)} < \infty.$$

This condition says that there is a uniform bound on the contraction (or dilatation) of any Borel set in B under the action of any compact part of G. It is satisfied in each example mentioned in the introduction: the boundary B is a compact space and the Radon–Nikodym derivative is given by a cocycle $c(g, b)$ continuous in (g, b).

The *Harish–Chandra function* Ξ associated with the action of G on (B, ν) is the coefficient of π defined by the characteristic function $\mathbb{1}_B \in L^2(B, \nu)$ of B. Namely

$$\Xi(g) = \langle \pi(g)\mathbb{1}_B, \mathbb{1}_B \rangle = \int_B c(g^{-1}, b)^{1/2}\, d\nu(b).$$

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Let $\varphi \in L^1(B, \nu)$. In some special cases (see §3.4 below) it is fruitful to consider

$$(\mathcal{P}_0\varphi)(g) = \frac{\langle \pi(g)\mathbb{1}_B, \overline{\varphi}\rangle}{\Xi(g)}$$

as *the normalized square root of a Poisson kernel*. It brings a useful link with potential theory.

2.2. *Averages of unitary operators.* Let $E \subset G$ be a relatively compact Borel subset and let

$$\mathrm{vol}(E) = \int_G \mathbb{1}_E(g)\, dg.$$

Let $e : E \to B$ be a Borel map. If π is as above, let $\mathcal{B}(L^2(B, \nu))$ denote the Banach space of the bounded operators on $L^2(B, \nu)$. Let $f : B \to \mathbb{C}$ be a Borel bounded function. If $\mathrm{vol}(E) \neq 0$ we may consider the average

$$M^f_{(E,e)} = \frac{1}{\mathrm{vol}(E)} \int_E f(e(g)) \frac{\pi(g)}{\Xi(g)}\, dg.$$

As the function

$$g \mapsto \frac{\mathbb{1}_E(g) f(e(g))}{\mathrm{vol}(E)\Xi(g)}$$

belongs to $L^1(G, dg)$ it follows that $M^f_{(E,e)} \in \mathcal{B}(L^2(B, \nu))$. Let $P_{\mathbb{1}_B} \in \mathcal{B}(L^2(B, \nu))$ be the orthogonal projection onto the complex line of constant functions on B and let $m(f) \in \mathcal{B}(L^2(B, \nu))$ be the multiplication operator defined by f.

Definition 2.1. (Ergodic quasi-regular representations) For each $n \in \mathbb{N}$ let (E_n, e_n) be as above. We say that the representation π is ergodic relative to the family (E_n, e_n) and the map f if

$$\lim_{n\to\infty} M^f_{(E_n,e_n)} = m(f) P_{\mathbb{1}_B}$$

in the weak operator topology. That is, if for all $\varphi, \psi \in L^2(B, \nu)$,

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} f(e_n(g)) \frac{\langle \pi(g)\varphi, \psi\rangle}{\Xi(g)}\, dg = \langle \varphi, \mathbb{1}_B\rangle\langle f, \psi\rangle.$$

THEOREM 2.2. (Fatou, counting, and uniform boundedness imply ergodicity of the representation) *Let $\pi : G \to \mathcal{U}(L^2(B, \nu))$ be a quasi-regular representation such that, for any relatively compact open set $U \in G$,*

$$\sup_{g\in U} \|\pi(g)\mathbb{1}_B\|_{L^\infty(B,\nu)} < \infty.$$

Suppose G is unimodular. Let L be a length function on G and let d be a distance on B inducing the topology of B. Let $E_n = E_n^{-1}$ be a sequence of relatively compact Borel subsets of G such that $\lim_{n\to\infty} \mathrm{vol}(E_n) = \infty$. Let $e_n : E_n \to B$ be a sequence of Borel maps. Assume the following two conditions hold:

(1) *for each $r > 0$ there is a non-increasing function $h_r : [0, \infty[\to [0, \infty[$ such that $\lim_{s\to\infty} h_r(s) = 0$ and such that for all $n \in \mathbb{N}$ and for all $g \in E_n$*

$$\frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_{\{b\in B : d(b, e_n(g)) \geq r\}}\rangle}{\Xi(g)} \leq h_r(L(g));$$

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(2) *for all Borel subsets* $U, V \subset B$ *such that* $\nu(\partial U) = \nu(\partial V) = 0$,

$$\limsup_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} \mathbb{1}_U(e_n(g^{-1}))\mathbb{1}_V(e_n(g))\, dg \le \nu(U)\nu(V).$$

- *Then for all Borel subsets* $U, V, W \subset B$ *such that* $\nu(\partial U) = \nu(\partial V) = \nu(\partial W) = 0$,

$$\lim_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)}\mathbb{1}_V, \mathbb{1}_W\rangle = \nu(U \cap W)\nu(V). \tag{1}$$

- *Assume moreover*

$$\sup_n \|M^{\mathbb{1}_B}_{E_n}\mathbb{1}_B\|_\infty < \infty.$$

Let $H \subset L^\infty(B, \nu)$ *be the smallest subspace containing all characteristic functions* $\mathbb{1}_U$ *with* $U \subset B$ *Borel such that* $\nu(\partial U) = 0$ *and let* $\overline{H} \subset L^\infty(B, \nu)$ *denote its closure. Then* π *is ergodic with respect to the family* (E_n, e_n) *and any* $f \in \overline{H}$.

Remark 2.3. Notice that if $f = \mathbb{1}_B$ then the choices of the maps e_n are irrelevant; in this case we say that the representation π is ergodic relative to the sets E_n. In the weak operator topology we have

$$\lim_{n\to\infty} M_{E_n} = P_{\mathbb{1}_B},$$

hence $P_{\mathbb{1}_B}$ belongs to the von Neumann algebra generated by $\pi(G)$ in the algebra $\mathcal{B}(L^2(B, \nu))$.

Remark 2.4. The definition of an ergodic quasi-regular representation generalizes the definition of an ergodic measure preserving transformation. Indeed, if T is a measure preserving transformation of the probability space (B, ν), the associated quasi-regular representation $\pi : \mathbb{Z} \to \mathcal{U}(L^2(B, \nu))$ is the Koopman representation, the constant function $\mathbb{1}_B$ is fixed by π, hence Ξ is a constant equal to one, the choice $E_n = \{k \in \mathbb{Z} : 0 \le k \le n-1\}$ defines the Birkhoff sum M_{E_n}, and it is well known that

$$\lim_{n\to\infty} M_{E_n} = P_{\mathbb{1}_B}$$

in the weak operator topology if and only if the measure preserving transformation T is ergodic.

PROPOSITION 2.5. *If the quasi-regular representation* π *of* G *on* $L^2(B, \nu)$ *is ergodic relative to a family* E_n*, then the action of* G *on* B *is ergodic.*

Proof. Let E be a Borel subset of B such that for all $g \in G$ we have $gE = E$. Let E^c denote the complement of E in B. Obviously

$$\langle \pi(g)\mathbb{1}_E, \mathbb{1}_{E^c}\rangle = 0$$

for all $g \in G$. Hence, if π is ergodic relative to a family E_n, then

$$0 = \langle M_{E_n}\mathbb{1}_E, \mathbb{1}_{E^c}\rangle \to \langle P_{\mathbb{1}_B}\mathbb{1}_E, \mathbb{1}_{E^c}\rangle = \nu(E)\nu(E^c). \qquad \square$$

Remark 2.6. If π is ergodic relative to a family of couples (E_n, e_n) and a Borel bounded function f, then

$$\lim_{n\to\infty} M^f_{(E_n,e_n)}\mathbb{1}_B = f$$

in the weak topology of $L^2(B, \nu)$. Hence if this holds for a family of Borel bounded functions f which is dense in $L^2(B, \nu)$, then $\mathbb{1}_B$ is a cyclic vector of π.

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Remark 2.7. If π is ergodic relative to a family of couples (E_n, e_n) for all continuous f, we may specialize the convergence to the case $\varphi = \psi = \mathbb{1}_B$ to obtain the equidistribution of $e_n(E_n) \subset B$ relative to ν:

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} f(e_n(g))\, dg = \int_B f(b)\, d\nu(b),$$

or, in the case that $G = \Gamma$ is discrete and the Haar measure is the counting measure,

$$\lim_{n\to\infty} \frac{1}{|E_n|} \sum_{\gamma\in E_n} f(e_n(\gamma)) = \int_B f(b)\, d\nu(b).$$

If $E \subset B$ is a Borel G-invariant subset with $0 < \nu(E) < 1$, then the orthogonal projection $P_{\mathbb{1}_E}$ commutes with $\pi(G)$, so $\ker(P_{\mathbb{1}_E})$ is a non-trivial closed invariant subspace for π. Hence *the ergodicity of the action is a necessary condition for the irreducibility of* π. *It is not a sufficient one*: the diagonal action of $\mathrm{SL}(2, \mathbb{Z})$ on the product of two copies of the circle at infinity of $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ equipped with the product of the angular measure is ergodic (see [**23**, Theorem 17] and the introduction of [**12**]) but the involution permuting the coordinates of the circles at infinity defines a non-trivial invariant subspace of the associated quasi-regular representation. The next proposition shows that *the ergodicity of the representation is strictly stronger than the ergodicity of the action because it implies the irreducibility of the representation.*

PROPOSITION 2.8. (Ergodic representations are irreducible) *Assume that for each bounded Borel function $f : B \to \mathbb{C}$ from a dense family in $L^2(B, \nu)$ containing $\mathbb{1}_B$ there is a sequence (E_n, e_n) relative to which π is ergodic. Then π is irreducible.*

Proof. As mentioned in the above remarks, the vector $\mathbb{1}_B$ is cyclic and the orthogonal projection $P_{\mathbb{1}_B}$ onto the complex line it generates belongs to the von Neumann algebra generated by $\pi(G)$. The irreducibility is then a consequence of the following classical lemma, which is easy to prove. □

LEMMA 2.9. *Let π be a unitary representation on a Hilbert space $\mathcal{H}$. Suppose there exists a cyclic vector $w \in \mathcal{H}$ such that the orthogonal projection P_w onto the line it generates belongs to the von Neumann algebra generated by $\pi(G)$. Then π is irreducible.*

2.3. *Ergodicity relative to balls, annuli, cones and subcones.*

PROPOSITION 2.10. (Ergodicity relative to cones and subcones) *Let π be a quasi-regular representation of G as above. Let $f : B \to \mathbb{C}$ be a Borel bounded map. Assume that for each $n \in \mathbb{N}$ we are given relatively compact Borel subsets B_n and $C_n \subset B_n$ of G of non-zero Haar measure and Borel maps $b_n : B_n \to B$ and $c_n = b_n|_{C_n} : C_n \to B$. Assume that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(B_n \backslash C_n)}{\mathrm{vol}(B_n)} = 0.$$

Assume moreover that the operators $M^f_{(B_n,b_n)}$ and $M^f_{(C_n,c_n)}$ are uniformly bounded. Then the following conditions are equivalent:

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(1) *the representation π is ergodic relative to the couples (B_n, b_n) and the map f;*
(2) *the representation π is ergodic relative to the couples (C_n, c_n) and the map f.*

Proof. As $L^\infty(B, \nu) \subset L^2(B, \nu)$ is dense and the operators are uniformly bounded, Lemma 1.3 applies. Hence it is enough to prove that, for all $\varphi, \psi \in L^\infty(B, \nu)$,

$$\lim_{n\to\infty} \langle (M^f_{(B_n,b_n)} - M^f_{(C_n,c_n)})\varphi, \psi\rangle = 0.$$

We have

$$M^f_{(B_n,b_n)} - M^f_{(C_n,c_n)} = \frac{1}{\mathrm{vol}(B_n)} \int_{B_n\setminus C_n} f(b_n(g)) \frac{\pi(g)}{\Xi(g)}\, dg + \frac{\mathrm{vol}(C_n) - \mathrm{vol}(B_n)}{\mathrm{vol}(B_n)\mathrm{vol}(C_n)} \int_{C_n} f(c_n(g)) \frac{\pi(g)}{\Xi(g)}\, dg.$$

Writing each bounded function f, φ, ψ as the sum of its positive and negative real part and its positive and negative imaginary part we obtain

$$\begin{aligned}
&|\langle (M^f_{(B_n,b_n)} - M^f_{(C_n,c_n)})\varphi, \psi\rangle| \\
&\quad\le \frac{4^3\|f\|_\infty\|\varphi\|_\infty\|\psi\|_\infty}{\mathrm{vol}(B_n)} \int_{B_n\setminus C_n} \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_B\rangle}{\Xi(g)}\, dg \\
&\qquad + 4^3\|f\|_\infty\|\varphi\|_\infty\|\psi\|_\infty \frac{\mathrm{vol}(B_n) - \mathrm{vol}(C_n)}{\mathrm{vol}(B_n)\mathrm{vol}(C_n)} \int_{C_n} \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_B\rangle}{\Xi(g)}\, dg \\
&\quad= 4^3\|f\|_\infty\|\varphi\|_\infty\|\psi\|_\infty 2\frac{\mathrm{vol}(B_n) - \mathrm{vol}(C_n)}{\mathrm{vol}(B_n)}.
\end{aligned}$$

□

On one hand there are good reasons to consider averages over 'coronas' or 'shells' or 'annuli' as in [**3**], as mentioned in the introduction of the present article (one technical advantage is that on a corona, both upper and lower estimates of the Harish–Chandra function are possible). On the other hand, averages over balls are natural (e.g. sums in von Neumann and Birkhoff theorems). The following proposition shows that, under a mild general hypothesis, the ergodicity of a representation with respect to 'balls' is equivalent to the ergodicity of a representation with respect to 'annuli'.

PROPOSITION 2.11. (Ergodicity relative to balls and annuli) *Let π be a quasi-regular representation of G as above. Let $f : B \to \mathbb{C}$ be a Borel bounded map. Assume that for each $n \in \mathbb{N}$ we are given relatively compact Borel subsets B_n and C_n of G of non-zero Haar measure and Borel maps $b_n : B_n \to B$ and $c_n : C_n \to B$. Assume that for each n the set*

$$B_n = \bigsqcup_{k=1}^{n} C_k$$

is the disjoint union of the sets $C_1, \ldots, C_k, \ldots, C_n$ and that b_n restricted to C_k equals c_k.

(1) *Assume $\lim_{n\to\infty}((\mathrm{vol}(B_{n-1}))/(\mathrm{vol}(B_n)))$ exists and is strictly smaller than one. If π is ergodic relative to the (B_n, b_n) and f, then it is ergodic relative to the (C_n, c_n) and f.*
(2) *Assume $\lim_{n\to\infty} \mathrm{vol}(B_n) = \infty$. If π is ergodic relative to the (C_n, c_n) and f, then it is ergodic relative to the (B_n, b_n) and f.*

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Proof. Let $\varphi, \psi \in L^2(B, \nu)$.

Suppose $\lim_{n\to\infty}((\mathrm{vol}(B_{n-1}))/(\mathrm{vol}(B_n))) = c < 1$ and that π is ergodic relative to the (B_n, b_n) and f. We have

$$\langle M^f_{(B_n,b_n)}\varphi, \psi\rangle = \frac{\mathrm{vol}(B_{n-1})}{\mathrm{vol}(B_n)}\langle M^f_{(B_{n-1},b_{n-1})}\varphi, \psi\rangle + \frac{\mathrm{vol}(C_n)}{\mathrm{vol}(B_n)}\langle M^f_{(C_n,c_n)}\varphi, \psi\rangle,$$

hence letting n tend to infinity we obtain

$$\langle m(f)P_{\mathbb{1}_B}\varphi, \psi\rangle = c\langle m(f)P_{\mathbb{1}_B}\varphi, \psi\rangle + (1-c)\lim_{n\to\infty}\langle M^f_{(C_n,c_n)}\varphi, \psi\rangle.$$

This proves the ergodicity of π relative to the (C_n, c_n) and f because by hypothesis $c \neq 1$.

Suppose $\lim_{n\to\infty}\mathrm{vol}(B_n) = \infty$ and that π is ergodic relative to the (C_n, c_n) and f. We have

$$\langle M^f_{(B_n,b_n)}\varphi, \psi\rangle = \frac{1}{\mathrm{vol}(B_n)}\sum_{k=1}^{n}\mathrm{vol}(C_k)\langle M^f_{(C_k,c_k)}\varphi, \psi\rangle.$$

Let $a_k = \langle M^f_{(C_k,c_k)}\varphi, \psi\rangle$ and let $a = \langle m(f)P_{\mathbb{1}_B}\varphi, \psi\rangle$. By hypothesis $\lim_{k\to\infty} a_k = a$. Let $s_k = \mathrm{vol}(C_k)$. The proof is finished thanks to the following lemma, whose proof is straightforward. □

LEMMA 2.12. *Let a_k be a sequence of complex numbers which converges to a. Let s_k be a sequence of non-negative numbers such that $\sum_{k=1}^{\infty} s_k = \infty$. Then*

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^{n} s_k}\sum_{k=1}^{n} s_k a_k = a.$$

2.4. *Uniform bounds for averages of unitary operators.* The following proposition is a corollary of the Riesz–Thorin theorem.

PROPOSITION 2.13. *Let (B, ν) be a probability space and let*

$$T_1 : L^1(B, \nu) \to L^1(B, \nu)$$

be a bounded operator such that its restriction T_2 to $L^2(B, \nu)$ preserves $L^2(B, \nu)$ and its restriction T_∞ to $L^\infty(B, \nu)$ preserves $L^\infty(B, \nu)$. If T_2 is bounded and self-adjoint then T_∞ is bounded,

$$\|T_\infty\|_{L^\infty(B,\nu)\to L^\infty(B,\nu)} = \|T_1\|_{L^1(B,\nu)\to L^1(B,\nu)},$$
$$\|T_2\|_{L^2(B,\nu)\to L^2(B,\nu)} \leq \|T_\infty\|_{L^\infty(B,\nu)\to L^\infty(B,\nu)}.$$

Let E be a Borel subset of G of positive measure. Let $b \in B$. Recall that by definition

$$M_E^{\mathbb{1}_B}\mathbb{1}_B(b) = \frac{1}{\mathrm{vol}(E)}\int_E \frac{c(g, b)^{1/2}}{\Xi(g)}\, dg.$$

Let $e : E \to B$ be a Borel map and $f : B \to \mathbb{C}$ a bounded Borel map.

LEMMA 2.14. *Assume E is relatively compact and symmetric. If G is unimodular then*

$$\|M^f_{(E,e)}\|_{L^2\to L^2} \leq \|f\|_\infty \|M_E^{\mathbb{1}_B}\mathbb{1}_B\|_\infty.$$

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Proof. To save notation we write M_E instead of $M_E^{\mathbb{1}_B}$. As E is relatively compact in G which is locally compact second countable and as we always assume that, for any relatively compact open set $U \in G$,

$$\sup_{g\in U} \left\| \frac{dg_*\nu}{d\nu} \right\|_{L^\infty(B,\nu)} < \infty,$$

it is easy to check that for $p \in \{1; \infty\}$,

$$M_E = \frac{1}{\mathrm{vol}(E)} \int_E \frac{\pi(g)}{\Xi(g)}\, dg \in \mathcal{B}(L^p(B, \nu)).$$

Obviously, $\|M^f_{(E,e)}\|_{L^p\to L^p} \le \|f\|_\infty \|M_E\|_{L^p\to L^p}$. As $E = E^{-1}$ and as G is unimodular we have $M_E = M_E^*$. According to Proposition 2.13,

$$\|M_E\|_{L^2\to L^2} \le \|M_E\|_{L^\infty\to L^\infty} = \|M_E \mathbb{1}_B\|_\infty. \qquad \square$$

2.5. *Stable functions on locally compact groups.*

Definition 2.15. A function $f : G \to \mathbb{C}$ on a locally compact group is *left stable* if there exist a constant $C \ge 1$ and a neighborhood V of the identity e in G such that, for all $g \in G$ and for all $v \in V$,

$$|f(g)|/C \le |f(vg)| \le C|f(g)|.$$

We write $f(g) \asymp f(vg)$ when we don't want to emphasize the actual value of the constant C.

PROPOSITION 2.16. *Assume that the cocycle $c : G \times B \to]0, \infty[$ associated with an action of a locally compact group G on a compact probability space (B, ν) is continuous. Let π be the corresponding unitary representation. Then for any $b \in B$ the function $g \mapsto c(g, b)$ is left stable. The generalized Harish–Chandra function $\Xi(g) = \langle \pi(g)\mathbb{1}_B, \mathbb{1}_B\rangle$ is left and right stable.*

Proof. Let V be a symmetric compact neighborhood of e in G. Since c is continuous, $c|_{V\times B}$ reaches its maximum M and its minimum m on the compact set $V \times B$. Let $v \in V$. The cocycle identity

$$1 = c(v^{-1}v, b) = c(v^{-1}, vb)c(v, b)$$

and the symmetry of V imply that $Mm = 1$. Since $c(vg, b) = c(v, gb)c(g, b)$ we obtain for all $g \in G$ and all $v \in V$,

$$\frac{c(g, b)}{\|c|_{V\times B}\|_\infty} \le c(vg, b) \le \|c|_{V\times B}\|_\infty c(g, b).$$

We have

$$\begin{aligned}\Xi(vg) = \Xi((vg)^{-1}) &= \int_B c(vg, b)^{1/2}\, d\nu(b) \asymp \int_B c(g, b)^{1/2}\, d\nu(b)\\ &= \Xi(g^{-1}) = \Xi(g),\end{aligned}$$

and hence

$$\Xi(gv) = \Xi(v^{-1}g^{-1}) \asymp \Xi(g^{-1}) = \Xi(g). \qquad \square$$

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LEMMA 2.17. *Assume the cocycle $c : G \times B \to]0, \infty[$ associated with an action of a locally compact group G on a compact probability space (B, ν) is continuous. Let π be the corresponding unitary representation. Let Γ be a discrete subgroup of G. For all small enough relatively compact neighborhoods U of e in G, there exists a constant $C > 0$ such that for any non-empty finite subset Λ of Γ and for any $b \in B$,*

$$\sum_{\gamma \in \Lambda} \frac{\pi(\gamma)\mathbb{1}_B(b)}{\Xi(\gamma)} \leq C \int_{\Lambda U} \frac{\pi(g)\mathbb{1}_B(b)}{\Xi(g)}\, dg.$$

Proof. According to Proposition 2.16 the cocycle is left stable. Hence, as $g \mapsto g^{-1}$ is a homeomorphism fixing $e \in G$, there exists a constant $C' > 0$ and a relatively compact neighborhood U of e in G such that for all $u \in U$, all $g \in G$ and all $b \in B$,

$$\pi(g)\mathbb{1}_B(b) = c(g^{-1}, b)^{1/2} \leq C' c(u^{-1}g^{-1}, b)^{1/2} = C'\pi(gu)\mathbb{1}_B(b).$$

According to the same proposition, the Harish–Chandra function is right stable. Hence we may assume that there is a constant $C'' > 0$ such that for all $u \in U$ and all $g \in G$,

$$\Xi(g) > \Xi(gu)/C''.$$

It follows that for any $\gamma \in \Gamma$,

$$\frac{\pi(\gamma)\mathbb{1}_B(b)}{\Xi(\gamma)} = \frac{1}{\mathrm{vol}(U)} \int_U \frac{\pi(\gamma)\mathbb{1}_B(b)}{\Xi(g)}\, du \leq \frac{C'C''}{\mathrm{vol}(U)} \int_U \frac{\pi(\gamma u)\mathbb{1}_B(b)}{\Xi(\gamma u)}\, du.$$

As Γ is discrete we may assume that U is small enough so that $UU^{-1} \cap \Gamma = \{e\}$. Putting $C = C'C''/\mathrm{vol}(U)$, we obtain

$$\sum_{\gamma \in \Lambda} \frac{\pi(\gamma)\mathbb{1}_B(b)}{\Xi(\gamma)} \leq C \int_{\Lambda U} \frac{\pi(g)\mathbb{1}_B(b)}{\Xi(g)}\, dg. \qquad \square$$

2.6. *General facts about metric Borel spaces.* Let (B, d) be a metric space and $E \subset B$ a subset. For $r > 0$ we denote

$$E(r) = \{b \in B : d(b, E) < r\}$$

the open r-neighborhood of E in B. Recall that the boundary ∂E of a subset E in a topological space B is the intersection of its closure with the closure of its complement

$$\partial E = \overline{E} \cap \overline{B \backslash E}$$

and that it is Borel in the case that E is Borel. For the sake of clarity we recall in the following proposition two (standard and easy to prove) facts that we will use about metric Borel spaces. To save space, we will recall the proof of the second statement only.

PROPOSITION 2.18. *Let (B, ν) be a Borel space with a probability measure. We assume the topology inducing the Borel structure is defined by a distance d. Let $E \subset B$ be a Borel set.*

(1) *If $\nu(\partial E) = 0$ then $\lim_{r \to 0} \nu(E(r)) = \nu(E)$.*
(2) *The set $\{r > 0 : \nu(\partial E(r)) \neq 0\}$ is at most countable.*

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Proof. To prove the second statement we notice that for any $r > 0$,

$$\partial E(r) \subset \{b \in B : d(b, E) = r\}.$$

Hence the boundaries of the open neighborhoods $\partial E(r)$ are disjoint for different values of r. This implies, because the measure of B is finite, that for any $n \in \mathbb{N}$ the set

$$\{r : \nu(\partial E(r)) > 1/n\}$$

is finite. Hence

$$\{r > 0 : \nu(\partial E(r)) \neq 0\} = \bigcup_{n\in\mathbb{N}} \{r : \nu(\partial E(r)) > 1/n\}$$

is at most countable. □

2.7. *Bounded operators preserving positive functions.* Let (B, ν) be a Borel space with a probability measure. We assume the topology inducing the Borel structure is defined by a distance d. Suppose for each Borel bounded function f on B we are given a bounded operator

$$M^f \in \mathcal{B}(L^2(B, \nu)),$$

and the following properties hold:

(1) $M^{af+bg} = aM^f + bM^g$ for all $a, b \in \mathbb{C}$ for all f, g Borel bounded;

(2) $f \geq 0 \Rightarrow M^f\varphi \geq 0$ for all $\varphi \in L^2(B, \nu)$ with $\varphi \geq 0$;

(3) $\langle M^{\mathbb{1}_B}\mathbb{1}_B, \mathbb{1}_B\rangle = 1$.

The following lemma is an abstract version of [**3**, Proposition 5.5]. The adaptation of the proof from [**3**] is not difficult and is left to the reader.

LEMMA 2.19. *Let (B, ν) and d be as above. Let M_n be a sequence of linear transformations from the space of bounded Borel functions on B to $\mathcal{B}(L^2(B, \nu))$ as above. Assume the following two conditions hold:*

(1) *for all Borel subsets $U, V \subset B$ with $\nu(\partial U) = \nu(\partial V) = 0$,*

$$\limsup_{n\to\infty} \langle M_n^{\mathbb{1}_U}\mathbb{1}_V, \mathbb{1}_B\rangle \leq \nu(U)\nu(V);$$

(2) *for all Borel subsets $U, W \subset B$ such that $d(U, W) > 0$,*

$$\lim_{n\to\infty} \langle M_n^{\mathbb{1}_U}\mathbb{1}_B, \mathbb{1}_W\rangle = 0.$$

Then for all Borel subsets $U, V, W \subset B$ with $\nu(\partial U) = \nu(\partial V) = \nu(\partial W) = 0$ we have

$$\lim_{n\to\infty} \langle M_n^{\mathbb{1}_U}\mathbb{1}_V, \mathbb{1}_W\rangle = \nu(U \cap W)\nu(V).$$

2.8. *Dirac-type peaks and counting.* Recall that a *length function* $L : G \to [0, \infty[$ on a locally compact group G is a locally bounded proper Borel map such that for all $g, h \in G$ we have $L(gh) \leq L(g) + L(h)$, $L(g^{-1}) = L(g)$ and $L(e) = 0$ where $e \in G$ denotes the neutral element. The following lemma is a generalization of [**3**, Proposition 5.1]. We give the proof for the convenience of the reader.

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LEMMA 2.20. *Let (B, ν) be a Borel probability space with a distance d inducing the topology of B. Let $\pi : G \to \mathcal{U}(L^2(B, \nu))$ be a quasi-regular representation. Let L be a length function on G. For each $n \in \mathbb{N}$ let $\mu_n \in L^1(G, dg)$ such that $\|\mu_n\|_1 \leq 1$, $\mu_n(g) \geq 0$ a.e. and*

$$\lim_{n\to\infty} \mu_n(g) = 0.$$

Let $\mathrm{supp}(\mu_n)$ denote the (essential) support of μ_n. Let $b_n : \mathrm{supp}(\mu_n) \to B$ be a Borel map. Assume that for each $r > 0$ there exists a function $h_r : [0, \infty[\to [0, \infty[$ with the following properties:

(1) *h_r is non-increasing;*
(2) *$\lim_{s\to\infty} h_r(s) = 0$;*
(3) *for all $n \in \mathbb{N}$ for all $g \in \mathrm{supp}(\mu_n)$,*

$$\frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_{\{b\in B : d(b, b_n(g)) \geq r\}}\rangle}{\Xi(g)} \leq h_r(L(g)).$$

Then for any Borel subset W in B and any $r > 0$,

$$\limsup_{n\to\infty} \int_G \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg \leq \limsup_{n\to\infty} \int_{\mathrm{supp}(\mu_n)} \mu_n(g)\mathbb{1}_{W(r)}(b_n(g))\, dg.$$

Proof. Let $W \subset B$ be a Borel subset and let $r > 0$. For each $n \in \mathbb{N}$ and each $s > 0$ we decompose the support of μ_n as

$$\begin{aligned}\mathrm{supp}(\mu_n) = \{g \in \mathrm{supp}(\mu_n) : L(g) < s\} \cup \{g \in \mathrm{supp}(\mu_n) : L(g) \geq s,\, b_n(g) \in W(r)\}\\ \cup \{g \in \mathrm{supp}(\mu_n) : L(g) \geq s,\, b_n(g) \notin W(r)\}.\end{aligned}$$

If g is such that $b_n(g) \notin W(r)$ then obviously $\mathbb{1}_W \leq \mathbb{1}_{\{b\in B : d(b, b_n(g)) \geq r\}}$. Hence we have

$$\begin{aligned}\int_G \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg &= \int_{\{g\in\mathrm{supp}(\mu_n) : L(g) < s\}} \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg\\ &\quad + \int_{\{g\in\mathrm{supp}(\mu_n) : L(g)\geq s, b_n(g)\in W(r)\}} \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg\\ &\quad + \int_{\{g\in\mathrm{supp}(\mu_n) : L(g)\geq s, b_n(g)\notin W(r)\}} \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg\\ &\leq \int_{\{g\in\mathrm{supp}(\mu_n) : L(g) < s\}} \mu_n(g)\, dg\\ &\quad + \int_{\{g\in\mathrm{supp}(\mu_n) : L(g)\geq s, b_n(g)\in W(r)\}} \mu_n(g)\, dg\\ &\quad + \int_{\{g\in\mathrm{supp}(\mu_n) : L(g)\geq s, b_n(g)\notin W(r)\}} \mu_n(g) h_r(s)\, dg.\end{aligned}$$

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As $\{g \in G : L(g) < s\}$ is relatively compact it has finite Haar measure, and Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} \int_{\{g\in G:L(g)<s\}} \mu_n(g)\, dg = \int_{\{g\in G:L(g)<s\}} \lim_{n\to\infty} \mu_n(g)\, dg = 0.$$

Hence, taking the limit superior in the above inequality we obtain

$$\limsup_{n\to\infty} \int_G \mu_n(g) \frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\, dg \le \limsup_{n\to\infty} \int_{\mathrm{supp}(\mu_n)} \mu_n(g)\mathbb{1}_{W(r)}(b_n(g))\, dg + h_r(s).$$

As the above inequality is true for any $s > 0$ and as $\lim_{s\to\infty} h_r(s) = 0$ the lemma is proved. □

PROPOSITION 2.21. (Away from the peak of the square root of the Poisson kernel) *Let $\pi : G \to \mathcal{U}(L^2(B, \nu))$ be a quasi-regular representation, and let L be a length function on G. For each n let $E_n \subset G$ be a relatively compact Borel subset and let $e_n : E_n \to B$ be a Borel map. Assume*

$$\lim_{n\to\infty} \mathrm{vol}(E_n) = \infty.$$

Assume that for each $r > 0$ there exists a function $h_r : [0, \infty[\to [0, \infty[$ with the following properties:

(1) *h_r is non-increasing;*
(2) *$\lim_{s\to\infty} h_r(s) = 0$;*
(3) *for all $n \in \mathbb{N}$ for all $g \in E_n$,*

$$\frac{\langle \pi(g)\mathbb{1}_B, \mathbb{1}_{\{b\in B:d(b,e_n(g))\ge r\}}\rangle}{\Xi(g)} \le h_r(L(g)).$$

Then for any Borel subset $U \subset B$ the operators

$$M_n^{\mathbb{1}_U} = M_{(E_n,e_n)}^{\mathbb{1}_U} = \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} \mathbb{1}_U(e_n(g)) \frac{\pi(g)}{\Xi(g)}\, dg$$

have the following properties:

(i) *if $U, W \subset B$ are Borel subsets such that $d(U, W) > 0$ then*

$$\lim_{n\to\infty} \langle M_n^{\mathbb{1}_U}\mathbb{1}_B, \mathbb{1}_W\rangle = 0;$$

(ii) *if G is unimodular and $E_n = E_n^{-1}$ (i.e. stable under taking inverses in G) then for all Borel subsets $U, V \subset B$ and all $r > 0$,*

$$\limsup_{n\to\infty} \langle M_n^{\mathbb{1}_U}\mathbb{1}_V, \mathbb{1}_B\rangle \le \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} \mathbb{1}_U(e_n(g^{-1}))\mathbb{1}_{V(r)}(e_n(g))\, dg.$$

Proof. To show the first claim we suppose $d(U, V) > 0$. We choose $0 < r < d(U, V)$ and define $\mu_n(g) = 0$ if $g \notin E_n$, and

$$\mu_n(g) = \frac{1}{\mathrm{vol}(E_n)} \mathbb{1}_{E_n}(g)\mathbb{1}_U(e_n(g))$$

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if $g \in E_n$. Applying Lemma 2.20 we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle M^{\mathbb{1}_U}_{(E_n,e_n)}\mathbb{1}_B, \mathbb{1}_W\rangle &= \limsup_{n\to\infty}\int_G \mu_n(g)\frac{\langle\pi(g)\mathbb{1}_B, \mathbb{1}_W\rangle}{\Xi(g)}\,dg\\ &\le \limsup_{n\to\infty}\int_{\mathrm{supp}(\mu_n)} \mu_n(g)\mathbb{1}_{W(r)}(e_n(g))\,dg\\ &= \limsup_{n\to\infty}\frac{1}{\mathrm{vol}(E_n)}\int_{E_n}\mathbb{1}_U(e_n(g))\mathbb{1}_{W(r)}(e_n(g))\,dg,\end{aligned}$$

but $\mathbb{1}_U(e_n(g))\mathbb{1}_{W(r)}(e_n(g)) = 0$ for all $g \in E_n$ because $U \cap W(r) = \emptyset$.

For the second claim we assume that G is unimoduar and that $E_n = E_n^{-1}$. The adjoint of $M^{\mathbb{1}_U}_{(E_n,e_n)}$ is

$$\begin{aligned}(M^{\mathbb{1}_U}_{(E_n,e_n)})^* &= \frac{1}{\mathrm{vol}(E_n)}\int_{E_n}\mathbb{1}_U(e_n(g))\frac{\pi^*(g)}{\Xi(g)}\,dg\\ &= \frac{1}{\mathrm{vol}(E_n)}\int_{E_n}\mathbb{1}_U(e_n(g))\frac{\pi(g^{-1})}{\Xi(g)}\,dg\\ &= \frac{1}{\mathrm{vol}(E_n)}\int_{E_n}\mathbb{1}_U(e_n(g^{-1}))\frac{\pi(g)}{\Xi(g)}\,dg.\end{aligned}$$

We define $\mu_n(g) = 0$ if $g \notin E_n$ and

$$\mu_n(g) = \frac{1}{\mathrm{vol}(E_n)}\mathbb{1}_{E_n}(g)\mathbb{1}_U(e_n(g^{-1}))$$

if $g \in E_n$. Applying Lemma 2.20 we obtain for any Borel subset $V \subset B$ and any $r > 0$,

$$\begin{aligned}\limsup_{n\to\infty}\langle M^{\mathbb{1}_U}_{(E_n,e_n)}\mathbb{1}_V, \mathbb{1}_B\rangle &= \limsup_{n\to\infty}\langle (M^{\mathbb{1}_U}_{(E_n,e_n)})^*\mathbb{1}_B, \mathbb{1}_V\rangle\\ &= \limsup_{n\to\infty}\int_G \mu_n(g)\frac{\langle\pi(g)\mathbb{1}_B, \mathbb{1}_V\rangle}{\Xi(g)}\,dg\\ &\le \limsup_{n\to\infty}\int_{\mathrm{supp}(\mu_n)} \mu_n(g)\mathbb{1}_{V(r)}(e_n(g))\,dg\\ &= \limsup_{n\to\infty}\frac{1}{\mathrm{vol}(E_n)}\int_{E_n}\mathbb{1}_U(e_n(g^{-1}))\mathbb{1}_{V(r)}(e_n(g))\,dg.\end{aligned}$$

□

2.9. *Proof of Theorem 2.2.*

Proof. In order to prove the first part of the theorem, namely

$$\lim_{n\to\infty}\langle M^{\mathbb{1}_U}_{(E_n,e_n)}\mathbb{1}_V, \mathbb{1}_W\rangle = \nu(U \cap W)\nu(V)$$

for Borel sets with zero measure boundary, it is enough to check that the hypotheses of Lemma 2.19 are satisfied. The operator $M^f_{(E_n,e_n)}$ is obviously positive if f is, and $\langle M^{\mathbb{1}_B}_{(E_n,e_n)}\mathbb{1}_B, \mathbb{1}_B\rangle = 1$.

It follows from Proposition 2.21 that the second assumption in Lemma 2.19 is satisfied, namely for all Borel subsets $U, W \subset B$ with $d(U, W) > 0$,

$$\lim_{n\to\infty}\langle M^{\mathbb{1}_U}_{(E_n,e_n)}\mathbb{1}_B, \mathbb{1}_W\rangle = 0.$$

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Let us check that the first assumption in Lemma 2.19 is satisfied, namely that for all Borel subsets U, $V \subset B$ with $\nu(\partial U) = \nu(\partial V) = 0$,

$$\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_B \rangle \leq \nu(U)\nu(V).$$

From Proposition 2.21 we deduce that for all $r > 0$,

$$\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_B \rangle \leq \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(E_n)} \int_{E_n} \mathbb{1}_U(e_n(g^{-1}))\mathbb{1}_{V(r)}(e_n(g))\, dg.$$

Hence, according to Proposition 2.18, for any $\epsilon > 0$ we may choose r such that $0 < r < \epsilon$ and such that $\nu(\partial(V(r))) = 0$. Applying the above inequality with the chosen r and the second assumption of the theorem we obtain

$$\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_B \rangle \leq \nu(U)\nu(V(r)).$$

As r may be chosen arbitrarily small in the above inequality and as $\nu(\partial V) = 0$ we deduce that

$$\limsup_{n\to\infty} \langle M^{\mathbb{1}_U}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_B \rangle \leq \nu(U)\nu(V),$$

since according to Proposition 2.18 $\lim_{r\to 0} \nu(V(r)) = \nu(V)$. This terminates the proof of the first part of the theorem.

To prove the second statement in the theorem, we notice first that linearity and the first part of the theorem imply that for any $h \in H$ and for all Borel sets V, W in B with $\nu(\partial V) = \nu(\partial W) = 0$ we have

$$\lim_{n\to\infty} \langle M^{h}_{(E_n,e_n)} \mathbb{1}_V, \mathbb{1}_W \rangle = \langle \mathbb{1}_V, \mathbb{1}_B \rangle \langle h, \mathbb{1}_W \rangle. \tag{2}$$

Applying Lemma 2.14 it is easy to deduce that (2) also holds for any $h \in \overline{H}$.

To finish the proof of the theorem we check that Lemma 1.3 applies.

The density hypothesis is satisfied: the topology on B is induced by a metric hence the probability Borel measure ν is regular, which implies that the smallest subspace of $L^2(B, \nu)$ containing the characteristic functions of Borel subsets of B whose boundaries have zero measure is dense.

The uniform boundedness hypothesis is satisfied thanks to Lemma 2.14 □

3. *Furstenberg–Poisson boundaries*

Let G be a non-compact semisimple connected Lie group with finite center. We use the notation introduced in §1.3.

3.1. *Cones around the barycenter of a Weyl chamber.* Let

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} \dim \mathfrak{g}_\alpha \cdot \alpha$$

be half the sum of the positive roots counted with multiplicity. The maximum of ρ on $\mathfrak{a}^+ \cap \overline{\mathfrak{a}_1}$ is reached at a unique vector $H_{\max}$ of norm one called the barycenter of $\mathfrak{a}^+$. Let $\theta > 0$. We define the open cone in $\mathfrak{a}^+$ of angle 2θ around $H_{\max}$ as

$$\mathfrak{a}^\theta = \{H \in \mathfrak{a}^+ : \measuredangle(H, H_{\max}) < \theta\},$$

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we denote the intersection (truncation) of this cone with the ball of radius T and center the origin in $\mathfrak{a}$ as

$$\mathfrak{a}_T^\theta = \mathfrak{a}^\theta \cap \mathfrak{a}_T$$

and its image in G under the exponential map as

$$A_T^\theta = \exp(\mathfrak{a}_T^\theta).$$

Eventually, we define

$$B_T^\theta = K A_T^\theta K.$$

3.2. *Volume estimates.* We normalize the Lebesgue measure dH on $\mathfrak{a}$ and the Haar measure dg on G so that for any continuous function f on G with compact support we have

$$\int_G f(g)\, dg = \int_K \int_{\mathfrak{a}^+} \int_K f(k \exp(H) l) J(H)\, dk\, dH\, dl,$$

where

$$J(H) = \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{\dim \mathfrak{g}_\alpha} \quad \text{for } H \in \mathfrak{a}^+.$$

Let E be a Borel subset of G. We denote its volume as

$$\mathrm{vol}(E) = \int_G \mathbb{1}_E(g)\, dg.$$

In particular, for any measurable subset L of $\mathfrak{a}^+$ we have, for $D = \exp L$,

$$\mathrm{vol}(KDK) = \int_L J(H)\, dH.$$

Let $x_0 \in X$ be a base point in the symmetric space (X, d_X) and let $B(x_0, r)$ be the ball of radius r around x_0. The exponential growth rate of balls (also called the volume entropy)

$$\delta = \lim_{r \to \infty} \frac{\log(\mathrm{vol}(B_r(x_0)))}{r}$$

of the symmetric space (X, d_X) is given by the formula

$$\delta = 2\rho(H_{\max}).$$

(See [**20**, §5 "Volume estimates"].) It is sometimes convenient to normalize the metrics d_G and d_X defined in §1.3. In the case that the real rank

$$r = \mathrm{rank}_{\mathbb{R}} G = \dim A$$

of G is one, the normalization condition

$$d_X(\exp(H)x_0, x_0) = \beta(H) \quad \text{for all } H \in \mathfrak{a}^+,$$

where β is the unique positive indivisible root, leads to a definition of δ in terms of the root system only:

$$2\rho = \delta\beta.$$

The values of δ are $n-1$ for $\mathrm{SO}(n, 1)$ and $2n$ for $\mathrm{SU}(n, 1)$. See [**30**, Lemma 2.8]. In the case that $X = \mathbb{H}^n$ is the hyperbolic n-space, the normalized metric $d_{\mathbb{H}^n}$ has constant curvature equal to -1.

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We now recall estimates (see [**20**, Lemma 5.4] for a proof) which we will apply in the proof of Proposition 3.3 below: the sets A_T^θ satisfy

$$\mathrm{vol}(K A_T^\theta K) \sim c T^{(r-1)/2} \cdot e^{\delta T}$$

with a constant $c > 0$ *independent* of the choice of θ. As usual, the notation $f \sim g$ for two functions $f, g : [0, \infty[\to \mathbb{R}$ means that $\lim_{T\to\infty} f(T)/g(T) = 1$. For

$$G_T = K \overline{A_T^+} K$$

we also have

$$\mathrm{vol}(G_T) \sim c T^{(r-1)/2} \cdot e^{\delta T}$$

with the same constant c as above.

3.3. *Ergodicity relative to cones.* If $\psi \in L^\infty(G/P, \nu)$, we denote $m(\psi) \in \mathcal{B}(L^2(G/P, \nu))$ the corresponding multiplication operator. Let $P_{\mathbb{1}_{G/P}} \in \mathcal{B}(L^2(G/P, \nu))$ be the orthogonal projection onto the complex line spanned by $\mathbb{1}_{G/P}$. Let Γ be a discrete subgroup of G,

$$\Gamma_T^\theta = B_T^\theta \cap \Gamma$$

and let $|\Gamma_T^\theta|$ denote the cardinality of this finite set. Let $f : G/P \to \mathbb{C}$ be a continuous function. In the case Γ_T^θ is non-empty we may consider the bounded operator

$$M_{\Gamma_T^\theta}^f = \frac{1}{|\Gamma_T^\theta|} \sum_{\gamma \in \Gamma_T^\theta} f(\mathbf{b}(\gamma)) \frac{\lambda_{G/P}(\gamma)}{\Xi(\gamma)}.$$

THEOREM 3.1. (Ergodicity of the quasi-regular representation of a lattice in a semisimple Lie group, relative to cones) *Let G be a non-compact connected semisimple Lie group with finite center. Let P be a minimal parabolic subgroup of G and Γ a lattice in G. Let $\theta > 0$ and let f be a continuous function on G/P. With the notation as above we have*

$$\lim_{T\to\infty} M_{\Gamma_T^\theta}^f = m(f) P_{\mathbb{1}_{G/P}}$$

in the weak operator topology of $\mathcal{B}(L^2(G/P, \nu))$. That is, for any $\varphi, \psi \in L^2(G/P, \nu)$,

$$\lim_{T\to\infty} \frac{1}{|\Gamma_T^\theta|} \sum_{\gamma \in \Gamma_T^\theta} f(\mathbf{b}(\gamma)) \frac{\langle \lambda_{G/P}(\gamma)\varphi, \psi\rangle}{\Xi(\gamma)} = \langle \varphi, \mathbb{1}_{G/P}\rangle \langle f, \psi\rangle.$$

3.4. *Convergence for the square root of the Poisson kernel.* It is convenient to identify the Furstenberg–Poisson boundary $B = G/P$ with the space K/M using the diffeomorphism

$$K/M \to G/P,$$

which sends kM to kP. We consider the unique action of G on K/M which makes this diffeomorphism G-equivariant. We denote ν the push-forward on K/M of the probability Haar measure on K. Let $G = KAN$ be the Iwasawa decomposition defined by $\mathfrak{a}^+$. If $g = kan$ we denote its $\mathfrak{a}$-component by $H_I(g) = \log(a)$. Notice that if $m \in M$ then $H_I(gm) = H_I(g)$. The Radon–Nikodym cocycle for the action of $g \in G$ at the point $kM \in K/M$ is

$$c(g, kM) = e^{-2\rho(H_I(gk))}.$$

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See for example [**16**, Proposition 2.5.4]. The quasi-regular representation of G on $\varphi \in L^2(K/M, \nu)$ is defined as

$$(\pi(g)\varphi)(kM) = \varphi(g^{-1}kM)e^{-\rho(H_I(g^{-1}k))}.$$

See for example [**16**, formula (3.1.12) p. 103]. The Harish–Chandra function is

$$\begin{aligned}\Xi(g) = \langle \pi(g)\mathbb{1}_{K/M}, \mathbb{1}_{K/M}\rangle &= \int_{K/M} e^{-\rho(H_I(g^{-1}k))}\, d\nu(kM)\\ &= \int_K e^{-\rho(H_I(g^{-1}k))}\, dk.\end{aligned}$$

Let $\varphi \in L^1(K/M, \nu)$. The normalized square root of the Poisson kernel is

$$\begin{aligned}(\mathcal{P}_0\varphi)(g) &= \frac{\langle \pi(g)\mathbb{1}_{K/M}, \overline{\varphi}\rangle}{\Xi(g)}\\ &= \frac{\int_{K/M} \varphi(kM)e^{-\rho(H_I(g^{-1}k))}\, d\nu(kM)}{\Xi(g)}.\end{aligned}$$

See [**32**]. Let d be a left K-invariant Riemannian distance on K/M. The formula $L(g) = d_X(gx_0, x_0)$ defines a left G-invariant and K-bi-invariant length function on G.

LEMMA 3.2. (Dirac sequences on Furstenberg–Poisson boundaries) *With the notation as above let $\pi : G \to \mathcal{U}(L^2(K/M, \nu))$ be the quasi-regular representation of G. Let $\theta > 0$ be small enough so that the intersection of $\overline{\mathfrak{a}^\theta}$ with the walls of $\overline{\mathfrak{a}^+}$ is reduced to the origin. For each $r > 0$ there exists a function $h_r : [0, \infty[\to [0, \infty[$ with the following properties:*

(1) *h_r is non-increasing;*
(2) *$\lim_{s\to\infty} h_r(s) = 0$;*
(3) *for all $g \in KA^\theta K$,*

$$\mathcal{P}_0\mathbb{1}_{\{x\in K/M : d(x, \mathbf{b}(g))\geq r\}}(g) \leq h_r(L(g)).$$

Proof. As the space K/M is normal, Urysohn's lemma applies so there exists a continuous function f_r on K/M with the following properties:

- $f_r \geq 0$;
- $f_r(x) = 1$, for all $x : d(x, eM) \geq r$;
- $f_r(x) = 0$, for all $x : d(x, eM) \leq r/2$.

According to [**32**, Theorem 5.1, p. 373] for any $r > 0$ and $s \geq 0$ the supremum

$$h_r(s) = \sup_{H\in\mathfrak{a}^\theta, \|H\|\geq s} \mathcal{P}_0 f_r(\exp(H))$$

is finite and

$$\lim_{s\to\infty} h_r(s) = f_r(eM) = 0.$$

The notation $H \to \infty$ in [**32**] means $\alpha(H) \to \infty$ for any positive root α; sequences contained in the cone $\mathfrak{a}^\theta$ with $\|H\| \to \infty$ obviously verify these conditions because by hypothesis θ is small enough so that the intersection of the closed cone $\overline{\mathfrak{a}^\theta}$ with the walls of $\overline{\mathfrak{a}^+}$ is reduced to the origin.

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Let $g \in KA^{\theta}K$. Let $k \in K$, $H \in \mathfrak{a}^{\theta}$, $l \in K$ such that $g = k\exp(H)l$. Hence $\mathbf{b}(g) = kM$. As the action of K on K/M preserves the measure and as d is K-invariant, we have

$$\pi(k^{-1})\mathbb{1}_{\{x\in K/M:d(x,kM)\geq r\}} = \mathbb{1}_{\{x\in K/M:d(x,eM)\geq r\}}.$$

It follows that

$$\begin{aligned}\mathcal{P}_0\mathbb{1}_{\{x\in K/M:d(x,\mathbf{b}(g))\geq r\}}(g) &= \frac{\langle \pi(k\exp(H)l)\mathbb{1}_{K/M},\, \mathbb{1}_{\{x\in K/M:d(x,kM)\geq r\}}\rangle}{\Xi(k\exp(H)l)}\\ &= \frac{\langle \pi(\exp(H))\mathbb{1}_{K/M},\, \mathbb{1}_{\{x\in K/M:d(x,eM)\geq r\}}\rangle}{\Xi(\exp(H))}\\ &= \mathcal{P}_0\mathbb{1}_{\{x\in K/M:d(x,eM)\geq r\}}(\exp(H)).\end{aligned}$$

As L is K-bi-invariant we conclude that it is enough to prove the lemma in the special case $g = \exp(H)$ with $H \in \mathfrak{a}^{\theta}$. But $\mathcal{P}_0$ preserves positive functions and

$$\mathbb{1}_{\{x\in K/M:d(x,eM)\geq r\}} \leq f_r.$$

This finishes the proof of the lemma. □

3.5. *Counting lattice points and the wave-front lemma.* Recall the notation $f \sim g$ for $\lim_{T\to\infty} f(T)/g(T) = 1$.

PROPOSITION 3.3. (Counting lattice points in sectors) *Let $\theta > 0$. As $T \to \infty$ we have*

$$|\Gamma_T^{\theta}| \sim |\Gamma_T| \sim |\Gamma \cap G_T|.$$

Proof. Notice that if $0 < \theta \leq \phi$ then for all $T > 0$

$$\Gamma_T^{\theta} \subset \Gamma_T^{\phi} \subset \Gamma_T \subset \Gamma \cap G_T.$$

Hence, in proving the proposition, we may assume θ is small. For all $T > 0$,

$$\frac{|(\Gamma \cap G_T)\backslash\Gamma_T^{\theta}| + |\Gamma_T^{\theta}|}{|\Gamma \cap G_T|} = 1.$$

Hence the proposition will be proved if we show that

$$\lim_{T\to\infty} \frac{|(\Gamma \cap G_T)\backslash\Gamma_T^{\theta}|}{|\Gamma \cap G_T|} = 0.$$

To that end we introduce the following notation. For any $\phi > 0$ and $R > 0$ we define

$$\mathfrak{a}^{\phi}(R) = \{H \in \mathfrak{a}^{\phi} : \|H\| \geq R\} = \mathfrak{a}^{\phi}\backslash\mathfrak{a}_R.$$

Let us choose $T_0 > 1/\sin(\theta/2)$. Elementary trigonometry shows that

$$\mathfrak{a}^{\theta/2}(T_0 + 1) + \mathfrak{a}_1 \subset \mathfrak{a}^{\theta}(T_0).$$

As θ is small, the closure of $\mathfrak{a}^{\theta/2}(T_0 + 1)$ in $\mathfrak{a}$ lies at positive distance from the walls of the Weyl chamber $\mathfrak{a}^+$. Hence we can apply the strong wave-front lemma from [**20**, Theorem 2.1] to the (closure) of the subset

$$A^{\theta/2}(T_0 + 1) = \exp(\mathfrak{a}^{\theta/2}(T_0 + 1))$$

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of $\overline{A^+}$ (warning: the closed set $\overline{A^+} = \exp(\overline{\mathfrak{a}^+})$ is denoted A^+ in [**20**]) and to the neighborhood $V = \exp(\mathfrak{a}_1)$ of e in A. The conclusion of [**20**, Theorem 2.1] is the existence of a neighborhood $\mathcal{O}$ of e in G with the following property: any $g = k\exp(H)l$ with $k, l \in K$ and $H \in A^{\theta/2}(T_0 + 1)$ satisfies

$$g\mathcal{O}^{-1} \subset K\exp(H)VK.$$

In other words

$$KA^{\theta/2}(T_0 + 1)K\mathcal{O}^{-1} \subset KA^{\theta/2}(T_0 + 1)VK. \tag{3}$$

Choosing $\mathcal{O}$ smaller if needed, we may assume that it satisfies the following additional properties:

$$\Gamma \cap (\mathcal{O} \cdot \mathcal{O}^{-1}) = \{e\}, \quad G_T\mathcal{O} \subset G_{T+1} \quad \text{for all } T > 0.$$

Combining the inclusion (3) with

$$A^{\theta/2}(T_0 + 1)V = \exp(\mathfrak{a}^{\theta/2}(T_0 + 1) + \mathfrak{a}_1) \subset \exp(\mathfrak{a}^{\theta}(T_0)) = A^{\theta}(T_0),$$

we deduce that

$$KA^{\theta/2}(T_0 + 1)K\mathcal{O}^{-1} \subset KA^{\theta}(T_0)K.$$

This in turn implies

$$((\Gamma \cap G_T)\backslash KA^{\theta}(T_0)K)\mathcal{O} \subset G_{T+1}\backslash KA^{\theta/2}(T_0 + 1)K.$$

As $\Gamma \cap (\mathcal{O} \cdot \mathcal{O}^{-1}) = \{e\}$ the union $\bigcup_{\gamma\in\Gamma} \gamma\mathcal{O}$ is disjoint, hence

$$\begin{aligned}
|(\Gamma \cap G_T)\backslash\Gamma_T^{\theta}| &\le |(\Gamma \cap G_T)\backslash KA^{\theta}(T_0)K| \\
&= \frac{1}{\mathrm{vol}(\mathcal{O})}\mathrm{vol}\Bigg(\bigcup_{\gamma\in(\Gamma\cap G_T)\backslash KA^{\theta}(T_0)K} \gamma\mathcal{O}\Bigg) \\
&\le \frac{1}{\mathrm{vol}(\mathcal{O})}\mathrm{vol}(G_{T+1}\backslash KA^{\theta/2}(T_0 + 1)K) \\
&\le \frac{1}{\mathrm{vol}(\mathcal{O})}(\mathrm{vol}(G_{T+1}\backslash KA^{\theta/2}K) + \mathrm{vol}(G_{T_0+1})).
\end{aligned}$$

This finishes the proof of the proposition because according to [**20**, Lemma 5.4], for all $\phi > 0$,

$$\lim_{T\to\infty} \frac{\mathrm{vol}(G_T\backslash KA^{\phi}K)}{\mathrm{vol}(G_T)} = 0,$$

and because there exists $C > 0$ such that for all $T > 1$

$$\mathrm{vol}(G_{T+1}) \le C\mathrm{vol}(G_T), \tag{4}$$

and since according to [**15**]

$$|\Gamma \cap G_T| \cdot \mathrm{vol}(G/\Gamma) \sim \mathrm{vol}(G_T) \tag{5}$$

as $T \to \infty$. □

In the following lemma we state an upper bound, needed in the proof of the main theorem, for the asymptotic distribution of couples of lattice points of the form (γ, γ^{-1}) in the Furstenberg–Poisson boundary. (The inequality in the lemma is in fact an equality but we don't need it.)

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LEMMA 3.4. *Let U, V be Borel subsets of K/M with $\nu(\partial U) = \nu(\partial V) = 0$. Let $\theta > 0$. Then*

$$\limsup_{T\to\infty} \frac{|\{\gamma \in \Gamma_T^\theta : \mathbf{b}(\gamma) \in U \text{ and } \mathbf{b}(\gamma^{-1}) \in V\}|}{|\Gamma_T^\theta|} \leq \nu(U)\nu(V).$$

Proof. Let μ be the probability Haar measure on K. Let $p : K \to K/M$ be the canonical projection. We have $p_*\mu = \nu$. Let $\tilde{U} = p^{-1}(U)$ and $\tilde{V} = p^{-1}(V)$. Let $N(A)$ be the normalizer of A in G and let $M' = N(A) \cap K$. Recall that the Weyl group is the quotient $W = M'/M$ and that it contains a unique element s_0 which sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Let $m_0 \in M'$ such that $m_0 M = s_0$.

We claim that for all $T > 0$,

$$\{\gamma \in \Gamma_T^\theta : \mathbf{b}(\gamma) \in U \text{ and } \mathbf{b}(\gamma^{-1}) \in V\} \subset \Gamma \cap \tilde{U}\overline{A_T^+}m_0^{-1}(\tilde{V})^{-1}. \tag{6}$$

Let $\gamma \in \Gamma_T^\theta$ such that $\mathbf{b}(\gamma) \in U$ and $\mathbf{b}(\gamma^{-1}) \in V$. According to the Cartan decomposition there exist $k, l \in K$ and $H \in \mathfrak{a}_T^+$ such that $\gamma = k\exp(H)l$. The proof will be complete if we show that $k \in \tilde{U}$ and $l \in m_0^{-1}(\tilde{V})^{-1}$. By definition $\mathbf{b}(\gamma) = kM = p(k)$ and by hypothesis $\mathbf{b}(\gamma) \in U$. Hence $k \in \tilde{U}$. Recall that the *opposition involution*

$$\iota : \mathfrak{a}^+ \to \mathfrak{a}^+, \quad H \mapsto -\mathrm{Ad}(m_0)H$$

satisfies $\exp(-H) = m_0^{-1}\exp(\iota H)m_0$ for all $H \in \mathfrak{a}^+$. We have

$$\gamma^{-1} = l^{-1}\exp(-H)k^{-1} = l^{-1}m_0^{-1}\exp(\iota H)m_0k^{-1}.$$

As $l^{-1}m_0^{-1} \in K$, $\iota H \in \mathfrak{a}^+$ and $m_0k^{-1} \in K$, this proves that $\mathbf{b}(\gamma^{-1}) = l^{-1}m_0^{-1}M = p(l^{-1}m_0^{-1})$. By hypothesis $\mathbf{b}(\gamma^{-1}) \in V$ hence $l^{-1}m_0^{-1} \in \tilde{V}$ and $l \in m_0^{-1}(\tilde{V})^{-1}$. This finishes the proof of the claim.

We apply [**20**, Theorem 1.6] with $\Omega_1 = \tilde{U}$ and $\Omega_2 = m_0^{-1}(\tilde{V})^{-1}$. Notice that for $i = 1, 2$, $\mu(\partial\Omega_i) = 0$ because $\nu(\partial U) = \nu(\partial V) = 0$. Since $\mu(\Omega_1 M) = \mu(\tilde{U}M) = \mu(\tilde{U}) = \nu(U)$,

$$\begin{aligned}\mu(M\Omega_2) &= \mu(Mm_0^{-1}(\tilde{V})^{-1}) = \mu(m_0Mm_0^{-1}(\tilde{V})^{-1})\\ &= \mu(M(\tilde{V})^{-1}) = \mu(\tilde{V}M) = \mu(\tilde{V}) = \nu(V)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}\overline{A_T^+}m_0^{-1}(\tilde{V})^{-1} &= \tilde{U}M\overline{A_T^+}m_0^{-1}(\tilde{V})^{-1} = \tilde{U}\overline{A_T^+}Mm_0^{-1}(\tilde{V})^{-1}\\ &= \Omega_1\overline{A_T^+}M\Omega_2,\end{aligned}$$

we obtain

$$|\Gamma \cap \tilde{U}\overline{A_T^+}m_0^{-1}(\tilde{V})^{-1}|\mathrm{vol}(G/\Gamma) \sim \nu(U)\nu(V)\mathrm{vol}(G_T)$$

as $T \to \infty$. Applying the equivalence

$$|\Gamma \cap G_T| \cdot \mathrm{vol}(G/\Gamma) \sim \mathrm{vol}(G_T)$$

as $T \to \infty$, Proposition 3.3 and the inclusion (6) finish the proof of the lemma. □

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3.6. *Uniformly bounded family of Markov operators.*

PROPOSITION 3.5. *Let $E \subset G$ be a Borel subset of finite non-zero Haar measure. Suppose $KE = E$. Then*

$$M_E^{\mathbb{1}_{K/M}} \mathbb{1}_{K/M}(x) = 1 \quad \text{for all } x \in K/M.$$

Proof. As E is K-invariant, a change of variables shows that the function is constant on any orbit of K. As the action of K is transitive the function is constant. Applying Fubini shows that its integral on the probability space $(K/M, \nu)$ equals one. □

PROPOSITION 3.6. *For any $\theta > 0$,*

$$\sup_{T>1} \|M_{\Gamma_T^\theta}^{\mathbb{1}_{K/M}} \mathbb{1}_{K/M}\|_\infty < \infty.$$

Proof. Let U be a neighborhood of e in G as in Lemma 2.17, small enough so that for all $T > 0$ we have $\Gamma_T^\theta U \subset G_{T+1}$, and let $C > 0$ as in Lemma 2.17. For any $x \in K/M$ we have

$$\begin{aligned} M_{\Gamma_T^\theta}^{\mathbb{1}_{K/M}} \mathbb{1}_{K/M}(x) &= \frac{1}{|\Gamma_T^\theta|} \sum_{\gamma \in \Gamma_T^\theta} \frac{\pi(\gamma)\mathbb{1}_{K/M}(x)}{\Xi(\gamma)} \\ &\le \frac{C}{|\Gamma_T^\theta|} \int_{\Gamma_T^\theta U} \frac{\pi(g)\mathbb{1}_{K/M}(x)}{\Xi(g)}\, dg \\ &\le \frac{C}{|\Gamma_T^\theta|} \int_{G_{T+1}} \frac{\pi(g)\mathbb{1}_{K/M}(x)}{\Xi(g)}\, dg. \end{aligned}$$

Applying the inequality (4) and the equivalence (5) as well as Proposition 3.3, we see that this integral is bounded above by

$$C' \frac{1}{\mathrm{vol}(G_{T+1})} \int_{G_{T+1}} \frac{\pi(g)\mathbb{1}_{K/M}(x)}{\Xi(g)}\, dg,$$

where $C' > 0$ is a constant which neither depends on $T > 1$ nor on $x \in K/M$. Since $KG_T = G_T$ for any $T > 0$, Proposition 3.5 applies and we obtain

$$\frac{1}{\mathrm{vol}(G_{T+1})} \int_{G_{T+1}} \frac{\pi(g)\mathbb{1}_{K/M}(x)}{\Xi(g)}\, dg = 1. \qquad \square$$

3.7. *Proof of Theorem 3.1 in the case that θ is small.*

Proof. We assume θ is small enough so that the intersection of $\overline{\mathfrak{a}^\theta}$ with the walls of $\overline{\mathfrak{a}^+}$ is reduced to the origin. We deduce Theorem 3.1 from Theorem 2.2. Let us check that all the hypotheses of Theorem 2.2 are satisfied in the case of a lattice in a non-compact connected semisimple Lie group with finite center. As θ is small we can apply Lemma 3.2. Hence the first condition is satisfied. Applying Lemma 3.4 we see that the second condition is satisfied. Proposition 3.6 implies that

$$\sup_{T>1} \|M_{\Gamma_T^\theta}^{\mathbb{1}_{K/M}} \mathbb{1}_{K/M}\|_\infty < \infty.$$

As K/M is a compact differentiable manifold (and ν is equivalent to the Lebesgue measure), characteristic functions of the type $\mathbb{1}_W$, with $W \subset K/M$ Borel such that $\nu(\partial W) = 0$ span a subset of $L^\infty(K/M, \nu)$ whose closure contains the continuous functions on K/M. □

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3.8. *Proofs of Theorems 1.2 and 3.1.* Notice that for ϕ large enough we have $\mathfrak{a}^{\phi} = \mathfrak{a}^{+}$, and therefore

$$\Gamma_T^{\phi} = B_T^{\phi} \cap \Gamma = K A_T^{\phi} K \cap \Gamma = K A_T K \cap \Gamma = \Gamma_T.$$

Hence Theorem 1.2 is a special case of Theorem 3.1. We now prove Theorem 3.1.

Proof. Applying Proposition 3.6 and Lemma 2.14 we deduce that for any $\phi > 0$ and any continuous function f on K/M,

$$\sup_{T>1} \|M^f_{\Gamma_T^{\phi}}\|_{\mathrm{op}} < \infty.$$

Suppose $\phi > 0$ is given. Let $0 < \theta < \phi$ be small enough so that we can apply the already proven special case of Theorem 3.1. Hence, for any continuous function f on K/M the quasi-regular representation π is ergodic relative to $(\Gamma_T^{\theta}, \mathbf{b}|_{\Gamma_T^{\theta}})$. It follows from Proposition 3.3 that

$$\lim_{T\to\infty} \frac{|\Gamma_T^{\phi} \backslash \Gamma_T^{\theta}|}{|\Gamma_T^{\phi}|} = 0.$$

Hence we can apply Proposition 2.10. We deduce that for any continuous function f on K/M the quasi-regular representation π is ergodic relative to $(\Gamma_T^{\phi}, \mathbf{b}|_{\Gamma_T^{\phi}})$. □

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