Proc. Camb. Phil. Soc. (1965), **61**, 621
Printed in Great Britain

On multilattice groups

By D. B. McALISTER
Queen's University, Belfast

Communicated by D. C. J. Burgess

(*Received* 12 *August* 1964)

Multilattice groups have been introduced by Benado ((1)) and considered by Vaida ((9)), who has initiated the study of the structure of a class of these groups. The purpose of this paper is to further the investigation begun by Vaida.

Basic in the work contained here is the set H consisting of the differences between the minimal upper bounds of pairs of elements of a multilattice group G. By means of conditions imposed on G and based on H, we determine the structure of the congruence relations on a multilattice group and study direct sums and lexicographic products and extensions of such groups (sections 2, 3).

In section 4 we apply the results obtained in previous sections to investigate the structure of certain distributive multilattice groups which obey

($\mathscr{F}$) *each strictly positive element is over at most a finite number of members of* H.

In this section it is shown that a distributive multilattice group which obeys ($\mathscr{F}$) is complete if and only if it is isomorphic to the direct sum of a lattice group and copies of the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$. A similar result is that a distributive multilattice group which obeys the minimal condition for positive elements is isomorphic to the direct sum of copies of $\mathbf{Z}$ and copies of the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$; the latter is a generalization of the main theorem of (9).

I should like to to thank Dr D. C. J. Burgess for his help and encouragement during the preparation of this paper and the Ministry of Education for Northern Ireland for the financial assistance given me while the work, on which this paper is based, was carried out.

1. *Introduction.* If P is a partially ordered set, and A and B are subsets of P then we write:

$A \leqslant B$ if and only if, for each $a \in A$, there exists $b \in B$ such that $a \leqslant b$;

$A \geqslant B$ if and only if, for each $a \in A$, there exists $b \in B$ such that $a \geqslant b$.

A partially ordered set P is a *multilattice* if and only if, for each $a, b \in P$, if $x \geqslant a$, $x \geqslant b$ then there exists $y \in P$ such that $x \geqslant y \geqslant a$ and $y \geqslant b$ and such that, if $y \geqslant u \geqslant a$ and $u \geqslant b$ then $u = y$; and dually for lower bounds. Thus P is a multilattice if, for each pair of elements $a, b \in P$, every upper bound of a, b is over a minimal upper bound, and dually.

The set of all minimal upper bounds of a and b is denoted by $a \vee b$; the set of all maximal lower bounds of a and b is denoted by $a \wedge b$. If A and B are subsets of a multi-

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

lattice P, then we denote by $A \vee B$ the set $\{x \in P: x \in a \vee b$ for some $a \in A$ and some $b \in B\}$; and dually for $A \wedge B$.

Multilattices have been extensively studied by Benado, and his paper ((2)) gives all the basic properties of such algebraic structures.

A partially ordered group G is said to be a *multilattice group* if and only if it is a multilattice under $\leqslant$. In this case, the operations $\vee$ and $\wedge$ are distributive with respect to $+$ and dual with respect to $-$:

$$a+(b \vee c)+d = (a+b+d) \vee (a+c+d), \quad -(a \vee b) = -a \wedge -b,$$
$$a+(b \wedge c)+d = (a+b+d) \wedge (a+c+d), \quad -(a \wedge b) = -a \vee -b.$$

By an obvious corollary to Theorem 4 of Burgess ((4)), an arbitrary multilattice group is an extension, of simple type, of a directed multilattice group, and hence, in order to find the structure of multilattice groups, it is sufficient to consider only directed multilattice groups. Throughout the remainder of this paper the terms 'multilattice group' and 'directed multilattice group' are synonymous.

Let G be a multilattice group, then we denote by H the set

$$\{x \in G: x \in (a \vee 0)-(a \vee 0) \quad \text{for some} \quad a \in G\}.$$

The set H plays an important part in our work; the following lemma gives some of its fundamental properties.

LEMMA 1. *Let G be a multilattice group, then H is a normal subset of G which contains, with any element h, its negative $-h$, and no non-zero member of H is comparable with* 0.

For $a, b \in G$ we have

$$(a \vee b)-(a \vee b) \subseteq H \quad \text{and} \quad (a \wedge b)-(a \wedge b) \subseteq H.$$

Proof. The properties all follow from the definition of H and the distributivity and duality of $\vee$ and $\wedge$ with respect to $+$ and $-$. Assuming the others, we prove, that, for $a, b \in G$, $a \wedge b - a \wedge b \subseteq H$.

Let
$$\begin{aligned} h \in a \wedge b - a \wedge b &= (a \wedge b - b)-(a \wedge b - b) \\ &= (a-b) \wedge 0-(a-b) \wedge 0 \\ &= -((b-a) \vee 0)+(b-a) \vee 0. \end{aligned}$$

Then $h = -c+d$, where $c, d \in (b-a) \vee 0$, and hence

$$c+h-c = d-c \in (b-a) \vee 0-(b-a) \vee 0 \subseteq H.$$

But H is normal, hence $h \in H$.

The set H may also be characterized as follows.

PROPOSITION 1. *Let G be a multilattice group, then*

$$H = \{x \in G: x \in a \wedge b \quad \text{for some} \quad a, b \in G \quad \text{with} \quad 0 \in a \wedge b\}.$$

Proof. Let $a, b \in G$ with $0 \in a \wedge b$, and let $h \in a \wedge b$ then, by Lemma 1, $h \in H$. Conversely, if $h \in H$, $h \neq 0$ then $h \in u \vee 0-u \vee 0$ for some $u \in G$ and, since $u \vee 0 = -(-u \wedge 0)$, $h = -c+d$ for some $c, d \in (-u \wedge 0)$. Let $a = -c-u$ and $b = -c$, then

$$0 = -c+c \in -c+(-u \wedge 0) = (-c-u) \wedge (-c) = a \wedge b$$

and
$$h = -c+d \in -c+(-u \wedge 0) = a \wedge b.$$

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

From Proposition 1, we have the following useful result.

PROPOSITION 2. *Let G be a multilattice group and let $h \in H$, $h \neq 0$, then* l.u.b. $(h, 0)$ *does not exist.*

Proof. Since $h \in H$ there exist $a, b \in G$ such that $0 \in a \wedge b$ and $h \in a \wedge b$. Since $0 \in a \wedge b$ and $h \in a \wedge b$ we have $a > h, a > 0$ $b > h$, $b > 0$. Hence, if l.u.b. $(h, 0) = h'$ existed, we should have $a \geqslant h' > 0$, $b \geqslant h' > 0$ which, since $0 \in a \wedge b$, is impossible.

A subset S of a multilattice group G is said to be a *submultilattice* of G if and only if, for each $a, b \in S$, $a \vee b \subseteq S$ and $a \wedge b \subseteq S$.

In any lattice group G, each directed convex subgroup of G is a sublattice of G; if K is a directed convex subgroup of a multilattice group G, then K need not be a submultilattice of G, see Example (A). However,

PROPOSITION 3. *Let G be a multilattice group, and let K be a directed convex subgroup of G. If K obeys the condition*:

$$(\mathscr{H}) \text{ if } 0 < a \in K \text{ and } a > h \text{ where } h \in H \text{ then } h \in K;$$

then K is a submultilattice of G.

To prove this we make use of a lemma.

LEMMA 2. *Let G be a multilattice group and let K be a directed convex subgroup of G, which obeys $(\mathscr{H})$. If $0 < a \in K$ and $0 < x \in G$, then $a \wedge x \subseteq K$.*

Proof. Let $0 \leqslant d \in a \wedge x$; then, by the convexity of K, $d \in K$. Let $d \neq m \in a \wedge x$; then $m - d = h \in H$. Since $d \geqslant 0$ and $a > m \in a \wedge x$, we have $a \geqslant m \geqslant h$. But $a \in K$ and K obeys $(\mathscr{H})$, whence $h \in K$ and $m = h + d \in K$. Thus $a \wedge x \subseteq K$.

Proof of proposition. It is clearly sufficient to show that, for each $a \in K$, $a \vee 0 \subseteq K$.

Since K is directed and convex, there exists $u \in (a \vee 0) \cap K$. Then $0 \leqslant u - a \in (-a) \vee 0$ and so, by Lemma 2, since $u \in K$, $u \wedge (u - a) \subseteq K$. But

$$u \wedge (u - a) = u + (0 \wedge -a) = u - (a \vee 0);$$

hence, since K is a subgroup, $a \vee 0 \subseteq K$.

A multilattice group G is said to be *distributive* if and only if for $a, b, c \in G$, the relations

$$(a \vee b) \cap (a \vee c) \neq \phi, \quad (a \wedge b) \cap (a \wedge c) \neq \phi,$$

together imply $b = c$.

(Not every multilattice group is distributive, see for example the first multilattice group in section 5, and Theorem 12.)

Positive elements a, b of a multilattice group G are said to be *weakly disjoint* if and only if $0 \in a \wedge b$; they are said to be *disjoint* or *orthogonal* if and only if $0 = a \wedge b$.

The *modulus* of an element a of a multilattice group G is defined to be the set $a \vee 0 + (-a \vee 0)$; it is denoted by $\|a\|$.

Jaffard ((7)) has proved the following results for disjoint elements of an arbitrary partially ordered group.

PROPOSITION 4. *Let a, b be disjoint elements of a partially ordered group G, then*:

(i) $a + b = b + a$;

(ii) $a \wedge (b + c) = a \wedge c$ *for each* $0 < c \in G$.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

For weakly disjoint elements of a multilattice group, neither of these results need be true. But

PROPOSITION 5 (*Benado* (1)). *Let a and b be weakly disjoint elements of a distributive multilattice group, then $a+b=b+a$.*

The following properties of the modulus of an element of a multilattice group are immediate from the definition.

PROPOSITION 6. *Let G be a multilattice group, and let $a, b \in G$, then:*

(i) $\|a\| = \|-a\|$;

(ii) $\|a\| \geqslant 0$ *while* $\|a\| = 0$ *if and only if* $a = 0$;

(iii) *if* $a > 0$ *then* $\|a\| = a$;

(iv) $\|b\| + \|a\| + \|b\| \geqslant \|a+b\|$; $b + \|a\| - b = \|b+a-b\|$.

Let G be a multilattice group, and let $0 < a \in G$, then we define

$$I(a) = \{x \in G: \text{ for some } n \in Z^+,\ na > \|x\|\}.$$

Let $X \subseteq G$, then we define

$$X^* = \{a \in G: \text{ for some } a' \in \|a\|,\quad a' \wedge x = 0 \text{ for each } x \in X^+ = X \cap G^+\},$$

where G^+ is the set of positive elements of G, i.e. $G^+ = \{x \in G: x \geqslant 0\}$.

PROPOSITION 7. *Let G be a multilattice group and let $0 < a \in G$, then $I(a)$ is the directed convex subgroup of G generated by a. Let X be a subset of G, then X^* is a directed convex subgroup of G; if X is a normal subset of G, then X^* is a normal subgroup of G.*

Proof. These results follow immediately from the definition of $I(a)$ and X^* by means of Proposition 6 and, in the case of X^*, Proposition 4.

COROLLARY. *Let G be a multilattice group, then G has no nontrivial directed convex subgroups if and only if it is strongly archimedean, that is, if and only if, for each $0 < a$ and each $b \in G$, there exists $n \in Z^+$ such that $na > b$.*

We have seen, Proposition 3, that, when a directed convex subgroup of a multilattice group G obeys ($\mathscr{H}$), then it is a submultilattice of G. In the sequel we shall be interested in multilattice groups which obey

($\mathscr{H}^*$) *each directed convex subgroup obeys* ($\mathscr{H}$).

Using Proposition 7, we may characterize such groups.

THEOREM 1. *Let G be a multilattice group, then G obeys ($\mathscr{H}^*$) if and only if, for each $0 < a \in G$:*

if $a > h$ for some $0 \neq h \in H$, then there exists $n \in Z^+$ such that $na \geqslant -h$.

Proof. If G obeys ($\mathscr{H}^*$), $0 < a \in G$ and $h < a$ for some $0 \neq h \in H$ then, by ($\mathscr{H}$), $h \in I(a)$. Hence, for some $n \in Z^+$, $na > \|h\| \geqslant -h$.

Conversely, let K be a directed convex subgroup of G and let $0 < a \in K$ with $a > h$ for some $0 \neq h \in H$. Then, under the hypothesis of the theorem, there exists $n \in Z^+$ such that $na > -h$. But this implies $a > h > -na$ whence, by the convexity of K, $h \in K$.

COROLLARY. *If every element of H is of finite order, then G obeys ($\mathscr{H}^*$); in particular, this is true if H is a finite subgroup of G.*

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

2. *Homomorphisms, congruences and multilattice ideals.* A mapping θ of a multilattice group G onto a multilattice group $\bar{G}$ is said to be a *homomorphism* of G onto $\bar{G}$ if and only if:

(i) θ is a homomorphism of the abstract group G onto the abstract group $\bar{G}$;

(ii) for $a, b \in G$ we have,

$$(a \vee b)\theta = a\theta \vee b\theta, \quad (a \wedge b)\theta = a\theta \wedge b\theta.$$

By (i) and the distributivity and duality of $\vee$ and $\wedge$ with respect to $+$ and $-$, it is clearly sufficient to replace (ii) by

(ii)′ for each $a \in G$, $(a \vee 0)\theta = a\theta \vee 0\theta$.

Under (i) and (ii) above, the equivalence R_θ on G, defined by

$$a \equiv b(R_\theta) \quad \text{if and only if} \quad a\theta = b\theta,$$

is a *congruence* on G in the following sense:

(iii) R_θ is a congruence on the abstract group G;

(iv) if $a \equiv b(R_\theta)$ and $c \equiv d(R_\theta)$ then,

$$a \vee c \equiv b \vee d(R_\theta) \quad \text{and} \quad a \wedge c \equiv b \wedge d(R_\theta).$$

(Where $A \equiv B(R_\theta)$ if and only if, for each $a \in A$, there exists $b \in B$ such that $a \equiv b(R_\theta)$, and conversely.)

It is well known that, if K is a convex normal subgroup of a partially ordered group G, then the group of cosets of K may be made into a partially ordered group by defining:

$$K + a \geqslant K + b \quad \text{if and only if, for some} \quad k \in K, \quad a - b \geqslant k.$$

We denote this group by G/K.

THEOREM 2. *Let θ be a homomorphism of a multilattice group G onto a multilattice group $\bar{G}$, and let K be the kernel of θ. Then K is a convex normal multilattice subgroup of G which obeys $(\mathscr{H})$ and $\bar{G} \simeq G/K$.*

Conversely, let K be a convex normal multilattice subgroup of G which obeys $(\mathscr{H})$, then the relation

$$a \equiv b(K) \quad \textit{if and only if} \quad K + a = K + b$$

is a congruence on G, G/K is a multilattice group and the natural mapping $a \to K + a$ is a homomorphism of G onto G/K.

Proof. Let θ be a homomorphism of G onto $\bar{G}$. From the definition of homomorphism, it is easy to verify that K is a convex normal multilattice subgroup of G which obeys $(\mathscr{H})$, and that $a\theta \geqslant 0\theta$ if and only if, for some $k \in K$, $a \geqslant k$. Hence, by the well-known theorem for partially ordered groups (see e.g. Fuchs ((6)), page 21) $\bar{G} \simeq G/K$.

Conversely, we show that G/K is a multilattice group and that the natural mapping is a homomorphism of G onto G/K. The remainder of the result then follows from the remarks before the enunciation of the theorem. To prove this we require two preliminary results.

(1) Let $u \equiv v(K)$ then $u \vee v \subseteq K + v$. Since $u \equiv v(K)$, $u - v \in K$; since K is a submultilattice of G, $(u - v) \vee 0 \subseteq K$. Hence $u \vee v = (u - v) \vee 0 + v \subseteq K + v$.

(2) Let $K + x \geqslant K + a$, $K + x \geqslant K$; then there exists $u \in a \vee 0$ such that

$$K + x \geqslant K + u.$$

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

Since $K+x \geqslant K+a$ there exists $x' \in K+x$ such that $x' \geqslant a$; since $K+x \geqslant K$ there exists $x'' \in K+x$ such that $x'' \geqslant 0$. Let $y \in x' \vee x''$, then, by (1), $y \in K+x''=K+x$ and also $y \geqslant a, y \geqslant 0$. Hence there exists $u \in a \vee 0$ such that $y \geqslant u$; for this u,

$$K+x \geqslant K+u.$$

Let $v \in a \vee 0$, then $K+v \geqslant K+a$ and $K+v \geqslant K$. Suppose that

$$K+v \geqslant K+x \geqslant K+a \quad \text{and} \quad K+x \geqslant K,$$

then, by (2), there exists $u \in a \vee 0$ such that $K+x \geqslant K+u$. Hence there exists $0 \leqslant k \in K$ such that $k \geqslant u-v$. But $u-v \in H$, and so, since K obeys $(\mathscr{H})$, $u-v \in K$. Thus $K+u=K+x=K+v$ and hence, since by (2), every upper bound of $K+a$ and K is over a member of $K+(a \vee 0)$, G/K is a multilattice group in which $(K+a) \vee K = K+(a \vee 0)$. It is immediate from this that the natural mapping of G onto G/K is a homomorphism, and so we have the result.

We say that a subgroup of a multilattice group G is a *multilattice ideal* of G if and only if it is the kernel of some homomorphism of G onto a multilattice group; that is, if and only if it is a convex normal multilattice subgroup of G, which obeys $(\mathscr{H})$, or equivalently, by Proposition 3, if and only if it is a directed convex normal subgroup of G which obeys $(\mathscr{H})$.

Note that we proved that, if θ is a homomorphism of a multilattice group G onto a multilattice group $\bar{G}$, then $\bar{H} = H\theta$. By means of this remark, we have

PROPOSITION 8. *Let G be a multilattice group, then G has a maximum lattice group homomorphic image, in the sense that, if K is the kernel of this homomorphism and L is the kernel of any homomorphism of G onto a lattice group, then $K \subseteq L$.*

Proof. Clearly $\bar{G} = G\theta$ is a lattice group if and only if $H\theta = 0\theta$, that is, if and only if H is contained in the kernel of θ. Let K be the intersection of all multilattice ideals of G which contain H; then K is the minimum such multilattice ideal, and G/K is the maximum lattice group homomorphic image of G.

Not only has a multilattice group G a universal lattice group homomorphic image but, as the next theorem shows, the, clearly non-empty, set of multilattice ideals of G which are also sublattices of G has a maximum number. Throughout this paper, this subgroup will be denoted by N.

To prove the theorem we require two lemmas.

LEMMA 3. *Let G be a multilattice group, and let $0 < a \in G$, then* g.l.b. (a, x) *exists for each $x \in G^+$ if and only if $a \not> h$ for any $0 \neq h \in H$.*

Proof. Suppose that $a \not> h$ for any $0 \neq h \in H$ and that, for some $x \in G^+$, g.l.b. (a, x) does not exist. Let $0 \leqslant d \in a \wedge x$, then any other member of $a \wedge x$ has, by Lemma 1, the form $h+d$ where $0 \neq h \in H$. Since $d \geqslant 0$ and $a > h+d$, we have $a > h$ which is a contradiction.

Conversely, let $0 < a \in G$, and suppose that $a > h$ for some $0 \neq h \in H$. Then $a-h > 0$ while g.l.b. $(a, a-h)$ does not exist; for $a \wedge (a-h) = a-(h \vee 0)$, and, by Proposition 2, l.u.b. $(h, 0)$ does not exist.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

LEMMA 4. *Let G be a multilattice group, and let $0 \neq h \in H$ and $e \in h \vee 0$. If $0 < x \leqslant e$ then $x > h'$ for some $0 \neq h' \in H$.*

Proof. Since $h \in H$ there are, by Proposition 1, elements a, b such that $0 \in a \wedge b$ and $h \in a \wedge b$. Hence, since G is a multilattice, there are elements $u, v \in h \vee 0$ such that $0 \in u \wedge v$ and $h \in u \wedge v$.

Let e, h, x be as in the enunciation of the lemma, and suppose that $x \not> h'$ for any $0 \neq h' \in H$. Then, by Lemma 3, g.l.b. (x, a) exists for each $a \in G^+$; in particular, g.l.b. $(x, e-h)$ exists. Hence, since $0 \in e \wedge (e-h)$ and $x \leqslant e, x \wedge (e-h) = 0$.

Let u, v be as in the first paragraph, and let $0 \leqslant d = x \wedge u$. Then, since $d \leqslant x$, $d \wedge (e-h) = 0$ and g.l.b. (d, a) exists for each $a \in G^+$. Since $0 \in u \wedge v$ and $d \leqslant u$, $d \wedge v = 0$; hence $e-h, v \in \{d\}^*$. Since $v > h$ and $e-h > -h$, it follows, from the convexity of $\{d\}^*$, that $h \in \{d\}^*$. Thus $e = (e-h)+h \in \{d\}^*$ which, since $d \leqslant e$, is impossible unless $d = 0$.

Hence $x \wedge u = 0$ and so, since $x \wedge (e-h) = 0$, we may deduce, as in the immediately preceding paragraph, that $x = 0$, which is a contradiction. Thus the lemma is proved.

THEOREM 3. *Let G be a multilattice group, then the, clearly non-empty, set of multilattice ideals of G which are also sublattices of G has a maximum member N; $0 < a \in N$ if and only if, for each $x \in G^+$,* g.l.b. (a, x) *exists or, equivalently, if and only if $a \not> h$ for any $0 \neq h \in H$. Further, N^* is a multilattice ideal of G which contains H.*

Proof. Let $\breve{H} = H \vee 0$ and let $N = (\breve{H})^*$; since H is a normal subset of G, so is $\breve{H}$, and hence, by Proposition 7, N is a directed convex normal subgroup of G. Let $0 < a \in N$ and suppose $a > h$ for some $0 \neq h \in H$; then $a \geqslant e$ for some $e \in h \vee 0$. But, by hypothesis, $a \wedge e = 0$ for each $0 \neq e \in \breve{H}$. Hence $a \not> h$ for any $0 \neq h \in H$, and so N is a multilattice ideal and, by Lemma 3, a submultilattice of G.

Suppose that $0 < a \in G$ and that g.l.b. (a, x) exists for all $x \in G^+$. Let $0 \neq e \in \breve{H}$ and let $d = a \wedge e$; if $d \neq 0$ then, by Lemma 4, $d > h$ for some $0 \neq h \in H$ which, since $d \leqslant a$ is impossible. Hence $a \wedge e = 0$ for each $e \in \breve{H}$, that is $a \in N$. Thus N^+ is as described in the statement of the theorem.

Let M be a sublatticed multilattice ideal of G, and let $0 < a \in M$. If $a > h$ for any $0 \neq h \in H$, then, since M obeys $(\mathscr{H})$, $h \in M$. But, by Proposition 2, l.u.b. $(h, 0)$ does not exist, which is a contradiction to the fact that M is a sublattice of G. Hence $N^+ \supseteq M^+$, and N is the maximum multilattice ideal of G which is also a sublattice of G.

Since N is a normal subgroup of G, N^* is a directed convex normal subgroup of G. Let $0 \neq h \in H$, and let $e \in h \vee 0$; then, by the definitions of N and N^*, both e and $e-h$ are in N^* and hence $h = -(e-h)+e \in N^*$. Thus $H \subseteq N^*$ and this clearly implies that N^* is a multilattice ideal of G.

It is a corollary to this result that any multilattice group is a subdirect union of a lattice group and a multilattice group which obeys

$$(\mathscr{O}) \text{ if } a > 0 \text{ then } a > h \text{ for some } 0 \neq h \in H.$$

In this paper we shall be concerned, however, with finding conditions for a multilattice group to be the direct sum of N and N^* in cases where N^* has a certain known structure.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

3. *Sums and extensions of multilattice groups.* The *direct sum* of a family $\{G_i: i \in I\}$ of partially ordered groups is defined to be their direct sum as abstract groups, partially ordered cardinally; the direct sum of the partially ordered groups $\{G_i: i \in I\}$ is denoted by $\Sigma \oplus \{G_i: i \in I\}$.

The following well-known theorem characterizes those partially ordered groups which are isomorphic to the direct sum of a family of their subgroups.

THEOREM 4. *Let G be a partially ordered group, then G is isomorphic to the direct sum of a family of its subgroups if and only if it contains a family $\{G_i: i \in I\}$ of subgroups such that:*

(i) *if $a_i \in G_i$ and $a_j \in G_j$ then $a_i + a_j = a_j + a_i$ when $i \neq j$;*

(ii) *G is generated by the groups $\{G_i: i \in I\}$;*

(iii) *for each $i \in I$, $G_i \cap \sum_{j \neq i} G_j = \{0\}$;*

(iv) *$a_1 + a_2 \ldots + a_n \geqslant 0$ if and only if, for each $\alpha = 1, 2, \ldots, n$, $a_\alpha \geqslant 0$ where distinct elements a_α belong to distinct groups G_i.*

Let A and B be subsets of a multilattice group G, then we say that A and B are *orthogonal* if and only if, for each $0 \leqslant a \in A$ and each $0 \leqslant b \in B$, $a \wedge b = 0$.

THEOREM 5. *Let $\{G_i: i \in I\}$ be a family of pairwise orthogonal directed subgroups of a multilattice group G. Then the subgroup of G generated by $\{G_i: i \in I\}$ is isomorphic to their direct sum $\Sigma \oplus \{G_i: i \in I\}$.*

Proof. We verify that conditions (i), (iii) and (iv) of Theorem 4 are satisfied. For (i), we note that, since each G_i is directed, every element $a_i \in G_i$ may be written as $a_i = a_i^+ - a_i^-$ where a_i^+ and a_i^- are in G_i^+. Hence, since distinct groups G_i are orthogonal, it follows, from Proposition 4, that (i) holds. It further follows from (i) that the subgroup of G generated by any subfamily of $\{G_i: i \in I\}$ is their sum.

Let $0 \leqslant a \in G_i \cap \sum_{j \neq i} G_j$; then $a = a_1 + a_2 + \ldots + a_n$ where each $a_\alpha (\alpha = 1, 2, \ldots, n)$ belongs to a different G_j. Now, for each α, $a_\alpha \leqslant a_\alpha^+$, and hence

$$a = a \wedge a \leqslant a \wedge (a_1^+ + a_2^+ + \ldots + a_n^+).$$

But, since distinct G_i are orthogonal, $a \wedge a_\alpha^+ = 0$ for each $\alpha = 1, 2, \ldots, n$ and so, by Proposition 4, $a \wedge (a_1^+ + a_2^+ + \ldots + a_n^+) = 0$. Thus $a = 0$ and, since the groups $G_i (i \in I)$ are directed, (iii) holds.

Let $0 \leqslant a = a_1 + a_2 + \ldots + a_n \in \Sigma G_i$ where each $a_\alpha (\alpha = 1, 2, \ldots, n)$ belongs to a different G_i. Then, since, by (i), the elements of distinct groups G_i are permutable,

$$0 \leqslant a_1^- + a_2^- + \ldots + a_n^- \leqslant a_1^+ + a_2^+ + \ldots + a_n^+.$$

Hence, by Proposition 4, we have, for each $\alpha = 1, 2, \ldots, n$,

$$a_\alpha^- = a_\alpha^- \wedge (a_1^- + a_2^- + \ldots + a_n^-), \quad a_\alpha^- \wedge a_\alpha^+ = a_\alpha^- \wedge (a_1^+ + a_2^+ + \ldots + a_n^+)$$

and thus $a_\alpha^- \leqslant a_\alpha^- \wedge a_\alpha^+$, that is, for each $\alpha = 1, 2, \ldots, n$, $a_\alpha \geqslant 0$. Hence (iv) holds, and the theorem is proved.

Note. If the subgroup $\bar{G}$ of a multilattice group G, generated by a family $\{G_i: i \in I\}$ of subgroups of G, is isomorphic to their direct sum we will often agree to denote $\bar{G}$ by $\Sigma \oplus \{G_i: i \in I\}$, or by $\Sigma \oplus G_i$.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

If G were a lattice group and each G_i $(i \in I)$ convex, then the subgroup of G generated by $\{G_i\colon i \in I\}$ would be convex; however this need not be true for multilattice groups, see example (B). Theorem 6 gives sufficient conditions for $\Sigma \oplus \{G_i\colon i \in I\}$ to be a convex subgroup of G when the latter is an arbitrary multilattice group.

THEOREM 6. *Let $\{G_i\colon i \in I\}$ be a family of pairwise disjoint directed convex subgroups of a multilattice group G, which obey $(\mathscr{H})$. Then the subgroup of G generated by $\{G_i\colon i \in I\}$ is isomorphic to their direct sum; it is directed and convex.*

Proof. We show that the groups $G_i (i \in I)$ are pairwise orthogonal, the first result then follows from Theorem 5. Let $0 \leqslant a_i \in G_i$ and $0 \leqslant a_j \in G_j$ then, by Lemma 2, $a_i \wedge a_j \subseteq G_i \cap G_j = \{0\}$, if $i \neq j$; hence G_i and G_j are pairwise orthogonal.

Let $0 \leqslant x \leqslant a = a_1 + a_2 + \ldots + a_n \in \Sigma \oplus G_i$; we use induction on n to show that $\Sigma \oplus G_i$ is convex.

If $n = 1$, the result is immediate from the convexity of each G_i. Suppose that the result is true for each $r < n$. Then, since $a_1 + a_2 + \ldots + a_n \geqslant x$, we have

$$a_1 + a_2 + \ldots + a_n = (a_1 + a_2 + \ldots + a_n) \vee a_n \geqslant x \vee a_n,$$

whence, since $x \vee a_n = x - (x \wedge a_n) + a$,

$$a_1 + a_2 + \ldots + a_{n-1} \geqslant x - d \geqslant 0 \quad \text{for some} \quad d \in x \wedge a_n.$$

Hence, by induction hypothesis, $x - d \in \Sigma \oplus G_i$. But, since each G_i obeys $(\mathscr{H})$,

$$d \in x \wedge a_n \subseteq G_j, \quad \text{where} \quad a_n \in G_j,$$

and so $x = (x - d) + d \in \Sigma \oplus G_i$. Thus the theorem is proved.

If a partially ordered group G is the direct sum of a family of its subgroups and $\mathbf{a}$ is a member of G, then we agree to identify $\mathbf{a}$ with the finite or zero sequence of non-zero elements, no two from the same direct summand of G, such that $\mathbf{a}$ is uniquely expressible as

$$\mathbf{a} = a_1 + a_2 + \ldots + a_n.$$

We write $\mathbf{a} \equiv (a_1, a_2, \ldots, a_n)$.

The following results may easily be deduced for partially ordered groups which are the direct sum of multilattice groups.

PROPOSITION 9. *Let G be the direct sum of a family $\{G_i\colon i \in I\}$ of multilattice groups, then G is a multilattice group and, if $\mathbf{a} \equiv (a_1, a_2, \ldots, a_n)$ is in G, then*

$$\mathbf{a} \vee \mathbf{0} \equiv (a_1 \vee 0, a_2 \vee 0, \ldots, a_n \vee 0).$$

Hence $\mathbf{h} \equiv (h_1, h_2, \ldots, h_n) \in H$ if and only if, for each $\alpha = 1, 2, \ldots, n$, $h_\alpha \in H_i$ for some $i \in I$.

Further:

(i) *G obeys $(\mathscr{H}^*)$ if and only if, for each $i \in I$, G_i obeys $(\mathscr{H}^*)$;*

(ii) *G is distributive if and only if, for each $i \in I$, G_i is distributive.*

By means of Theorems 5 and 6 and Proposition 9, we may immediately infer

THEOREM 7. *Let G be a multilattice group, then G is isomorphic to the direct sum of a family of its subgroups if and only if it contains a family $\{G_i\colon i \in I\}$ of subgroups which*

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

together generate G, as an abstract group, and which obeys one of the following sets of conditions:

(i) *the G_i are a family of pairwise disjoint multilattice ideals of G*;

(ii) *the G_i are a family of pairwise disjoint directed convex subgroups of G which obey* $(\mathscr{H})$;

(iii) *the G_i are a family of pairwise orthogonal directed subgroups of G.*

Another useful concept in the study of the structure of multilattice groups is that of lexicographic extension. We shall make use of this concept and that of direct sum in the next section.

Let K and L be partially ordered groups, and let G be the direct sum as abstract groups, of K and L. Then G is said to be the *lexicographic product* of K by L if and only if G is partially ordered by

$$(l_1, k_1) \geqslant (l_2, k_2) \quad \text{if and only if} \quad l_1 > l_2 \quad \text{or} \quad l_1 = l_2 \quad \text{and} \quad k_1 \geqslant k_2,$$

where

$$l_1, l_2 \in L \quad \text{and} \quad k_1, k_2 \in K.$$

THEOREM 8. *Let K and L be non-trivial partially ordered groups and let G be the lexicographic product of K by L. Then G is a multilattice group if and only if one of the following conditions is satisfied*:

(i) *K is a multilattice group and L is simply ordered*;

(ii) *K is trivially ordered and L is a multilattice group in which each strictly positive element is over an atom.*

Proof. Suppose that G is a multilattice group and that K is not trivially ordered. If L is not simply ordered, there exists $a \| 0 \in L$. Suppose that (u, v) is a minimal upper bound of $(a, 0)$ and $(0, 0)$; then u is a minimal upper bound of a and 0 in L. But, for any $0 > k \in K$, $(u, v) > (u, v + k)$ and, by the lexicographic ordering, $(u, v + k)$ is an upper bound of $(a, 0)$ and $(0, 0)$. Hence L must be simply ordered.

By the lexicographic ordering, $(x, y) \geqslant (0, b)$ and $(x, y) \geqslant (0, 0)$ if and only if $x > 0$ or $x = 0$ and $y \geqslant 0$, $y \geqslant b$; since K is not trivially ordered, any minimal upper bound must be of the form $(0, y)$. Hence G is a multilattice only if, for each $b \in K$, each member of the set

$$\{(0, p) \colon p \geqslant 0 \quad \text{and} \quad p \geqslant b\}$$

is over a minimal member of the set, and K is directed. Thus G is a multilattice only if (i) is satisfied.

Suppose that G is a multilattice group and that K is trivially ordered. Let $0 \neq a \in K$, and let (p, q) be a minimal upper bound of $(0, 0)$ and $(0, a)$; then p is an atom of L. Hence, by the lexicographic ordering, every strictly positive element of L is over an atom.

Since G is a multilattice group, L is directed. Let $0 \| a \in L$, then $x > 0$, $x > a$ if and only if, for some $k \in K$, $(x, k) > (0, 0)$ and $(x, k) > (a, 0)$. Hence G is a multilattice group only if L is a multilattice group; that is, since each strictly positive element of L is over an atom, only if (ii) is satisfied.

Conversely, we may easily verify that, if either (i) or (ii) holds, then G is a multilattice group.

For a multilattice group we may extend the concept of lexicographic product as follows.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

Let G be a multilattice group, and let K be a multilattice ideal of G. Then G is said to be the *lexico extension* of K by G/K if and only if G/K is simply ordered, and every strictly positive element of $G\backslash K$ is greater than every positive element of K.

In this case we shall write $G = \langle K \rangle$.

Let G be a multilattice group, and let K be a trivially ordered normal subgroup of G. Then G is said to be the *p-lexico extension* of K by G/K if and only if G/K is a multilattice group, each strictly positive element of which is over an atom, and every strictly positive element of $G\backslash K$ is greater than each element of K.

In this case we write $G = \langle K \rangle_p$.

Let the multilattice group G be the lexicographic product of K by L. Then G is isomorphic to the lexico extension of K by G/K if (i) of Theorem 8 holds, while G is isomorphic to the p-lexico extension of K by G/K if (ii) is satisfied.

THEOREM 9. *Let G be a multilattice group, and let K be a normal subgroup of G. Then G is the lexico extension of K by G/K if and only if K is directed and every strictly positive element of $G\backslash K$ is greater than each element of K.*

Proof. From the definition of lexico extension, the conditions are obviously necessary; we now show that they are sufficient.

We note first that, under the conditions of the theorem, K must be convex. Let $0 < a \in K$ and suppose that $a > h$ for some $0 \neq h \in H$, then, by convexity, $e \in K$ for some $e \in h \vee 0$. If $e - h \notin K$ then, by hypothesis, $e - h > e$; but this is impossible since $h \in H$. Hence K obeys $(\mathscr{H})$ and so is a multilattice ideal of G.

If G/K is not simply ordered, there are positive elements $a, b \in G$ such that

$$K \in (K + a) \wedge (K + b);$$

that is, such that $(a \wedge b) \cap K \neq \phi$. But a and similarly b is greater than each element of K; hence K must have at least one maximal element. Since K is directed, this is impossible and so G/K must be simply ordered. Thus $G = \langle K \rangle$.

THEOREM 10. *Let G be a multilattice group, and let K be a normal subgroup of G. Then G is the p-lexico extension of K by G/K if and only if K is trivially ordered and every strictly positive element of G is greater than each element of K.*

In this case, each strictly positive element of G is over an atom; the atoms of G/K are the images of the atoms of G.

Proof. The conditions are obviously necessary; we now show their sufficiency. Clearly each non-zero member of $K \vee 0$ is an atom of G; in fact the set of non-zero members of $K \vee 0$ is precisely the set of atoms of G. Hence each strictly positive member of G is over an atom of G.

Let $a \in G$ be such that $K + a > K$, then, for some $k \in K$, $a > k$ and so $a - k > 0$. Hence, by hypothesis, $a - k$ is greater than each element of K; in particular, $a - k > -k$. Hence $K + a > K$ if and only if $a > 0$.

It follows easily from the immediately preceding paragraph that G/K is a multilattice group in which, for non-comparable cosets K and $K + a$,

$$K \vee (K + a) = K + (a \vee 0).$$

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

It is also easy to see that the atoms of G/K are the cosets containing the atoms of G. Hence, since the mapping $x \to K + x$ is isotone, every strictly positive element of G/K is over an atom, and $G = \langle K \rangle_p$.

An important class of multilattice groups is that consisting of those which are p-lexico extensions of trivially ordered groups by simply ordered groups; we shall make use of these in section 4. Such groups may easily be characterized as we shall now show.

A subset X of a partially ordered set P is said to be *completely diverse* if and only if:

(i) X is *diverse*, that is $a, b \in X$ and $a \geqslant b$ imply $a = b$;

(ii) if $x \in X$, $a \in P$ and $a \| x$ then $a \in X$.

THEOREM 11. *Let G be a multilattice group, then the following four conditions on G are equivalent*:

(i) *G is the p-lexico extension of a trivially ordered group by a simply ordered group*;

(ii) *G has a completely diverse subset*;

(iii) *H is a completely diverse subset of G*;

(iv) *H is a subgroup, every strictly positive element of G is greater than each element of H.*

Proof. Let $G = \langle K \rangle_p$, where G/K is simply ordered, and let $a \in G$ with $a \| k$ for some $k \in K$. If $a \notin K$, then, since G/K is simply ordered, either $K + a > K$ or $K + a < K$. But, as in Theorem 10, this implies that a is comparable with each member of K. Hence $a \in K$ and K is a completely diverse subset of G.

We next show that any completely diverse subset of G which contains 0 is a subgroup of G. Let X be such a subset of G and let a, b be distinct elements of X neither of which is 0. Since $0 \in X$, $a + b \| a$ and hence $a + b \in X$, also $a \| 0$ implies $-a \in X$. Thus X is a subgroup of G.

Let Y be any completely diverse subset of G, and let $y \in Y$. Then the set

$$\{x \in G\colon x = z - y \quad \text{for some} \quad z \in Y\}$$

is clearly a completely diverse subset of G which contains 0; we show that it is H, let us denote it by D. Let $a > 0$ and $d \in D$ then, since D is completely diverse and $0 \in D$, $a > d$.

Let $e \in d \vee 0$ for some $0 \neq d \in D$, then $e + d > 0$; for otherwise $e + d$ and hence $e \in D$. Thus $e + d \geqslant f$ for some $f \in d \vee 0$. Also $f - d \geqslant g$ for some $g \in d \vee 0$. But then $e \geqslant f - d \geqslant g$ which implies, since $e,\ g \in d \vee 0$, $e + d = f \in d \vee 0$. Hence $D \subseteq H$, but clearly $H \subseteq D$ and so $H = D$.

Thus (ii) implies that H is a completely diverse subset and, since $0 \in H$, this implies that H is a subgroup of G and that every strictly positive element of G is greater than each element of H.

If (iv) is satisfied then, by Theorem 10, $G = \langle H \rangle_p$, and the atoms of G/H are the images of the atoms of G. Let e, f be atoms of G, and let $0 \neq h \in H$, then, since by (iv) $e, f \in h \vee 0$, $e - f \in H$. Hence G/H has exactly one atom and this clearly implies, as in the theorem of Teh ((8)), that G/H is simply ordered.

4. *Distributive multilattice groups of finite type.* In this section, we shall be concerned with the structure of certain distributive multilattice groups which obey:

($\mathscr{F}$) *each strictly positive element is over at most a finite number of members of H.*

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

PROPOSITION 10. *Let G be a multilattice group which obeys $(\mathscr{F})$, and let $0 < a \in G \setminus N$. Then a is over a minimal non-zero member of $\check{H} = H \vee 0$; the minimal non-zero members of $\check{H}$ are the atoms of $G \setminus N$.*

Proof. Let $0 \neq e_0 \in \check{H}$; if e_0 is not over a minimal non-zero member of $\check{H}$, then we can construct an infinite strictly decreasing sequence,

$$e_0 > e_1 > e_2 > \ldots > e_n > \ldots,$$

of non-zero members of H. Suppose that $e_n \in h_n \vee 0$ where $h_n \in H$; then, clearly, $h_i \neq h_j$ for $i \neq j$, and hence e_0 is over an infinite number of members of H contrary to $(\mathscr{F})$.

Since, by Theorem 3, $G \setminus N$ obeys $(\mathscr{O})$, it follows that each strictly positive element of $G \setminus N$ is over a minimal non-zero member of $\check{H}$. Further it is clear that the minimal non-zero members of $\check{H}$ are the atoms of $G \setminus N$.

Distributive multilattice groups have many properties which are not enjoyed by arbitrary multilattice groups, most of these are consequences of the next theorem.

THEOREM 12. *Let G be a distributive multilattice group, then every element of H has order two.*

Proof. Let $0 \neq h \in H$ and $e, f \in h \vee 0$ with $0 \in e \wedge f$, $h \in e \wedge f$; by Proposition 1, this is possible. Then $0 \in e - (h \vee 0) = (e - h) \wedge e$ and

$$e - h + f \in e - h + (h \vee 0) = (e - h) \vee e.$$

But
$$e - h + f \in e - (e \wedge f) + f = e \vee f.$$

Hence, by distributivity, $e - h = f$. Similarly $e = f - h$ and so $2h = 0$.

COROLLARY 1. *Let G be a distributive multilattice group, then G obeys $(\mathscr{H}^*)$.*

COROLLARY 2. (*Vaida* ((9))). *Let G be a distributive multilattice group and let p, q be distinct atoms of G then either $p \wedge q = 0$ or $2(p - q) = 0$.*

Proof. If $p \wedge q \neq 0$ then there exists $0 \neq h \in p \wedge q$. Since p, q are atoms, $p, q \in h \vee 0$. Hence $p - q \in H$.

From Theorem 12, we may also immediately deduce

THEOREM 13. *Let G be a multilattice group, then G is a lattice group if and only if*

(i) *G is distributive, and*

(ii) *G contains no non-zero elements of order two.*

As a special case of Theorem 13 we have the main result of (9).

COROLLARY. *Let G be a multilattice group, then G is isomorphic to the direct sum of copies of Z if and only if*

(i) *G is distributive,*

(ii) *G obeys the minimal condition, and*

(iii) *G contains no non-zero elements of order two.*

LEMMA 5. *Let P be any set of atoms of a distributive multilattice group G, and let ρ be the following relation on P:*

$$a \equiv b(\rho) \quad \text{if and only if} \quad 2(a - b) = 0.$$

Then ρ is an equivalence on P and each ρ-class has at most two members.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

Proof. Clearly ρ is reflexive and, since $2(a-b) = 0$ implies

$$a-b = -(a-b) = b-a,$$

symmetric. Suppose that $a, b, c \in P$ and that $a \equiv b(\rho)$ and $b \equiv c(\rho)$ where $a-b = h \neq 0$, $b-c = k \neq 0$. Then $a-b = h$ implies, since $2h = 0$, that $b > h$ and, since b is an atom, that $b \in h \vee 0$; similarly $b \in k \vee 0$. Thus $b \in (h \vee 0) \cap (k \vee 0)$. But since $h = -h$ and $k = -k$, this implies $-b \in (h \wedge 0) \cap (k \wedge 0)$, and hence, by distributivity, that $h = k$; that is $a-c = h+k = 0$. Thus ρ is an equivalence and each ρ-class has at most two members.

A multilattice group G is said to be *almost ordered* if and only if it contains no orthogonal elements; it is said to be *properly almost ordered* if it is almost ordered but not simply ordered.

THEOREM 14. *Let G be a distributive multilattice group which obeys $(\mathscr{F})$ then G is almost ordered if and only if it is either simply ordered or is isomorphic to the p-lexico extension of $\frac{1}{2}\mathbf{Z}$ (the additive group of integers modulo two), by a simply ordered group.*

Proof. If G is almost ordered, then, since N and N^* are orthogonal, G is either a lattice group, and hence simply ordered, or obeys $(\mathscr{O})$.

Suppose that G obeys $(\mathscr{O})$. Then, by Proposition 10, Theorem 12, Corollary 2 and Lemma 5, G has exactly two atoms, p, q; let $p-q = k$. It is easily verified that $K = \{0, k\}$ is a normal subgroup of G and that every strictly positive element of G is greater than each element of K. Hence $G = \langle K \rangle_p$.

Further the atoms of G/K are the cosets $K+p$, $K+q$, hence G/K has a minimum strictly positive element, and so is simply ordered. The necessity of the conditions is thus verified; their sufficiency is obvious.

THEOREM 15. *Let G be a distributive multilattice group which obeys $(\mathscr{F})$. Then G is strongly archimedean if and only if it is isomorphic either to a subgroup of the additive reals or to the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$.*

Proof. Let G be strongly archimedean, then G is almost ordered, and we may apply Theorem 14.

Suppose that G is isomorphic to the p-lexico extension of $\frac{1}{2}\mathbf{Z}$ by a simply ordered group, and let p, q be the atoms of G where $p-q = h$; then $G = \langle H \rangle_p$ and $H = \{0, h\}$.

Since G is strongly archimedean, and the natural mapping η of G onto G/H has the property that

$$x\eta > 0\eta \quad \text{if and only if} \quad x > 0,$$

G/H is strongly archimedean. Since it has an atom, it must be isomorphic to $\mathbf{Z}$.

Since $a, b \in G$ and $a \| b$ implies $a-b = h$, every element of G has the form $np + \epsilon h$ where $n \in \mathbf{Z}$ and $\epsilon = 0$ or 1. Let $Z(p)$ be the cyclic subgroup of G generated by p; it is easy to see that G is, as an abstract group, isomorphic to the direct sum of $Z(p)$ and H. Finally, since $G = \langle H \rangle_p$, $np + \epsilon h > 0$ if and only if $np > 0$; that is if and only if $n > 0$. Hence G is isomorphic to the lexicographic product of $\mathbf{Z} \simeq Z(p)$ and $\frac{1}{2}\mathbf{Z} \simeq H$.

If G is simply ordered then (c.f. (6), page 45) G is isomorphic to a subgroup of the additive reals; hence the necessity of the conditions. Again their sufficiency is obvious.

We shall now proceed to the proofs of our main structure theorems; to prove them, we need a sequence of lemmas. From now on, unless the contrary is stated, G denotes a distributive multilattice group which obeys $(\mathscr{F})$.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

Let $\overline{P}$ be the set of minimal non-zero members of $\check{H}$, and let $\{\rho_i: i \in I\}$ be the set of ρ-classes of $\overline{P}$. For each $i \in I$, define $G_i = \{x \in G$: for some $x' \in \|x\|$, $x' \wedge e_j = 0$ for each $e_j \in \overline{P} \setminus \rho_i$ and $x' \wedge e = 0$ for each $0 < e \in N\}$.

It is clear, from Theorem 12, Corollary 2 and Lemma 5, that, for each $i \in I$, $\rho_i \subseteq G_i$ and hence no G_i is trivial.

LEMMA 6. *For each $i \in I$, G_i is a directed convex subgroup of G; it is isomorphic to the p-lexico extension of $\frac{1}{2}\mathbf{Z}$ by a simply ordered group.*

Proof. For each $i \in I$, $G_i = \cap\{\rho_j^*: j \in I \setminus \{i\}\} \cap N^*$; hence, since each of these is, by Proposition 7, a directed convex subgroup of G, G_i is directed and convex and so, by Theorem 12, Corollary 1, obeys ($\mathscr{H}$).

Clearly, the atoms of G_i are the members of ρ_i, and, by Theorem 12, Corollary 2, no two of these are orthogonal. Since G_i is convex, it follows from Proposition 10, that G_i is almost ordered, and, since it obeys ($\mathscr{H}$), it is not simply ordered. The result is now immediate from Theorem 14.

LEMMA 7. *Let $\overline{N}$ be the subgroup of G generated by N and $\{G_i: i \in I\}$, then $\overline{N}$ is isomorphic to their direct sum $N \oplus \Sigma \oplus \{G_i: i \in I\}$.*

Proof. Since, $\cup\{G_i: i \in I\} \subseteq N^*$, it is clear that distinct groups G_i, G_j are disjoint from one another and from N. Hence, from Theorem 12 and Theorem 6, the subgroup of G generated by N and $\{G_i: i \in I\}$ is isomorphic to their direct sum.

LEMMA 8. *Let $0 < a \in G \setminus \overline{N}$; then, for some $i \in I$, a is greater than each atom of G_i.*

Proof. Since $N \subseteq \overline{N}$, if $0 < a \in G \setminus \overline{N}$, then $a > h$ for some $0 \neq h \in H$, and hence $a > e$ for some $e \in \overline{P}$. For such an e, $a = d + e$ where $0 < d \in G \setminus \overline{N}$. Since $d < a$, if $f < d$ for some $f \in \rho_i$ for some $i \in I$, then $f < a$. We show that

(*) If d is over a member of ρ_i if and only if a is over a member of ρ_i then the conclusions of the lemma are satisfied for a.

Under these latter conditions, d is greater than at least one member of the ρ-class $\{e, e+h\}$ containing e. Hence either $a > 2e$ or $a > 2e + h$; in any case, since $2e = 2(e+h)$, $a > e + h$. Thus the conclusions of the lemma are satisfied for a.

We now use induction on the statement:

If $0 < a \in G \setminus \overline{N}$ is over a member of ρ_j for each of n distinct ρ-classes of $\overline{P}$, then, for some $i \in I$, a is greater than each member of ρ_i.

Suppose that a is greater than a member of ρ_i for exactly one ρ-class then, with the notation above, d is greater than a member of that ρ-class and hence, by (*), the conclusions of the lemma are satisfied for a.

Suppose that a is greater than a member of ρ_j for each of r distinct ρ-classes, and suppose that the result is true for each $0 < b \in G \setminus \overline{N}$ which is over a member of ρ_j for each of s ρ-classes and some $s < r$. Then, since $a = d + e$ where $0 < d \in G \setminus \overline{N}$ is over a member of ρ_j for at most r distinct ρ-classes, either by the induction hypothesis or by (*), a is greater than each member of ρ_i for some $i \in I$.

Since G obeys ($\mathscr{F}$) and, for each $i \in I$, the members of ρ_i are the atoms of G_i, the lemma is proved.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

LEMMA 9. *Let $0 < a \in G\backslash\overline{N}$; then, for some $i \in I$, $a > a_i$ for each $a_i \in G_i$.*

Proof. Let $0 < a \in G\backslash\overline{N}$ and let $i \in I$ be such that a is greater than each atom of G_i. Let $a_i \in G_i$ be such that a_i exceeds each atom of G_i. Then, for $0 \leqslant d \in a \wedge a_i$, we have

$$a = d + a', \quad a_i = d + a_i' \quad \text{where} \quad 0 \in a' \wedge a_i'.$$

By the convexity of G_i, $d \in G_i$ and $a_i' \in G_i$; hence $a' \in G\backslash\overline{N}$.

An induction proof as in Lemma 8, using a' as d was used there, completes the proof.

THEOREM 16. *Let G be a distributive multilattice group which obeys $(\mathscr{F})$. Then G is isomorphic to the direct sum of a lattice group and p-lexico extensions of $\frac{1}{2}\mathbf{Z}$ by simply ordered groups if and only if no maximal convex, properly almost ordered, subgroup of G is bounded above.*

Proof. Let K be a convex, properly almost ordered, subgroup of G; clearly $K \subseteq N^*$. Since by Theorem 14, K is isomorphic to the p-lexico extension of $\frac{1}{2}\mathbf{Z}$ by a simply ordered group, $K \cap \overline{P} = \rho_i$ for some $i \in I$.

Let $0 < a \in K$ and let $e_j \in \rho_j$ for $i \neq j$, then, by Lemma 2, $a \wedge e_j \subseteq K \cap G_j$. Since by Theorem 11, $0 \| h \in K$ implies $h \in \rho_i - \rho_i \subseteq G_i$ it follows that $a \wedge e_j = 0$. Thus $K \subseteq G_i$ and the groups G_i, $i \in I$ are the maximal convex, properly almost ordered subgroups of G.

It is thus immediate, from Lemma 9, that the conditions of the theorem are sufficient; their necessity is easily verified.

A multilattice group G is said to be *complete* if and only if it is a complete multilattice in the sense of Benado ((3)). That is, if and only if each subset of G which is bounded above by an element $a \in G$ has a minimal upper bound in G, under a; and dually for lower bounds.

Such multilattice groups have many properties analogous to those of complete lattice groups. However, we are here concerned only to notice that no non-trivially ordered subgroup of such a group G may be bounded above.

Suppose that G is a complete multilattice group and that, for some $0 \leqslant b \in G$, the cyclic subgroup $Z(b)$ of G generated by b is bounded above. Let a be a minimal upper bound of $Z(b)$, then, since $b \geqslant 0$, $a - b \leqslant a$; but $Z(b) \leqslant a - b$, hence $a = a - b$ and $b = 0$.

THEOREM 17. *Let G be a distributive multilattice group which obeys $(\mathscr{F})$. Then G is complete if and only if it is isomorphic to the direct sum of a complete lattice group and copies of the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$.*

Proof. Suppose that G is complete. Then, by convexity, each directed convex subgroup of G is complete. Let $\rho_i = \{e_i, e_i + h_i\}$ and let $0 < a \in G_i$. Since G_i is complete, there exists $n \in \mathbf{Z}^+$ such that $a \not> ne_i$. But, since G_i is the p-lexico extension of $\{0, h_i\}$ by a simply ordered group, $a \| ne_i$ implies $a = ne_i + h_i$. It thus follows that G_i is strongly archimedean and so, by Theorem 15, is isomorphic to the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$.

Since N is directed and convex, it is a complete multilattice group; since it latticed, it must be a complete lattice group.

Since G is complete, we may apply Theorem 16, obtaining the necessity of the conditions; their sufficiency may easily be verified.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

COROLLARY. *Let G be a distributive multilattice group which obeys $(\mathscr{F})$ if G is complete, then G is commutative.*

Although every complete lattice group is commutative, this need not be the case for arbitrary complete multilattice groups, as may easily be seen by considering the lexicographic product of a non-commutative, trivially ordered, group by $\mathbf{Z}$.

It is also a corollary to Theorem 17 that a distributive multilattice group, which obeys $(\mathscr{F})$ and the minimal condition for positive elements ((6), page 71), is isomorphic to the direct sum of copies of $\mathbf{Z}$ and copies of the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$. This latter result may however be strengthened by omitting the requirement that G obeys $(\mathscr{F})$, as we shall now indicate.

Let G be such a group, let P be the set of all atoms of G, and let $\{\rho_i : i \in I\}$ be the set of ρ-classes of P. For each $i \in I$, define

$$G_i = \{x \in G: \text{for some } x' \in \|x\|,\ x' \wedge e_j = 0 \text{ for each } e_j \in P \backslash \rho_i\}.$$

Then each $G_i\,(i \in I)$, is an almost ordered convex subgroup of G, and, by the minimal condition, is either isomorphic to $\mathbf{Z}$ or to the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by $\mathbf{Z}$. Further, distinct G_i are pairwise orthogonal and generate G, hence we have, from Theorem 7:

THEOREM 18. *Let G be a multilattice group, then G is isomorphic to the direct sum of copies of $\mathbf{Z}$ and copies of the lexicographic product of $\frac{1}{2}\mathbf{Z}$ by Z if and only if it is distributive and obeys the minimal condition for positive elements.*

Note. Using methods almost identical to those of Conrad ((5)), it may be shown that if a distributive multilattice group obeys $(\mathscr{F})$ then N^* is a lexico sum of p-lexico extensions of $\frac{1}{2}\mathbf{Z}$ by a simply ordered group. However, in this case it is not necessarily true that G is isomorphic to $N \oplus N^*$; see Example 2.

5. *Examples.* (1) Let G be the group of all triples of integers (m, n, p), such that $m+n+p$ is even, under componentwise addition, ordered by

$$(m, n, p) \geqslant (0, 0, 0) \quad \text{if and only if} \quad m \geqslant 0,\ n \geqslant 0,\ p \geqslant 0.$$

Clearly G is directed and obeys the minimal condition, hence it is a multilattice group.

In G, H consists of all triples (i, j, k) of G where each of i, j, k is one of $0, +1, -1$ and $i+j+k = 0$.

(A) Let K be the set of all triples $(0, n, p) \in G$. Then K is a directed convex subgroup of G, but is not a submultilattice of G. For $(1, 1, 0) \in (0, 1, -1) \vee (0, 0, 0)$, but $(1, 1, 0) \notin K$ while $(0, 1, -1) \in K$.

(B) Let G_1 consist of all triples $(m, 0, 0)$, G_2 of all triples $(0, n, 0)$ and G_3 of all triples $(0, 0, p)$ of G. Then the groups $G_i\,(i = 1, 2, 3)$ are pairwise orthogonal directed convex subgroups of G. The subgroup $\bar{G}$ of G that they generate is isomorphic to their direct sum; it consists of all triples (m, n, p) where each of m, n, p is even. However $\bar{G}$ is not convex, for $(2, 2, 0) \in \bar{G}$ and

$$(2, 2, 0) > (1, 1, 0) > (0, 0, 0)$$

while $(1, 1, 0) \notin \bar{G}$.

(C) It is well known that a commutative lattice group has no non-trivial lattice ideals if and only if it is strongly archimedean; for a multilattice group to have no non-trivial multilattice ideals, this condition need not be necessary unless the group obeys $(\mathscr{H}^*)$.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press

Suppose that M is a non-trivial multilattice ideal of G, then M contains a positive element (m, n, p) where one of m, n, p is at least two. By symmetry, we may suppose $m \geqslant 2$, then, by convexity, $(2, 0, 0) \in M$. Hence since $(+1, -1, 0) \in H$ and M obeys $(\mathscr{H})$, $(+1, -1, 0) \in M$. But this implies $(0, 2, 0) = (2, 0, 0) + (-2, 2, 0) \in M$. Similarly, $(0, 0, 2) \in M$ and $H \subseteq M$. But, clearly, G is generated by $\bar{G}$ and H; hence $M = G$.

(2) Let L be the complete direct sum of an infinite set $\{\mathbf{Z}_i : i \in I\}$ of copies of the integers. Let K be the p-lexico extension of $\frac{1}{2}\mathbf{Z}$ by a simply ordered group, and let M be the lexicographic product of K by the integers. Let G be the subgroup of $L \oplus M$ consisting of all elements (a, b) where $a \in L$ and $b \in M$ such that, if $b = (m, k)$ where $m \in \mathbf{Z}$ and $k \in K$, then, except possibly for a finite number, each component of a is equal to m. Then G is a distributive multilattice group in which H contains only one non-zero element. However, N consists of all elements of the form $(a, 0)$ where a belongs to $\Sigma \oplus \{\mathbf{Z}_i : i \in I\}$ and N^* consists of all elements of the form $(0, b)$ where $b = (0, k)$ for some $k \in K$. Hence G is not isomorphic to the direct sum of N and N^*.

REFERENCES

(1) BENADO, M. Sur la théorie de la divisibilité. *Acad. R.P. Romîne. Bul. Sti. Sect. Sti. Mat.-Fiz.* **6**, no. 2 (1954), 263–270.

(2) BENADO, M. Les ensembles partiellement ordonnées et le théorème de raffinement de Schreier, II. *Czechoslovak Math. J.* (5) **80** (1955), 308–344.

(3) BENADO, M. Bemerkungen zur Theorie der Vielverbande, IV. *Proc. Cambridge Philos. Soc.* **56** (1960), 291–317.

(4) BURGESS, D. C. J. Generalized intervals in partially ordered groups. *Proc. Cambridge Philos. Soc.* **55** (1959), 165–171.

(5) CONRAD, P. Some structure theorems for lattice groups. *Trans. American Math. Soc.* **99** (1961), 212–240.

(6) FUCHS, L. *Partially ordered algebraic systems* (Pergamon Press; London, 1963).

(7) JAFFARD, P. Contribution a l'étude des groupes ordonnées. *J. Math. Pures Appl.* **32** (1953), 203–280.

(8) TEH, H. H. A note on l-groups. *Proc. Edinburgh Math. Soc.* **13** (Series II), Part I (1962), 123–24.

(9) VAIDA, D. Groupes ordonnées dont les éléments admettent une décomposition jordanienne généralisée. *C.R. Acad. Sci. Paris*, **257** (1963), 2053–55.

https://doi.org/10.1017/S0305004100038962 Published online by Cambridge University Press