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Geometric graphs

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In this chapter we address a colourful area of geometric graph theory: finding the chromatic number of the plane and related problems. In addition to results, we present open problems of a classical kind that are easy to understand, but hard to solve.

1. The chromatic number of the plane

In the definition of a graph, edges symbolize the adjacency of points and nothing else: we ignore their shape and length. However, the past century has witnessed a great deal of interest in *geometric graphs*, where geometrical considerations such as distance define the adjacency. The wealth of material related to geometric graphs is so vast that one can easily imagine a book written on this topic alone. In view of space and time limitations and our emphasis on chromatic graph theory,we have chosen to go deep and address a small but colourful area of geometric graphs: the problem of finding the chromatic number of the plane and related problems. My monograph [45]

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contains further material on this topic, although there is some new material here that did not appear in that book.

We can create a graph G from the Euclidean plane E^2 by taking all of its points as vertices, and joining two vertices by an edge if and only if they are at distance 1 apart. More generally, we call a graph *unit-distance* when any two vertices are adjacent if and only if they are at distance 1 apart. The main open problem in the subject is as follows.

Problem *Find the chromatic number of the above graph G.*

This number is called the *chromatic number of the plane* (CNP) and is denoted by $\chi(E^2)$. As outlined in [45], this problem was created in late 1950 by the 18-year-old Edward Nelson, who determined a lower bound; his 20-year-old friend John Isbell found an upper bound:

$$4 \leq \chi(E^2) \leq 7 - \text{that is, } \chi(E^2) = 4, 5, 6 \text{ or } 7.$$

There are similarities between the problem of finding the chromatic number of the plane (CNP) and the four-colour problem (4CP); both refer to plane colourings and both withstood all assaults for a long time, but whereas the latter problem was eventually solved, the former one remains open. Both problems had their distinguished promoters who kept the problems alive in their infancy: in the case of the 4CP it was Augustus De Morgan, and in the case of the CNP problem it was Paul Erdős. Yet, the problems differ vastly in their history. In 1890 P. J. Heawood proved the five-colour theorem, showing that the answer to the four-colour problem is 4 or 5. After over sixty years of intensive work, using tools from geometry, graph theory, abstract algebra, topology and measure theory, we still today have no improvement in the general case on the above bounds for $\chi(E^2)$, a surprisingly wide spread.

In a 2002 lecture, Ronald L. Graham offered a reward for the first solution to a special case that he called 'another four-colour conjecture'.

Problem *Is it possible to* 4*-colour the plane so as to forbid a monochromatic distance* 1*?*

Graham now believes that the chromatic number of the plane is 5 or 6 (see [19] and [20]). He cited a theorem of O'Donnell [31] showing the existence of 4-chromatic unit-distance graphs of arbitrarily large girth (see Theorem 3.2 below) as 'perhaps, the evidence that χ is at least 5'. Erdős [13], [14] believed that the chromatic number of the plane is 5, 6 or 7. Somewhat disjointly from Graham and Erdős, I predict that the answer is either 4 or 7.

2. The polychromatic number: lower bounds

Numerous problems related to the chromatic number of the plane have been posed. Let us look at one of them here. We say that a point set S *realizes the distance* d if S contains two points at distance d apart. In 1958 Erdős posed the following question.

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Problem *What is the smallest number of colours needed to colour the plane so that no colour realizes all distances?*

In 1992 Soifer [40] named this invariant *the polychromatic number of the plane*, and denoted it by $\chi_p(E^2)$. Clearly $\chi_p(E^2) \leq \chi(E^2) \leq 7$. In 1970 upper and lower bounds for $\chi_p(E^2)$ were published by a Russian high-school student, Dmitry Raiskii [37]. We look at the lower bound here and return to the upper bound in Section 4.

In 1961 Leo and Willie Moser produced the seven-point plane configuration that we now call the *Mosers spindle* (see Fig. 1): each edge in the spindle has length 1. In [29] they proved the following result.

Theorem 2.1 *Any three points of the Mosers spindle contain two points at distance* 1 *apart. Consequently, in a colouring of the Mosers spindle that forbids monochromatic distance* 1*, at most two points can be of the same colour.*

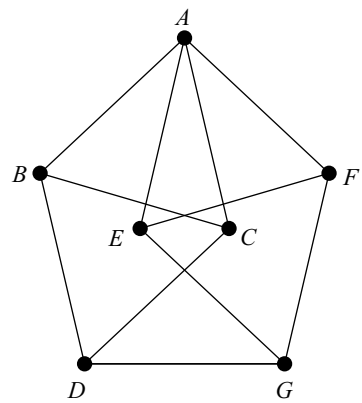


Fig. 1. The Mosers spindle

Raiskii's lower bound was as follows. A striking proof was found in 1997 by Alexei Merkov, another Russian high-school student, and appeared in an obscure brochure [34]. The following proof is due to Merkov, with modifications by the author.

Theorem 2.2 $\chi_p(E^2) \geq 4$.

Proof We assume that the plane is coloured in three colours, red, white and blue, and that each colour forbids a distance r, w and b, respectively. Equip the 3-coloured plane with Cartesian coordinates with origin O, and construct three 7-point sets S_r, S_w and S_b, each being the Mosers spindle (Fig. 2), in such a way that all three spindles share O as one of their seven vertices and have edges coloured r, w and b, respectively. This construction defines eighteen vectors: six 'red' ones, $v_1, v_2, \ldots, v_6$ from the origin O to each remaining point of S_r, six 'white' ones, $v_7, v_8, \ldots, v_{12}$ from O to the points of S_w, and six 'blue' ones $v_{13}, v_{14}, \ldots, v_{18}$ from O to the points of S_b.

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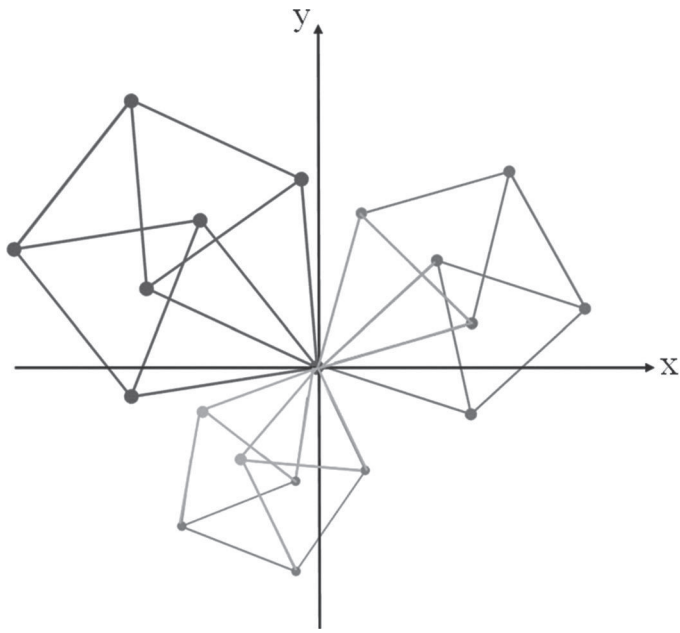


Fig. 2. The Cartesian plane with three Mosers spindles

We now introduce 18-dimensional Euclidean space E^{18} and a function M: $E^{18} \to E^2$ that maps each vector $(a_1, a_2, \ldots, a_{18})$ to $a_1v_1 + a_2v_2 + \ldots + a_{18}v_{18}$. This function induces a 3-colouring of E^{18} by assigning to a point of E^{18} the colour of the corresponding point of the plane. In accordance with the colours of the vectors v_i,we call the first six axes of E^{18} 'red', the next six axes 'white', and the last six axes 'blue'.

Let W be the subset of E^{18} consisting of all points whose coordinates include at most one equal to 1 for each of the three colours of the axes, and the remaining (fifteen or more) coordinates equal to 0; it is easy to verify that W has $7^3 = 343$ points. For any fixed array of coordinates allowable in W on white and blue axes, we get the 7-element set A of points in W with fixed coordinates on white and blue axes. The image $M(A)$ forms in the plane a translate of the original 7-point set S_r. If we fix another array of white and blue coordinates we obtain another 7-element set in E^{18}, whose image under M in the plane would form another translate of S_r. Thus, the set W gets partitioned into 49 subsets, each of which maps into a translate of S_r.

By Theorem 2.1, any translate of S_r has at most two red points among its 7 points. Since W has been *partitioned* into translates of S_r, at most $^2/_7$ of the points of Warered. We can now start all over again, and show similarly that at most $^2/_7$ of the points of W are white, and at most $^2/_7$ of the points are blue. But $^2/_7 + ^2/_7 + ^2/_7 < 1$. This contradiction implies that at least one of the colours realizes all distances, as required. ■

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3. The de Bruijn–Erdős reduction to finite sets

We can expand the notion of the chromatic number to any subset S of the plane. The *chromatic number* $\chi(S)$ is the smallest number of colours sufficient to colour the points of S in such a way that forbids monochromatic pairs of points at distance 1 apart.

In 1951 de Bruijn and Erdős [11] published a powerful lemma that implies the following 'compactness theorem'; it assumes the axiom of choice.

Theorem 3.1 *The chromatic number of the plane is equal to the maximum chromatic number of its finite subsets.*

Accordingly, Erdős used to claim that the problem of finding the chromatic number of the plane is a problem about finite sets in the plane. In 1975 he posed the following problem [12].

Problem *Let S be a subset of the plane which contains no equilateral triangle of side length* 1. *Join two points of S if their distance is* 1. *Does this graph have chromatic number at most* 3*? If the answer is no, assume that the graph defined by S contains no cycles C_l of length l, for $3 \leq l \leq t$, and ask the same question.*

Erdős was unsure of the outcome: he expected triangle-free unit distance graphs to have chromatic number at most 3, or else that a chromatic number of 3 can be forced by prohibiting all small cycles up to C_t, for sufficiently large t.

In 1979 Wormald [51] disproved the first (easier) triangle-free conjecture above, and Erdős promptly reported the result in a lecture and later in print (see [15]):

In a recent paper (still unpublished), Wormald found a set S for which the unit distance graph $G_1(S)$ has girth 5 *and chromatic number* 4. *His construction involved elaborate computations and is fairly complicated. Indeed, aided by a computer, he had proved in* [51] *the existence of a set S of* 6448 *points with chromatic number* 4 *and without triangles or quadrilaterals with all sides of length* 1.

In a talk in 1992 the author shared this problem of Erdős, in the form of a competition:

Problem *Find the smallest number σ_4 of points in a plane set with chromatic number* 4 *without unit equilateral triangles, and classify all such sets S of σ_4 points.*

A number of young mathematicians entered the race, and the graphs obtained by the record setters were as mathematically significant as they were beautiful (see [45]). The two record-holding graphs, created by Hochberg and O'Donnell [22], are shown in Figs. 3 and 4; both graphs are 4-chromatic unit-distance graphs.

The problem of finding a 5-chromatic unit distance graph – or proving that none exists – remains open. However, much has been learned about 4-chromatic unit-distance graphs. The best of these results appeared in O'Donnell's Ph.D. thesis [31], where he answered Erdős's problem above in the negative (see [45, Theorem 48.4]). O'Donnell's result can be stated as follows.

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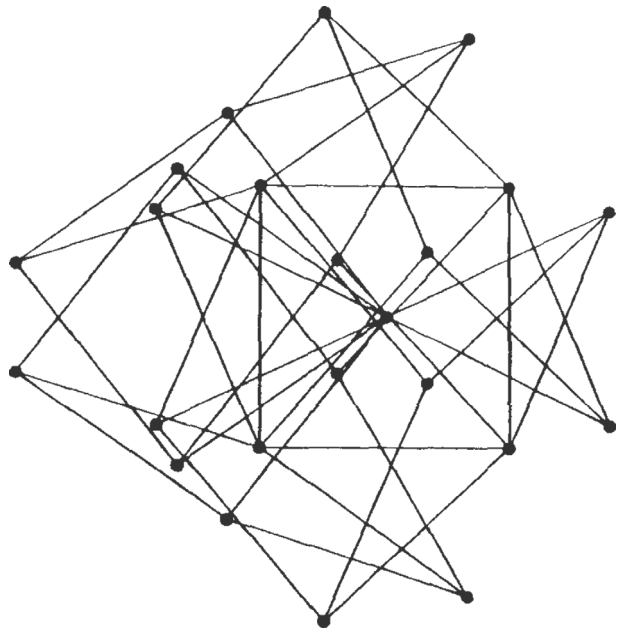


Fig. 3. The Hochberg-O'Donnell fish graph, of girth 4 and order 23

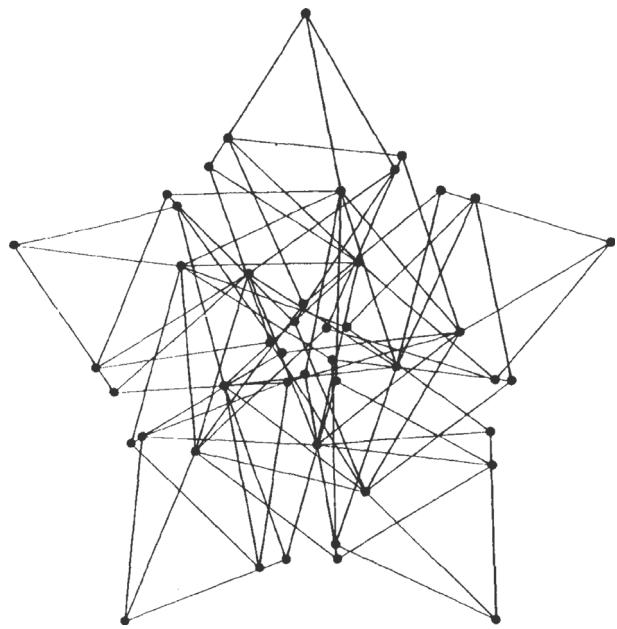


Fig. 4. The Hochberg-O'Donnell star graph, of girth 5 and order 45

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Theorem 3.2 *There exist* 4*-chromatic unit-distance graphs with arbitrarily large finite girth.*

4. The polychromatic number: upper bounds

Raiskii's paper [37] also contained the upper bound $\chi_p(E^2) \leq 6$. An example proving this bound was found by Stechkin and published by Raiskii. We present Stechkin's construction (see Fig. 5).

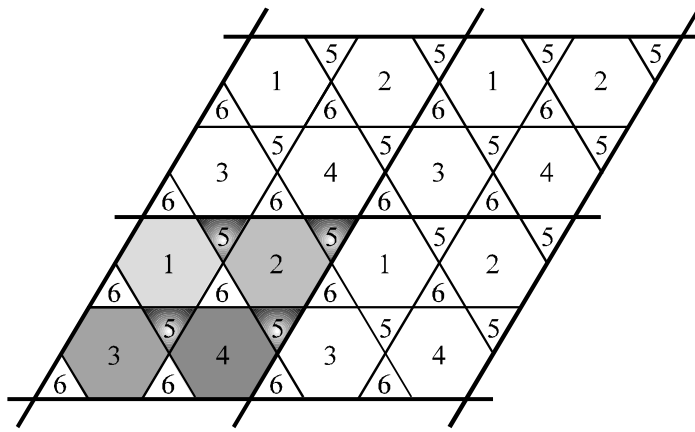


Fig. 5. Stechkin's 6-colouring of the plane

The 'unit of construction' is a parallelogram consisting of four regular hexagons and eight equilateral triangles, all of side-length 1. We colour the hexagons with colours 1, 2, 3 and 4. We then partition the triangles into two types: assign colour 5 to the triangles with a vertex below their horizontal base, and colour 6 to those with a vertex above their horizontal base. While colouring, assume that every hexagon includes its entire boundary, except for its one right-most and two lowest vertices, and that every triangle includes none of its boundary points. We can now tile the entire plane with translates of the 'unit of construction'.

If our ultimate goal is to find the chromatic number of the plane, or at least to improve the known bounds of 4 and 7, it might be worthwhile to 'measure' how close a given colouring of the plane is to achieving this goal. In 1992 such a measurement was introduced by the author and named the *colouring type* (see Soifer [40], [41]): given an n-colouring of the plane for which colour i does not realize the distance d_i (for $1 \leq i \leq n$), we say that this colouring is of *type* $(d_1, d_2, \ldots, d_n)$.

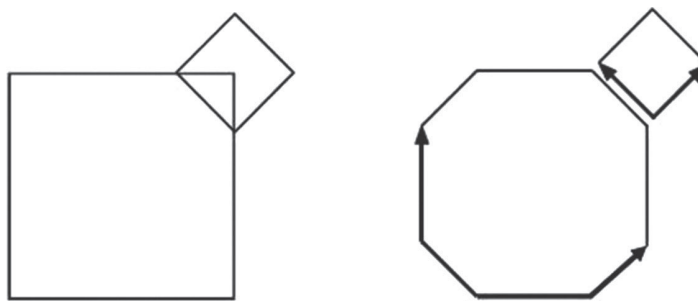
It would greatly improve our search for the chromatic number of the plane if we could find a 6-colouring of type (1,1,1,1,1,1), or show that one does not exist. With an appropriate choice of unit, Stechkin's colouring in Fig. 5 has type (1,1,1,1,$^1/_2$, $^1/_2$). In 1973 Woodall [50] found a second 6-colouring of the plane with no colour

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realizing all distances; his colouring had the special property that each of the six monochromatic sets is closed. His example, however, had three 'missing distances': it had type $(1,1,1,\ 1/\sqrt{3},\ 1/\sqrt{3},\ 1/2\sqrt{3})$.

In 1991 a new 6-colouring was found by the author [51], using a tiling of the plane by squares and non-regular octagons; it had type $(1,1,1,1,1,\ 1/\sqrt{5})$. To construct it, we start with two squares, one with side 2 and the other with diagonal 1 (see Fig. 6), and use them to create a tiling of the plane with squares and non-regular octagons (see Fig. 8). We use colours 1–5 for the octagons and colour 6 for all the squares. With each octagon and each square we include half of its boundary (bold lines in Fig. 7) without the endpoints of that half. It is easy to verify that $\sqrt{5}$ is not realized by any of the colours 1–5, and that 1 is not realized by colour 6. By shrinking all linear sizes by a factor of $\sqrt{5}$, we obtain the 6-colouring of type $(1,1,1,1,1,\ 1/\sqrt{5})$.

To simplify the verification, we define the unit of construction as the region bounded by the bold line in Fig. 8; its translates tile the plane.



Figs. 6 and 7

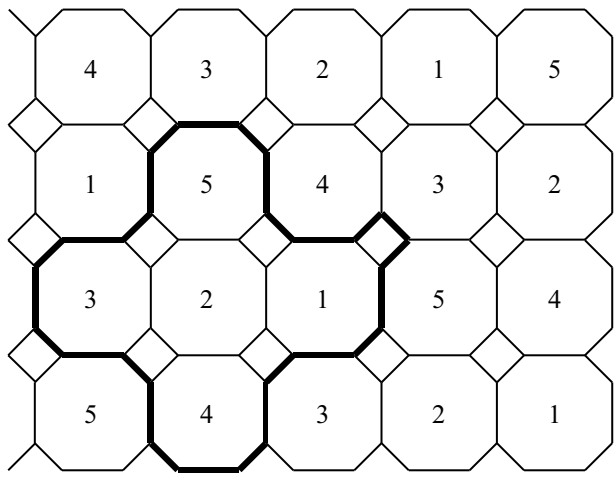


Fig. 8. Soifer's 6-colouring of the plane

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The above 6-colouring gave birth to a new definition (see [23]). The *almost chromatic number* $\chi_a(E^2)$ of the plane is the minimum number of colours that are required to colour the plane so that all but one of the colours forbid distance 1, and the remaining colour forbids a distance that is not necessarily 1. Note that $\chi_a(E^2) = 4$, 5 or 6; the lower bound follows from Raiskii [37], and the upper bound follows from the above 6-colouring. The problem of determining $\chi_a(E^2)$ is still open (see [45]).

5. The continuum of 6-colourings

In 1993 Hoffman and the author [23], [24] found another 6-colouring, with type $(1, 1, 1, 1, 1, \sqrt{2} - 1)$; the story of its discovery can be found in [45]. To construct it we first tile the plane with squares of diagonals 1 and $\sqrt{2} - 1$(see Fig. 9). We use colours 1–5 for the larger squares and colour 6 for all the small squares. With each square we include the left and lower sides of its boundary without the endpoints of this half (see Fig. 10). To verify that this colouring does the job, define the unit of construction that is bounded by the bold line in Fig. 9; its translates tile the plane.

In 1993 the above two examples prompted the introduction of the new terminology and the translation of earlier results and problems into this new language (see [42],

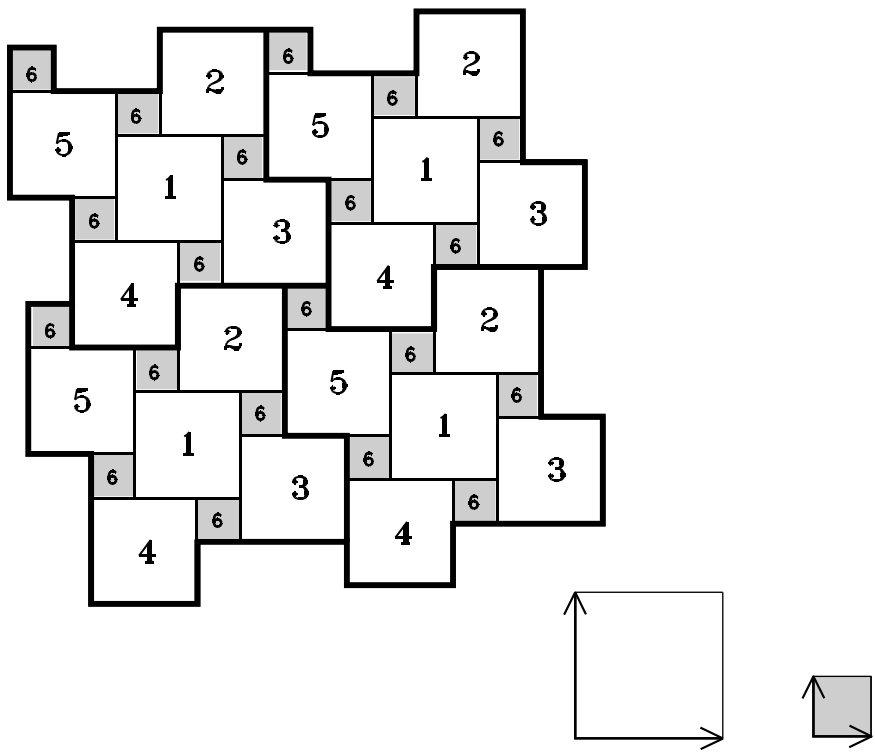


Fig. 9. Hoffman–Soifer's 6-colouring of the plane

Fig. 10.

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[43]). We let X_6 denote the 6-*realizable set* of all positive numbers α for which there exists a 6-colouring of the plane of type $(1, 1, 1, 1, 1, \alpha)$.The problem, which is still open and extremely difficult, is to find X_6.

We know from our above discussion that $1/\sqrt{5}$ and $\sqrt{2} - 1$ both lie in X_6. As shown in [42] and [43](see also [45]), these are extreme examples of the general case, which includes a continuum of 'working' 6-colourings: for each α between $\sqrt{2} - 1$ and $1/\sqrt{5}$, there is a 6-colouring of type $(1, 1, 1, 1, 1, \alpha)$.

Theorem 5.1 *X_6 contains the closed interval* $[\sqrt{2} - 1, 1/\sqrt{5}]$.

Outline of proof Partly cover a unit square by a smaller square that cuts off the unit square's vertical and horizontal segments (say, of lengths x and y, respectively) and forms with it an angle ω (see Fig. 11). These squares induce a tiling of the plane that consists of congruent non-regular octagons and 'small' squares (see Fig. 12).

We now colour this tiling in six colours. Denote by F the unit of our construction, bounded by a bold line and consisting of five octagons and four 'small' squares. Use colours 1–5 for the octagons inside F and colour 6 for all 'small' squares, and include in the colours of octagons and 'small' squares those parts of their boundaries that are shown in bold in Fig. 13. Translates of F tile the plane and thus determine the 6-colouring of the plane.

We now wish to select parameters to guarantee that each colour forbids a distance. In fact, a stronger result was proved (see [42], [43]) – namely that, for each angle ω between the small and the large squares in Fig. 11, there are uniquely determined sizes of the two squares, and uniquely determined parameters x and y, for which the constructed 6-colouring has type $(1, 1, 1, 1, 1, \alpha)$ for a uniquely determined α.

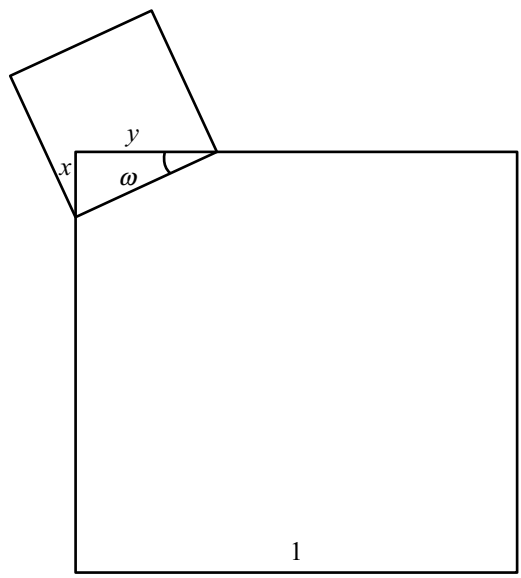


Fig. 11.

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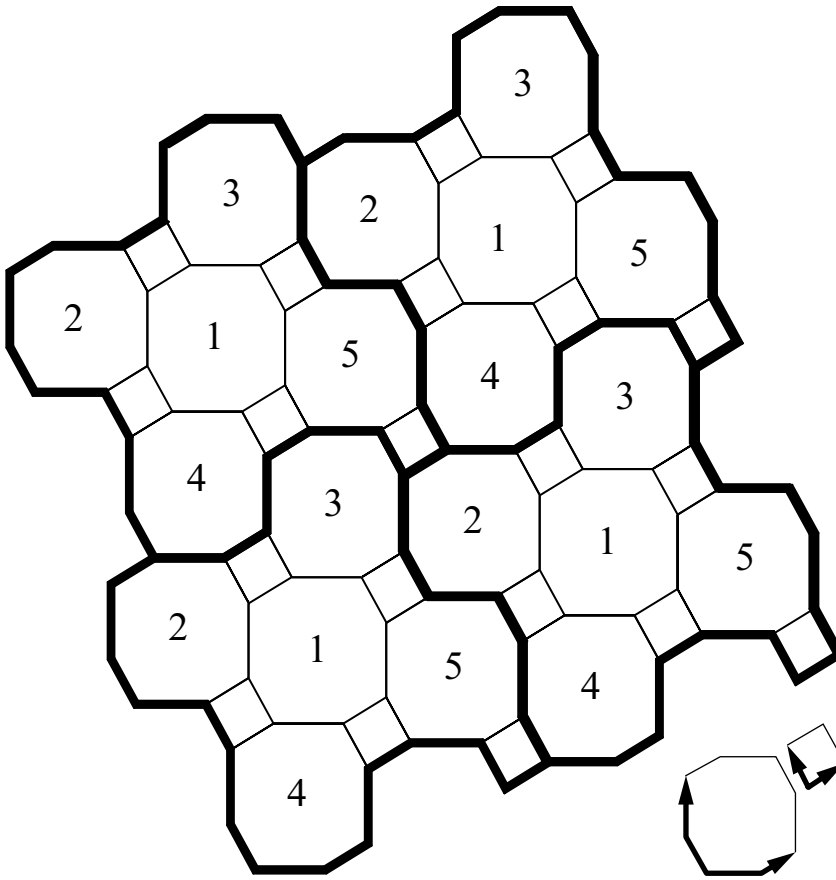


Fig. 12. Continuum of 6-colorings of the plane

Fig. 13.

Note that the 'working' solutions barely exist – they comprise something of a curve in three-dimensional space of the angle ω and two linear variables x and y. We have thus found a continuum of permissible values for α and a continuum of 'working' 6-colourings of the plane.

We remark that the problem of finding the 6-realizable set X_6 is closely related to the problem of finding the chromatic number $\chi(E^2)$ of the plane: finding X_6 would shed light on – if not solve – the latter problem, for if 1 is in X_6 then $\chi(E^2) \leq 6$, and if 1 is not in X_6 then $\chi(E^2) = 7$.

6. Special circumstances

In 1973 Woodall [50] attempted to prove a lower bound for the chromatic number of the plane for the special case of 'map-type colourings' of the plane. However, in 1979

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Stephen Townsend constructed a counter-example that showed that one essential idea of Woodall's proof was incorrect. By that time Townsend had already proved the same result, and his proof was much more elaborate than Woodall's attempt. For decades Townsend's proof was unavailable until he produced a clear version of it where the complete definition of the map-type colouring can also be found (see [45]). The following result was thus conjectured by Woodall and proved by Townsend.

Theorem 6.1 *The chromatic number of the plane under map-type colouring is* 6 *or* 7.

Woodall [50] also showed that this result implies another one worth mentioning.

Theorem 6.2 *The chromatic number of the plane under colouring with closed monochromatic sets is* 6 *or* 7.

In 1993–94 three American undergraduate students, Nathanial Brown, Nathan Dunfield and Greg Perry, proved that a similar result is true for colouring with open monochromatic sets (see [3], [4] and [5]).

Theorem 6.3 *The chromatic number of the plane under colouring with open monochromatic sets is* 6 *or* 7.

Meanwhile, Falconer [16], while still a graduate student, proved the following important result, that the *measurable chromatic number* $\chi_m(E^2)$ of the plane is 5, 6 or 7.

Theorem 6.4 *If* $E^2 = \bigcup_{i=1}^{4} A_i$ *is a covering of the Euclidean plane* E^2 *by four disjoint measurable sets, then one of the sets* A_i *realizes distance* 1.

Recently, Falconer wrote a more detailed and self-contained exposition (see [45]).

Theorem 6.4 implies that if a 4-colouring of the plane forbids monochromatic distance 1, then one of the classes will not be 'nice' (it will be non-measurable). There is a supplementary result by Thomassen [49], who proved that if the colour classes are sufficiently 'nice' and forbid monochromatic distance 1, then seven colours are needed.

7. Space explorations

Around 1961 Erdős generalized the problem of finding the chromatic number of the plane to n-dimensional Euclidean space E^n. He was interested both in asymptotic behaviour as n increases, and in exact values of the chromatic number $\chi(E^n)$ for small n (especially $n = 2$ and 3).

In 1970 Raiskii [37] proved that $\chi(E^n) \geq n + 2$, for all $n > 1$; thus, for $n = 3$ we have $\chi(E^3) \geq 5$. This lower bound for E^3 lasted until 2000, when Oren Nechushtan

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proved that $\chi(E^3) \geq 6$ (see [30]). For upper bounds Coulson [9], [10] proved that $\chi(E^3) \leq 15$, by using a face-centred cubic lattice (see Conway and Sloane [8] for more on cubic lattices). Coulson informally conjectured that his upper bound of 15 is best possible for lattice-based colouring. For higher dimensions, Cantwell [6] proved in 1996 that $\chi(E^4) \geq 7$ and $\chi(E^5) \geq 9$; these remain the best results known. Then, in 2008, Josef Cibulka [7] proved that $\chi(E^6) \geq 11$.

Many years ago, Erdős conjectured that the chromatic number $\chi(E^n)$ increases exponentially with n. This conjecture was settled in the affirmative by two results, an exponential upper bound, found in 1972 by Larman and Rogers [26], and an exponential lower bound obtained in 1981 by Frankl and Wilson [18]:

$$\text{for all } n,\ (1 + o(1))1.2^n \leq \chi(E^n) \leq (3 + o(1))^n.$$

Asymptotically, Larman and Rogers's upper bound remains the best possible today. In 2000 Frankl and Wilson's asymptotic lower bound was improved by Raigorodskii [36] to $(1.239\ldots + o(1))^n$. It would be desirable to narrow the gap between these bounds.

The polychromatic number $\chi_p(E^2)$ also generalizes to higher dimensions. Raiskii [37] was the first to obtain a result, $\chi_p(E^n) \geq n + 2$, for all $n > 1$. Larman and Rogers's upper bound implies that $\chi_p(E^n) \leq (3 + o(1))^n$. They conjectured that $\chi_p(E^n)$ grows exponentially with n, and this was proved by Frankl and Wilson [18], who showed that $(1 + o(1))1.2^n \leq \chi_p(E^n)$.

8. Rational spaces

Another approach to the chromatic number of the Euclidean plane E^2 is to use Cartesian coordinates. As usual, E^2 is the set of all ordered pairs (x, y) with real coordinates x and y, and the distance between two points is defined in the usual Euclidean way.

By de Bruijn and Erdős's Theorem 3.1 it suffices to deal with finite subsets of E^2, and so we can restrict the coordinates to some subset C of E. The problem is: which subset should we choose?

Problem *Find a countable subset C of the set of real numbers E for which the chromatic number $\chi(C^2)$ equals that of the plane.*

The set $\mathbb{Q}$ of all rational numbers does not work, as shown by Woodall [50] in 1973.

Theorem 8.1 $\chi(\mathbb{Q}^2) = 2$.

Proof We need to colour the points of the rational plane $\mathbb{Q}^2$ – that is, the set of ordered pairs (r_1, r_2), where r_1 and r_2 are rational numbers. We first partition $\mathbb{Q}^2$ into disjoint classes by putting two pairs (r_1, r_2), and (q_1, q_2) into the same class if and

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only if $r_1 - q_1$ and $r_2 - q_2$ both have odd denominators when written in their lowest terms.

This partition of $\mathbb{Q}^2$ into subsets has an important property: if the distance between two points of $\mathbb{Q}^2$ is 1, then both points belong to the same subset of the partition. Indeed, let the distance between (r_1, r_2), and (q_1, q_2) be equal to 1. Then

$$(r_1 - q_1)^2 + (r_2 - q_2)^2 = 1.$$

Let $r_1 - q_1 = a/b$ and $r_2 - q_2 = c/d$ be these differences, written in their lowest terms. Then

$$(a/b)^2 + (c/d)^2 = 1 - \text{that is, } a^2d^2 + b^2c^2 = b^2d^2.$$

Thus b and d must both be odd and so, by our definition above, (r_1, r_2) and (q_1, q_2) must belong to the same class.

Since any class of our partition can be obtained from any other class by a translation, it suffices to colour just one class and extend the colouring to the whole of $\mathbb{Q}^2$ by translations. Let us colour the class that contains the point (0, 0). This class consists of the points (r_1, r_2), where (in their lowest terms) the denominators of r_1 and r_2 are both odd. We colour red the points of the form $(o/o, o/o)$ and $(e/o, e/o)$, and colour blue the points of the form $(o/o, e/o)$ and $(e/o, o/o)$, where o stands for an odd number and e for an even number. In this colouring, two points of the same colour cannot be at distance 1 apart. ■

In 1975 there appeared a 'legendary unpublished manuscript', as P. D. Johnson, Jr. referred to a manuscript by Miro Benda and Micha Perles. This admired and widely circulated manuscript was called *Colorings of Metric Spaces*; Johnson tells its story in *Geombinatorics*, where in January 2000 the Benda–Perles paper was finally published [2]. Their results include the following.

Theorem 8.2 $\chi(\mathbb{Q}^3) = 2$ *and* $\chi(\mathbb{Q}^4) = 4$.

Benda and Perles [2] then posed some important open problems.

Problem *Find* $\chi(\mathbb{Q}^5)$ *and, in general,* $\chi(\mathbb{Q}^n)$.

Problem *Find the chromatic number of* $\mathbb{Q}(\sqrt{2})^2$ *and, in general, of any algebraic extension of* $\mathbb{Q}^2$.

This direction was developed by P. D. Johnson, Jr., Joseph Zaks, Klaus Fischer, Kiran B. Chilakamarri, Michael Reid, Douglas Jungreis, David Witte and Timothy Chow. In 2006 Johnson [25] published in *Geombinatorics* 'A tentative history and compendium' of this direction of inquiry.

More recently Matthias Mann [27] proved that $\chi(\mathbb{Q}^5) \geq 7$. This jump from $\chi(\mathbb{Q}^4) = 4$ explains the difficulty of finding $\chi(\mathbb{Q}^5)$, whose exact value is still

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unknown. Mann [28] then obtained further lower bounds: $\chi(\mathbb{Q}^6) \geq 10$, $\chi(\mathbb{Q}^7) \geq 13$ and $\chi(\mathbb{Q}^8) \geq 16$. In 2008 Cibulka [7] obtained new lower bounds for these chromatic numbers, improving some of Mann's results: $\chi(\mathbb{Q}^5) \geq 8$ and $\chi(\mathbb{Q}^7) \geq 15$.

9. One odd graph

In 1994 Moshe Rosenfeld [38] defined the odd-distance graph E_{odd} to be the graph with vertex-set E^2 in which two vertices are adjacent whenever the distance between them is an odd integer. He showed that E_{odd} does not have a subgraph K_4, and asked whether the chromatic number of E_{odd} is finite. In fact, while the problem was new, the absence of K_4-subgraphs was not, following from the following much more general result of Graham, Rothschild and Straus [21].

Theorem 9.1 *In E^n there exist $n + 2$ points, the distance between any two of which is an odd integer if and only if $n \equiv 14 (\text{mod } 16)$.*

In the necessary part of the proof, the authors used an old result about determinants by Arthur Cayley. The main problem of this section remains wide open.

Problem *Find $\chi(E_{\text{odd}})$.*

We do not even know whether $\chi(E_{\text{odd}})$ is finite. In 2009 Ardal, Manuch, Rosenfeld, Shelah and Stacho [1] improved the lower bound to $\chi(E_{\text{odd}}) \geq 5$.

We denote the measurable chromatic number of the odd-distance graph by $\chi_m(E_{\text{odd}})$. In 1986 Falconer and Marstrand [17] proved that plane sets with positive density at infinity contain all large distances. This implies that $\chi_m(E_{\text{odd}}) \geq \aleph_0$. In 2009 the MIT undergraduate Jacob Steinhardt [48] found an alternative proof of this result using tools of spectral graph theory, which may be beneficial in solving other colouring problems.

10. Influence of set theory axioms

In this section, we need some ideas from set theory. We denote the standard Zermelo–Fraenkel choice system of axioms for set theory by **ZFC**, the countable axiom of choice by $\mathbf{AC}_{\aleph_0}$ and the principle of dependent choices by **DC**. We will use one further axiom, **LM**: every set of real numbers is Lebesgue measurable.

Our first task is to extend the definition of the chromatic number of a graph. Without the axiom of choice, the chromatic number of a graph may not exist. When allowing a system of axioms for set theory to exclude the axiom of choice, we need to create a much broader definition of the chromatic number than the usual one, if we want it to exist. In fact, instead of the chromatic number we ought to talk about *the set of chromatic cardinalities*. There are several meaningful ways to define this. Here is one that I chose in [45].

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Let G be a graph and let A be a system of axioms for set theory. The *set of chromatic cardinalities* $\chi^A(G)$ of G is the set of all cardinal numbers $\tau \leq |V(G)|$ for which there is a proper colouring of the vertices of G in τ colours, and τ is minimum with respect to this property. As can be seen, the set of chromatic cardinalities need not have just one element as was the case with $A = \mathbf{ZFC}$. It can also be empty. The advantage of this definition is its simplicity. Best of all, we can use inequalities on sets of chromatic cardinalities as follows.

Let τ be a cardinal number. The inequality $\chi^A(G) > \tau$ means that $\sigma > \tau$ for every $\sigma \in \chi^A(G)$; the inequalities $<$, $\leq$ and $\geq$ are defined analogously. We also agree that the empty set is greater than or equal to any other set of cardinal numbers. Finally, if τ is a cardinal number and $\chi^A(G) = \{\tau\}$ is a one-element set of chromatic cardinalities (as is the case with the chromatic number when $A = \mathbf{ZFC}$), then we will simplify our notation by omitting parentheses and writing $\chi^A(G) = \tau$.

An infinite cardinal $\aleph_\alpha$ is *regular* if $\mathrm{cf}\ \omega_\alpha = \omega_\alpha$, and κ is a *strong limit cardinal* if, for every cardinal λ, $\lambda < \kappa$ implies that $2^\lambda < \kappa$. A cardinal κ is *inaccessible* if $\kappa > \aleph_0$ and κ is regular and a strong limit cardinal. Assuming the existence of an inaccessible cardinal, and using Paul Cohen's forcing, Solovay [47] constructed in 1964 (and published in 1970) a model that proved a remarkable theorem. In his honour the author [45] defined the *Zermelo–Fraenkel–Solovay system of axioms* **ZFS** for set theory by

$$\mathbf{ZFS} = \mathbf{ZF} + \mathbf{AC}_{\aleph_0} + \mathbf{LM} \text{ and } \mathbf{ZFS+} = \mathbf{ZF} + \mathbf{DC} + \mathbf{LM}.$$

Solovay's theorem can now be formulated very concisely.

Theorem 10.1 **ZFS**+ *is consistent.*

Shelah and Soifer [39] constructed the following example. Define a graph G as follows: the vertex-set is the set of real numbers, and the set of edges is $\{(s, t) : s-t-\sqrt{2} \in \mathbb{Q}\}$. They then proved that, for this graph, $\chi^{\mathbf{ZFC}}(G) = 2$, while $\chi^{\mathbf{ZFS}}(G) > \aleph_0$.

Similar examples with the plane E^2 and (in general) E^n as the vertex-set were constructed in [46] and [44], respectively. These examples illuminate the influence of the system of axioms for set theory on combinatorial results. They also suggest that the chromatic number of E^n may not exist 'in the absolute' (that is, in **ZF**), but may depend upon the system of axioms chosen for set theory. These examples naturally prompt the following open problem.

Problem *For which values of n is the chromatic number* $\chi(E^n)$ *uniquely defined 'in the absolute' – that is, in* **ZF***– regardless of the addition of the axiom of choice or its relative?*

An important example came from the Australian student Michael Payne [32], who started with the unit-distance graph G_1 with vertex-set $\mathbb{Q}^2$, where two vertices are adjacent if and only if they are at distance 1 apart. He then showed that the desired

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unit-distance graph G on the vertex-set E^2 is obtained by tiling the plane by translates of G_1- that is, its edge-set is $\{(p_1, p_2) : p_1, p_2 \in E^2, p_1 - p_2 \in \mathbb{Q}^2$ and $|p_1 - p_2| = 1\}$, and proved that $\chi^{\mathbf{ZFC}}(G) = 2$ and $\chi^{\mathbf{ZFS}}(G) = 3, 4, 5, 6$ or 7.

Payne proved first that any measurable set S of positive Lebesgue measure contains the endpoints of a path of length 3 in G; of course, this ruled out a 2-colouring of S. Payne continued: 'We can then proceed in a similar fashion to Shelah and Soifer's proof'[39]. In 2009 he constructed a new class of unit distance graphs on the vertex-set E^n whose chromatic number depends upon the system of axioms for set theory [33].

11. Predicting the future

In 2003 Shelah and Soifer [39] obtained the following surprising result.

Theorem 11.1 *Assume that any finite unit-distance plane graph has chromatic number not exceeding 4. Then* $\chi^{\mathbf{ZFC}}(E^2) = 4$, *but* $\chi^{\mathbf{ZFS+}}(E^2) \geq 5$.

Can we obtain any results unconditionally? Yes, we can (see [45]), but not yet in **ZFC**.

Theorem 11.2 $\chi^{\mathbf{ZFS+}}(E^2) \geq 5$.

We conclude with the author's conjectures [45] of the expected value of the chromatic number of the plane, and more generally of E^n, in **ZFC**.

Conjecture $\chi(E^2) = 4$ *or* 7.

If the chromatic number of the plane were 4, then Theorem 11.1 would imply that the chromatic number of the plane does not exist in absolute, but depends upon the choice of the system of axioms for set theory. However, if I were limited to conjecturing a single value for the chromatic number of the plane, I would choose the value 7.

If the last conjecture is true, then a unit-distance 7-chromatic finite graph must exist in the plane. In 1998 Pritikin [35] published a lower bound for the order of such a graph.

Theorem 11.3 *Any unit-distance* 7*-chromatic graph G has at least* 6198 *vertices.*

In fact, the order of the smallest such graph may have to be much larger.

For 3-dimensional space the author [45] conjectured that $\chi(E^3) = 15$. My main conjecture is as follows.

Main conjecture For any positive integer $n > 1$, $\chi(E^n) = 2^{n+1} - 1$.

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To paraphrase Paul Erdős's words about some of his conjectures, we can say that this conjecture is likely to withstand centuries, but we shall see!

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