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The closing lemma for holomorphic maps

JOHN ERIK FORNÆSS†§ and NESSIM SIBONY‡

† *Mathematics Department, The University of Michigan, Ann Arbor, MI 48104, USA*
‡ *Universite Paris Sud, Bat 425. Mathematiques, 91405 Orsay, France*

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Abstract. The closing lemma is proved for various classes of holomorphic self-maps on $\mathbb{C}^k$. These classes include those consisting of biholomorphic maps, endomorphisms and symplectomorphisms.

1. *Introduction*

Let f be an endomorphism of a smooth manifold M. Let d be a distance compatible with the topology. A point x is recurrent if

$$\underline{\lim}_{n\mapsto\infty} d(f^n(x), x) = 0.$$

A remarkable result of Pugh [**P**], see also Pugh and Robinson [**PR**], states that for any diffeomorphism f of a compact manifold and any $\mathcal{C}^1$ neighborhood V of f, there is a diffeomorphism $g \in V$ such that x is periodic for g.

The question, in this form, is open for the $\mathcal{C}^k$ topology for $k \geq 2$. A similar question can be asked for symplectic diffeomorphisms, flows, etc. When the answer is affirmative we will say that the closing lemma holds.

There are various ways of asking for a stronger result than simply that recurrent orbits can be closed. Instead of considering recurrent points, one can consider non-wandering points, i.e. points p so that for any neighborhood U of p there is an $n \geq 1$ such that $f^n(U)$ intersects U. Then one can ask if the orbit of p can be closed up. Another way of strengthening the result is to require that if the orbit of the recurrent point lies in a compact set K and U is some neighborhood of K then the orbit of the approximating periodic point can be required to be contained in U.

In the case of non-compact manifolds one can demand that the approximating maps are close on compacts or in the Whitney topology of better approximation at infinity. In the holomorphic case, only the first topology makes sense.

Herman [**H1**], see also the survey by Yoccoz [**Y**], has recently constructed a counter-example to a closing lemma in the $\mathcal{C}^k$ topology for k large enough. More precisely, he has a symplectic form ω_A on the $2p$ dimensional torus T^{2p}, $p \geq 2$, and a Hamiltonian H_0

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such that for H close enough to H_0 (in the $\mathcal{C}^k$ topology, k large enough) and c in an open interval, the orbits of the Hamiltonian flow on $H^{-1}(c)$ are dense. In particular, there are no periodic points.

In the survey by Yoccoz [**Y**] the following question is asked: Is the closing lemma true for biholomorphisms.

Following methods introduced in [**FS1–3**] we prove here that the closing lemma is true for biholomorphisms of $\mathbb{C}^k$, endomorphisms of $\mathbb{C}^k$ and symplectic biholomorphisms of $\mathbb{C}^{2p}$. We also need approximation results from Andersén [**A**] and Forstneric and Rosay [**FR**]. To prove the closing lemma for Hamiltonian vector fields in $\mathbb{C}^2$ we use a minor modification of a technique developed in [**FG**].

There are two opposite questions to consider: one is whether closed orbits can be broken, the other is whether almost closed orbits can be closed. The previous papers [**FS1–3**] and [**FG**] have been concerned with the question of breaking closed orbits and showing that, generically, orbits in the holomorphic category are unbounded. This means that the closing lemma in the holomorphic case can be reduced to showing that orbits can be closed up near infinity. This leaves open the question whether orbits can be closed in small neighborhoods of the given recurrent orbit. This is already unknown in the case of quadratic polynomial maps in $\mathbb{C}$.

2. *Holomorphic symplectomorphisms*

The simplest case to deal with is that of biholomorphic maps which preserve the symplectic form. There, most orbits are unbounded according to previous work by the authors [**FS3**].

We first recall some definitions. In $\mathbb{C}^{2k}$ we consider the complex symplectic form

$$\Omega := \sum_{j=1}^{k} dz_j \wedge dw_j,$$

where $(z, w), z = (z_1, \ldots, z_k), w = (w_1, \ldots, w_k)$ denote coordinates in $\mathbb{C}^{2k}$. A holomorphic map $\Phi : \mathbb{C}^{2k} \mapsto \mathbb{C}^{2k}$ is symplectic if $\Phi^*(\Omega) = \Omega$.

Let $z_j = p_{2j-1} + ip_{2j}$ and $w_j = q_{2j-1} - iq_{2j}$, where p_j and q_j are real coordinates. Then the real part of Ω, $\Re\Omega = \sum_1^{2k} dp_j \wedge dq_j =: \omega_0$. It follows that if we consider Φ as a map on $\mathbb{R}^{4k}$ with the previous coordinates, then Φ is a symplectomorphism with respect to the standard symplectic structure ω_0.

Let S denote the space of biholomorphic symplectomorphisms of $\mathbb{C}^{2k}$, endowed with the topology of convergence on compact sets. Then S is a Baire space.

Given $f \in S$, we define

$$K_f := \{(z, w);\ (f^n(z, w)) \text{ is bounded}\}.$$

Then K_f is an F_σ set.

Solving a conjecture by Herman, the authors proved the following result in [**FS3**].

THEOREM 2.1. *There is a G_δ dense set $S' \subset S$ such that for $f \in S'$, K_f is an F_σ set with empty interior.*

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Using the above result we are going to prove a strong form of the closing lemma for the class S.

THEOREM 2.2. *Let $f \in S$. Given any $x \in \mathbb{C}^{2p}$ and any neighborhood V of f in S there is a $g \in V$ such that x is a periodic point for g.*

The result is a consequence of the following lemma.

LEMMA 2.3. *Let $f_0 \in S$. Assume (p, q) are two points in $\mathbb{C}^{2k} \setminus \{K_{f_0} \cup K_{f_0^{-1}}\}$. In any given neighborhood V of f_0 in S there is an f such that p, q belong to a periodic orbit of f.*

Proof. Let $B(0, R)$ be the ball of radius R in $\mathbb{C}^{2k}$. Assume

$$V = \{f;\ f \in S, |f - f_0| < \epsilon \text{ in } B(0, R)\}.$$

Choose R_1 such that $f_0(B(0, R)) \subset B(0, R_1)$. We can assume $p_j := f_0^j(p) \in \overline{B(0, 2R_1)}$ if $-M < j < N$ and that p_{-M}, $p_N \notin \overline{B(0, 2R_1)}$. Similarly, assume $q_j := f_0^j(q) \in \overline{B(0, 2R_1)}$ if $-M' < j < N'$ and that $q_{-M'}$, $q_{N'} \notin \overline{B(0, 2R_1)}$.

The idea is to construct $s, \sigma_1, \sigma_2 \in S$ satisfying the following conditions.

(i) s, σ_1, σ_2 are close to the identity on $B(0, 2R_1)$.

(ii) $\sigma_1(p_N) = p'_N$ where all coordinates of p'_N are strictly larger than R, $\sigma_1(p_j) = p_j$, $-M \le j < N$, and $\sigma_1(q_{N'}) = q'_{N'}$ where all coordinates of $q'_{N'}$ are strictly larger than R, $\sigma_1(q_j) = q_j$, $-M' \le j < N'$.

(iii) $\sigma_2(q'_{-M'}) = q_{-M'}$, $\sigma_2(p'_{-M}) = p_{-M}$ where all coordinates of $q'_{-M'}$, p'_{-M} are very large, $\sigma_2(p_j) = p_j$, $-M < j \le N$, $\sigma_2(q_j) = q_j$, $-M' < j \le N'$. The coordinates of p'_N, q'_N, p'_{-M} and $q'_{-M'}$ are all distinct.

(iv) $s(p_j) = p_j$, $-M \le j \le N$, $s(q_j) = q_j$, $-M' \le j \le N'$.

(v) $s(p'_N) = q'_{-M'}$, $s(q'_{N'}) = p'_{-M}$.

The point (p_0, q_0) will be on a periodic orbit of

$$s \circ \sigma_1 \circ f_0 \circ \sigma_2.$$

We now construct s, σ_1, σ_2.

We introduce the following symplectomorphism, a so-called shear. Write $Z = (z, w)$. Define

$$\tau(Z) = Z + h(\Omega(Z, \xi))\xi,$$

where h is an entire function. It is easy to check that τ is a symplectomorphism. Observe that τ moves points along the ξ direction.

Choose ξ of norm 1 such that $|\Omega(p_N, \xi)| > 2R_1$, and assume all coordinates of ξ are different from 0. Let h be an entire function vanishing at all $\Omega(p_j, \xi)$, $-M \le j < N$, and on $\Omega(q_j, \xi)$, $-M' \le j \le N'$ and small on the disc $D(0, R_1)$, and $h(\Omega(p_N, \xi))$ very large. Let τ_ξ be the corresponding symplectomorphism.

Choose another vector η such that τ_η fixes p_j, $-M \le j \le N$, and $p'_N := \tau_\xi(p_N)$ and q_j, $-M' \le j < N'$, but such that $\tau_\eta(q_{N'}) =: q'_{N'}$ has all coordinates very large.

Define $\sigma_1 = \tau_\eta \circ \tau_\xi$. Similarly, we construct σ_2, satisfying the given condition.

Now define

$$s_i(z, w) = (z_1, w_1, \ldots, z_i + \alpha_i(w_i), w_i, \ldots)$$
$$s'_i(z, w) = (z_1, w_1, \ldots, z_i, w_i + \beta_i(z_i), \ldots),$$

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where α and β are entire functions. The symplectomorphism s will be a composition of $2k$ shears of type s_i or s_i'. For example, s_1 will fix all points in the construction, will be close to the identity on $B(0, R_1)$, and will be such that $s_1(p_N')$ and $q_{-M'}'$ also have the same z_1 coordinate. We choose the other s_i, s_i' in order that the z_i coordinate (respectively w_i) of the above points coincide.

Proof of the theorem. Let $f_0 \in S$. Let V be a neighborhood of f_0. Given any x we can choose f in V such that $x \notin \{K_f \cup K_{f^{-1}}\}$ which is an F_σ of empty interior. The lemma then implies that x is in a periodic orbit for a symplectomorphism $g \in V$. □

Recall the following definition.

Definition 2.4. If f is a self-map on a locally compact space Y, a point $p \in Y$ is wandering if there is a neighborhood $U(p) \subset Y$ such that $U(p) \cap f^n(U(p)) = \emptyset \ \forall n \neq 0$. The non-wandering set $\Omega(f)$ consists of all points that are not wandering.

Remark 2.5. It is an observation due to Herman [**H2**] that there are biholomorphic maps on $\mathbb{C}^2$ with a dense orbit. The above construction shows that for $\Phi \in \mathcal{S}$, where $\mathcal{S}$ is a dense G_δ set in S, generically the orbits of Φ are dense. Fix p and a polydisc Δ. Let

$$G_\Delta := \{\Phi;\ \Phi \in S,\ p \in \cup_{n \geq 0} \Phi^{-n}(\Delta)\}.$$

Then by the lemma, G_Δ is a dense open subset of S. If (Δ_l) is a basis for the topology of $\mathbb{C}^{2k}$, then $G = \cap G_{\Delta_l}$ is a dense G_δ set and it is easy to check that the set of points whose orbit is dense for $\Phi \in G$ is already a dense G_δ set. If Φ belongs to G, then the non-wandering set is $\mathbb{C}^{2k}$.

Remark 2.6. The lemma can be refined to give the following. Given $f_0 \in \mathcal{S}$, and $p^1, \ldots, p^l$, l points such that for $1 \leq i \leq l$, $p^1, \ldots, p^l \notin \{K_{f_0} \cup K_{f_0^{-1}}\}$ then there is an $f \in S$ arbitrarily close to f_0 such that $p^1, \ldots, p^l$ belong to a periodic orbit for f.

PROPOSITION 2.7. *There is a dense G_δ set $D \subset \mathcal{S}$ such that for any $g \in D$, hyperbolic periodic points are dense in $\mathbb{C}^{2k}$.*

LEMMA 2.8. *Let $p_1, \ldots, p_l$ be distinct points in $\mathbb{C}^{2k}$. Let $\xi \in \mathbb{C}^{2k}$ be a non-zero vector. Then there exist two shears, τ_1, τ_2, arbitrarily close to the identity, such that if $\tau := \tau_2 \circ \tau_1$, then $\tau(p_r) = p_r$, $1 \leq r \leq l$, $\tau'(p_r) = \mathrm{Id}$, $1 < r \leq l$, and $\tau'(p_1)\xi = \alpha\xi$, $\alpha > 1$.*

Proof of Lemma 2.8. We consider the set A of $\eta \in \mathbb{C}^{2k}$ satisfying the following conditions:
(1) $\Omega(p_j, \eta)$ are all distinct for $1 \leq j \leq l$;
(2) $\Omega(\xi, \eta) \neq 0$.

It is clear that A is open and dense in $\mathbb{C}^{2k}$. We choose a vector $\xi_1 \in A$. Let $\xi_2 := \xi - \xi_1$. Wiggling ξ_1 we can assume $\xi_1, \xi_2 \in A$. We define $\tau_1(z) := z + h(\Omega(z, \xi_1))\xi_1$ and $\tau_2(z) := z + k(\Omega(z, \xi_2))\xi_2$, where h and k are one variable entire functions. We can choose them to vanish at $\Omega(p_j, \xi_s)$, $1 \leq j \leq l$, $s = 1, 2$, $h'(\Omega(p_j, \xi_1)) = k'(\Omega(p_j, \xi_2)) = 0$, $1 < j \leq l$.

We only need to check the last condition in the lemma. Computing, we get

$$\begin{aligned}
\tau'(p_1)(\xi) &= \tau_2'(p_1)(\tau_1'(p_1)\xi) \\
&= \tau_2'(p_1)(\xi + h'(\Omega(p_1, \xi_1))\Omega(\xi, \xi_1)\xi_1) \\
&= \xi + h'(\Omega(p_1, \xi_1))\Omega(\xi, \xi_1)\xi_1 \\
&\quad + k'(\Omega(p_1, \xi_2))\Omega([\xi + h'(\Omega(p_1, \xi_1))\Omega(\xi, \xi_1)\xi_1], \xi_2)\xi_2.
\end{aligned}$$

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Choose h so that $\lambda := h'(\Omega(p_1, \xi_1))\Omega(\xi, \xi_1)$ is a small positive number. Then $\Omega([\xi + \lambda\xi_1, \xi_2]) \neq 0$ and we can choose k so that $\lambda = k'(\Omega(p_1, \xi_2))\Omega(\xi + \lambda\xi_1, \xi_2)$.

Then $\tau'(p_1)\xi = (1 + \lambda)\xi$. □

Proof of Proposition 2.7. Fix a polydisc Δ. Define U_Δ as the set of $\Phi \in \mathcal{S}$ so that Φ has a hyperbolic periodic orbit passing through Δ. Clearly U_Δ is open. We show it is dense. By Lemma 2.3, for any $f_0 \in \mathcal{S}$ and $p \in \Delta$ we can find f in a given neighborhood of f_0 such that f has a periodic orbit through p. Then we can inductively use Lemma 2.8 to modify all eigenvalues of modulus 1 into hyperbolic eigenvalues.

If Δ_n is a basis for the topology of $\mathbb{C}^{2k}$ then $D := \cap\Delta_n$ satisfies the requirements. □

3. *Volume preserving automorphisms*

We denote by $\mathcal{V}$ the set of volume preserving biholomorphic maps on $\mathbb{C}^k$. To avoid trivial cases, we assume that $k \geq 2$. Note that as opposed to the previous section, we also consider odd complex dimensions. In the case of even dimension, note that $\mathcal{V}$ contains all biholomorphic symplectomorphisms.

We prove a version of Theorem 2.1 for volume preserving biholomorphisms. For $f \in \mathcal{V}$ define as above

$$K_f := \{z; (f^n(z)) \text{ is bounded}\}.$$

THEOREM 3.1. *There exists a dense G_δ set $\mathcal{V}_1$ in $\mathcal{V}$ such that for every $f \in \mathcal{V}_1$, K_f is an F_σ set with empty interior.*

Proof. Define

$$K = \left\{(z, f) \in \mathbb{C}^k * \mathcal{V}, \sup_m |f^m(z)| < \infty\right\}.$$

It suffices to show that K has empty interior in $\mathbb{C}^k * \mathcal{V}$. Suppose not. Let $U = \operatorname{int} K$. For every C let

$$U_C := \left\{(z, f) \in U, \sup_m |f^m(z)| \leq C\right\}.$$

By a category argument we can assume that for some C, U_C has a non-empty interior U_C^0.

The map $\psi : \mathbb{C}^k * \mathcal{V} \mapsto \mathbb{C}^k * \mathcal{V}, \psi(z, f) = (f(z), f)$ maps U_C^0 to itself. For each slice with fixed f, the map is volume preserving on a bounded region. Hence each component of a slice is periodic.

Fix $(z_0, f_0) \in U_C^0$. The Jacobian determinant of the derivative of f_0 is a constant of modulus one. Composing with an arbitrarily small rotation, we can assume that the Jacobian determinant is a root of unity.

Let Ω_0 be a component of the slice $C_{f_0}^k$ containing z_0. Let Ω be the cycle generated by Ω_0, and let G be the closed subgroup generated by f_0. It is a commutative compact Lie group, hence it is isomorphic to $T^l * A$, where A is a finite group and T^l is the torus of dimension l. Moreover, $l \leq k$. For example, if $l = 0$ we have a periodic point and we can change it into a saddle point by a small volume preserving perturbation (see the proof of Theorem 4.1 in [**FS3**]). That $l \leq k$ is a consequence of the fact that orbits are totally real manifolds.

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Assume $G = T^l * A$, $1 \le l \le k$. Let X_q be the closure of the orbit of q under G. Let $G_0 := T^l * \mathrm{Id}$.

We want to show that there is a q in the slice of U_C^0 over f_0 such that X_q is polynomially convex. Recall that if F is compact in $\mathbb{C}^k$, the polynomially convex hull $\hat{F}$ of F is defined by

$$\hat{F} := \{z \in \mathbb{C}^k; |P(z)| \le \sup_{\zeta \in F} |P(\zeta)| \ \forall \text{ holomorphic polynomials } P\}.$$

LEMMA 3.2. *If $q_1 \in \hat{X}_{q_0} \setminus X_{q_0}$, then $\hat{X}_{q_1} \subset \hat{X}_{q_0} \setminus X_{q_0}$.*

Proof. Suppose $q_1 \in \hat{X}_{q_0} \setminus X_{q_0}$. Then q_1 is not on the orbit X_{q_0} of q_0. Hence $X_{q_1} \cap X_{q_0} = \emptyset$. We show that $X_{q_1} \subset \hat{X}_{q_0}$. This follows since $\hat{X}_{q_0}$ is closed under the group action. It remains to show that $\hat{X}_{q_1} \cap X_{q_0} = \emptyset$.

Let $A(\hat{X}_{q_0})$ be the closure in the sup norm of the holomorphic polynomials. A point $p \in \hat{X}_{q_0}$ is a peak point if there is an $h \in A(\hat{X}_{q_0})$ such that $h(p) = 1$, $|h| < 1$ on $\hat{X}_{q_0} \setminus \{p\}$. It is well known ([**G**, p. 60]) that $\hat{X}_{q_0}$ is the hull of its peak points. By the group action, all points in X_{q_0} are peak points for $A(\hat{X}_{q_0})$. One uses Bishop's $1/4 - 3/4$ criterion (see [**G**, p. 52]). Hence no point in X_{q_0} can be in the hull of X_{q_1}. So $\hat{X}_{q_1} \cap X_{q_0} = \emptyset$ as we wanted to show. □

By Zorn's lemma there is a point q in $\hat{X}_{q_0}$ such that $\hat{X}_q = X_q$. It follows from a result of Serre (see [**Ho**, p. 67]) that $\dim X_q \le k - 1$; this is due to the polynomial convexity of the boundaryless manifold X_q. If $\dim X_q = 0$, q is a periodic point and we can again perturb f_0 to create a saddle point which contradicts the fact that q is in the interior of K. Hence we can assume that $\dim X_q \ge 1$.

Let $\tilde{X}_q$ be the complexification of X_q; it is a germ of a complex manifold of dimension $1 \le r \le k - 1$. The automorphisms in G preserve X_q and hence also $\tilde{X}_q$. In fact, after dividing out with the isotropy group of q, the resulting quotient group still preserves $\tilde{X}_q$. It follows that $\tilde{X}_q$ is foliated by orbits of G which are all of the same dimension.

We are going to construct a volume preserving vector field on a neighborhood of $\tilde{X}_q$ pointing away from X_q. We can assume that G acts on $\tilde{X}_q$ without fixed point. Let s be such that f_0^s is in the identity component of G. The map f_0^s is in a one-parameter subgroup (f_t) in G with $f_0^s = f_1$.

Let ξ be the vector field $(d/dt) f_t|_{t=0}$; it is defined on $U_C^0 \cap C_{f_0}^k$ and is tangential to each orbit under G.

Since the Jacobians of all iterates of f_0^s are powers of a root of unity and since they generate G_0, it follows that all elements in G_0 have Jacobian 1.

LEMMA 3.3. *Let $\eta = (h_1, \dots, h_k)$ be a holomorphic vector field with holomorphic coefficients and let g_t be the associated flow. Then g_t is volume preserving of Jacobian* 1 *if and only if* $\sum \partial h_j / \partial z_j = 0$.

Proof. Define the *complex* divergence of η by

$$\mathrm{div}_{\mathbb{C}}(\eta) := \sum_{j=1}^{k} \frac{\partial \eta_j}{\partial z_j}.$$

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Assume $\eta_j = u_j + iv_j$ and let $\tilde{\eta}_j$ be the vector field $\tilde{\eta} = (u_1, v_1, \ldots, u_k, v_k)$. Then the (real) divergence is given by

$$\operatorname{div}_{\mathbb{R}}(\tilde{\eta}) = \sum_{j=1}^{k} \left(\frac{\partial u_j}{\partial x_j} + \frac{\partial v_j}{\partial y_j} \right).$$

It is easy to check that

$$\operatorname{div}_{\mathbb{R}}(\tilde{\eta}) = \operatorname{div}_{\mathbb{C}}(\eta) + \overline{\operatorname{div}_{\mathbb{C}}(\eta)}.$$

So if η is complex divergence free then the flow associated to it is volume preserving. Conversely, if the flow is of Jacobian 1, then computing to first order

$$dg_t^1 \wedge \cdots \wedge dg_t^k$$

we get that η is complex divergence free. □

It follows from the lemma that $i\xi$ is the generator of a germ of an automorphism of Jacobian 1.

LEMMA 3.4. *There is a Runge neighborhood U of X_q such that the vector ξ (and hence $i\xi$) is uniformly approximable on compact sets of U by $\mathbb{C}$-divergence free polynomial vector fields.*

Proof. To the vector field ξ we associate the $(k-1, 0)$-form

$$\tilde{\xi} := \sum_{j=1}^{k} (-1)^{j-1} \xi_j \, dz_1 \wedge \cdots \wedge \hat{dz_j} \wedge \cdots \wedge dz_k.$$

Since ξ is divergence free, then $\partial\tilde{\xi} = 0$.

We want to show that the cohomology class of $\tilde{\xi}$ in a tubular neighborhood W of X_q is zero. We consider here the de Rham cohomology space $H^{n-1}(W, \mathbb{C})$.

Using the fact that X_q is a deformation retract of W it is enough to show that $j^*\tilde{\xi} = 0$; here $j : X_q \mapsto W$ is the canonical injection. When $\dim X_q < k-1$ this is clear. Assume $\dim X_q = k-1$.

Let h be a locally defined holomorphic function vanishing on X_q with $dh \neq 0$. Since ξ is tangential to X_q, $\langle dh, \xi \rangle = 0$ and hence $\tilde{\xi} \wedge dh = 0$.

Choose coordinates $w_1, \ldots, w_k$ with $w_k = h$. Let

$$\tilde{\xi} = \sum \lambda_j \, dw_1 \wedge \cdots \wedge \hat{dw_j} \wedge \cdots \wedge dw_k.$$

Then $\lambda_k \, dw_1 \wedge \cdots \wedge dw_k = \tilde{\xi} \wedge dh = 0$, hence $j^*\tilde{\xi} = \lambda_k \, dw_1 \wedge \cdots \wedge dw_{k-1} = 0$. This proves our first claim since we can apply Serre's result [**Ho**, Theorem 2.7.10].

We can choose a Runge neighborhood U of X_q where $\tilde{\xi} = \partial v$, where v is a holomorphic $(k-2, 0)$-form. We then approximate v by forms with polynomial coefficients (v_k). The vector fields ξ_r corresponding to the forms ∂v_r approximate ξ and are $\mathbb{C}$-divergence free. □

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Completion of the proof of Theorem 3.1. Let $l = (l_1, \dots, l_k)$ be a divergence free vector field in $\mathbb{C}^k$ with polynomial coefficients close enough to $i\xi$ on a neighborhood Ω_1 of the component of X_q containing q and close to 0 on a neighborhood Ω_2 of the remaining components. Set $\Omega = \Omega_1 \cup \Omega_2$. This l can be written as a sum of complete divergence free polynomial vector fields (see Andersén [**A**]). Hence it is possible to approximate the flow Φ_t associated to l on Ω by a flow ψ_t of volume preserving automorphisms in $\mathbb{C}^k$. Hence for t small enough $\psi_t \circ f_0$ is in the given neighborhood of f_0. We extend the flow to complex time. It is defined for $|t| < \delta$.

So using the fact that $(X_q, \psi_t \circ f_0) \subset U_C^0$ we get by Schwarz's lemma

$$|(\psi_t \circ f_0)^n - f_0^n| \leq C|t|.$$

This contradicts that ψ_t moves away from X_q in a transverse direction. □

Remark 3.5. A similar proof to that of Theorem 2.2 shows for any $f \in \mathcal{V}$, given any $x \in \mathbb{C}^k$ and a neighborhood V of f in $\mathcal{V}$ there is $g \in V$ such that x is periodic for g. We also have that hyperbolic periodic points are dense in $\mathbb{C}^k$ for a dense G_δ set of volume preserving biholomorphic maps.

4. *Closing lemma for holomorphic endomorphisms*

Let $f : \mathbb{C}^k \mapsto \mathbb{C}^k$ be a holomorphic endomorphism of $\mathbb{C}^k$. We do not assume that f is surjective. We put the topology of uniform convergence on compact sets on the space $\mathcal{E}$ of such endomorphisms.

THEOREM 4.1. *The closing lemma holds for $\mathcal{E}$.*

Proof. Fix $f_0 \in \mathcal{E}$. Let $p \in \mathbb{C}^k$. Assume first that $(f_0^n)_n$ is not locally bounded near p. We claim that given a neighborhood V of f_0 in $\mathcal{E}$ there is $f \in V$ such that p is periodic for f. We assume that

$$V = \{f; |f - f_0| < \epsilon \text{ on } B(0, R)\}.$$

By assumption, given any $0 < r \ll 1$ and any R_1, there is a point $q \in B(p, r)$ whose orbit is not contained in $B(0, R_1)$. Assume $f_0(B(0, R)) \subset B(0, R_1)$ with $R_1 \gg R$. Translating coordinates we can assume that $q = 0$. We can assume that there is an N such that $q_n = f_0^n(0) \in \overline{B(0, 2R_1)}$ for $0 \leq n < N$ and $q_N = f_0^N(0) \notin \overline{B(0, 2R_1)}$. We can assume also that $q_N = (q_N^1, 0, \dots, 0)$.

Choose an entire function h, small enough on $B(0, R_1)$. Define

$$s(z) = (z_1 + h(z_1), z_2, \dots, z_k).$$

If h is small enough on $B(0, R)$ and $h(q_N^1) = -q_N^1$, $h(q_n) = 0, 0 \leq n < N$, then $s \circ f_0 \in V$. Moreover, 0 is periodic for $s \circ f_0$.

Now we consider the case where p is recurrent for f_0 and $(f_0^n)_n$ is locally bounded near p. By the previous argument we can even assume that there exist $r > 0$ and $C > 0$ such that for every $f \in V$, $(f^n)_n$ is bounded on $B(p, r)$ by C. We are going to show that this is possible only if p is on an attracting periodic orbit for f_0.

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Define

$$U_C := \{(q, f);\ f \in V, (\tilde{f}^n(\tilde{q})) \text{ is bounded by } C \text{ for all } (\tilde{q}, \tilde{f}) \text{ in some neighborhood of } (q, f)\}.$$

We know that $B(p, r) * V \subset U_C$. Let U_C^f be the slice of U_C, $U_C^f := \{q; (q, f) \in U_C\}$ and let $U_C^{f,0}$ denote the connected component of U_C^f containing p. It is easy to verify that $U^0 := U_C^{f,0}$ is a Runge domain in $\mathbb{C}^k$. Clearly f maps U_C^f into itself.

Since p is recurrent there is a smallest integer $l \geq 1$ such that $f_0^l(U^0) \subset U^0$ and there is a subsequence n_i such that $f_0^{n_i l}(p) \mapsto p$. Let us denote $g := f_0^l$. We can assume that the g^{n_i} converge uniformly on compact sets of U_0 to a limit h. Clearly $h(p) = p$. Define $\Sigma := h(U^0)$.

The map g commutes with h so the set of fixed points for h is invariant under g. If $\operatorname{rank} h = 0$ then p is a fixed point for g, indeed it is an attracting fixed point so the closing lemma holds.

We assume now that $\operatorname{rank} h \geq 1$, for any such limit h. Choose h_1 with maximal possible rank. This rank is reached on a dense open set in U_0. Let $\Sigma_1 := h_1(U_0)$.

Since g commutes with h_1 we have $g(\Sigma_1) \subset \Sigma_1$. We have $g^{n_{i+1}-n_i}(g^{n_i}) = g^{n_{i+1}}$, so if g^{m_i} is a convergent subsequence of $g^{n_{i+1}-n_i}$ with limit h_2, then $h_2(h_1) = h_1$ on $h_1^{-1}(U_0)$. Hence h_2 is the identity on $\Sigma_1 \cap U_0$. Define

$$M := \{q; q \in U_0, h_2(q) = q\}.$$

Then M is a closed complex subvariety of U_0, hence a countable union of connected components. Observe that since g commutes with h_2, $g(M) \subset M$.

Let M_0 be the connected component of M through p. Since $g^{m_i}(p) \mapsto p$, it follows that for some $l' \geq 1$, $g^{l'}(M_0) \subset M_0$. For r small enough, $h_1(B(p, r))$ is a connected subset of U_0 containing p. Hence $h_1(B(p, r)) \subset M_0$. It follows that M_0 contains an irreducible branch N_0 through p and the dimension of N_0 equals the maximal rank of h_1. Notice that M_0 is a complex manifold at every point of N_0 and hence $M_0 = N_0$. This uses a consideration of the rank of $h_2 - \mathrm{Id}$. It follows that $g^{l'}$ is an automorphism of M_0.

The group $\mathrm{Aut}(M_0)$ of holomorphic automorphisms of M_0 has the structure of a Lie group [**Na**] since M_0 is contained in a bounded domain. Since $g^{m_i} \mapsto \mathrm{Id}$ on M^0 then $G := \overline{(g^{l'n})_{n\geq 0}}$ is a compact Lie subgroup of $\mathrm{Aut}(M^0)$. If this group is finite then p is periodic and with at least one eigenvalue of modulus 1. We make this eigenvalue larger than 1 with a small perturbation of f_0 contradicting that we are in the interior of U_C. Consequently the closure X_q of the orbit under g of any $q \in M_0$ is a compact totally real submanifold in M_0.

Since the closure of the orbit of p under f is a compact subset X_p of the Runge domain U_C^f, it follows that $\hat{X}_p \subset U_C^f$.

As in §3 we can assume that there is a $q \in \hat{X}_p$ such that $\hat{X}_q = X_q$.

If X_q is finite, then q is periodic for f_0. If q is not attracting, then by an arbitrarily small perturbation we can make it into a saddle point, contradicting that (f^n) is locally bounded in a neighborhood of q for all small perturbations of f_0. So X_q is a totally real polynomially convex set of dimension $1 \leq r \leq k$. By the structure of the compact Lie

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group G, X_q is a finite collection of tori.

Let T be one of these tori. There is an integer $m \geq 1$ such that $f_0^m(T) = T$. The action of f_0^m on T is real analytic and conjugate to multiplication by $(e^{i\theta_1}, \ldots, e^{i\theta_r})$ in the standard torus T^r in $\mathbb{C}^r$. We denote that map by R_θ. More precisely, there is a real analytic map ψ such that

$$f_0^m \circ \psi = \psi \circ R_\theta.$$

We extend the map ψ to a holomorphic map, which we still denote by ψ, in a neighborhood of T^r in $\mathbb{C}^r$.

Define $Z_1 = (\partial/\partial z_1, \ldots, \partial/\partial z_r)$. The derivative of f_0^m on T read in the chart ψ is just R_θ. Let $Z := \psi_* Z_1$. The holomorphic vector field Z is defined in a small neighborhood of T. We use the map f_0 to transport it to the other tori of X_q. The local flow of Z_1 pushes away from T^r. We want to construct a global vector field with the same effect.

Since X_q is polynomially convex we can assume that Z is defined in a Runge neighborhood U_q of X_q. By Oka's theorem we can approximate Z uniformly on any compact of U_q by holomorphic vector fields in $\mathbb{C}^k$. Let Z' be such a vector field which we assume close enough to Z on a neighborhood of X_q.

Let Φ_t be the flow of Z' defined on $B(0, R)$. If $|t| < \delta$, provided δ is small enough, $\Phi_t(B(0, R))$ is convex and hence Runge. It is a result due to Andersén and Lempert [**AL**] that we can then approximate Φ_t by (ψ_t), $|t| < \delta$, where ψ_t are biholomorphic maps in $\mathbb{C}^k$. Since for δ' small enough $\psi_t \circ f_0$ are in V we should have, using the holomorphic dependence on t and Schwarz's lemma, that

$$|(\psi_t \circ f_0)^n - f_0^n| \leq 2C|t|.$$

There is a contradiction because $\operatorname{dist}((\psi_t \circ f_0)^{mn}(q), T) > cn$, for some constant $c > 0$. □

Remark 4.2. Indeed, we have shown that if p is non-wandering for f_0, then p is a periodic point for a nearby f.

5. *The biholomorphic case*

Let $\mathcal{B}$ denote the group of biholomorphic maps in $\mathbb{C}^k$. We consider that $\mathcal{B}$ is endowed with the topology of uniform convergence on compact sets.

THEOREM 5.1. *The closing lemma holds for biholomorphic maps.*

Proof. As above, let $K_f = \{z; z \in \mathbb{C}^k, (f^n(z)) \text{ is bounded}\}$. Let $f_0 \in \mathcal{B}$. If $p \notin K_{f_0} \cup K_{f_0^{-1}}$ then for any neighborhood V of f_0 in $\mathcal{B}$ there is $g \in V$ such that p is periodic for g. Indeed, the proof is the same as for Lemma 2.3, since the maps s, σ_1, σ_2 we constructed were independent of the symplectic nature of f_0.

Suppose now that p is recurrent for f_0. Let $V = \{f; f \in \mathcal{B}, |f - f_0|_{B(0,R)} < \epsilon\}$. Assume first that $p \in K_{f_0} \setminus K_{f_0^{-1}}$. If f is close enough to f_0 then the orbit $f^{-n}(p)$ leaves the ball of radius R_1, $R_1 \gg R$. If under a small perturbation of f_0 the orbit of a point near p goes out of a given ball, we will be in the previous case. So we can assume that there is $r > 0$ such that $f^n(B(p, r))$ is uniformly bounded by C for all f in V. In this case the proof of Theorem 4.1 applies since the maps ψ_t constructed there are biholomorphic.

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In the last case we assume $p \in K_{f_0} \cap K_{f_0^{-1}}$. We can even assume that there exists $r > 0$ such that for every $f \in V$, $f^n(B(p,r)), n \in \mathbb{Z}$, is uniformly bounded by C, in which case we show as in Theorem 4.1 that this is possible only if p belongs to an attracting periodic orbit. We use again the fact that the maps ψ_t constructed in Theorem 4.1 are biholomorphic. But then $f^{-n}, n > 0$, cannot be bounded in a neighborhood of p. □

Remark 5.2. Indeed, we have shown that if p is non-wandering for f_0, then p is a periodic point for a nearby f.

6. *Generic density of periodic orbits in the non-wandering set*

Let $\Gamma(f)$ denote the closure of the periodic points for f and let $\Omega(f)$ denote the non-wandering set.

We show, following general arguments, see [**PR**], that generically, periodic points are dense in the non-wandering set for the maps we have considered so far.

THEOREM 6.1. *Let $\mathcal{D}$ denote either $\mathcal{S}$, $\mathcal{V}$, $\mathcal{E}$ or $\mathcal{B}$. Then there exists a G_δ dense set $\mathcal{G}$ in $\mathcal{D}$ such that for $f \in \mathcal{G}$, $\Gamma(f) = \Omega(f)$.*

Proof. Assume at first that $\mathcal{D}$ is $\mathcal{S}$ or $\mathcal{V}$. Then the result follows since we have shown that there is a dense G_δ set in $\mathcal{D}$ for which periodic points are dense.

Assume next that $\mathcal{D}$ is $\mathcal{E}$ or $\mathcal{B}$. Let $\{\Delta_n\}$ be a basis for the topology on $\mathbb{C}^k$. For each n, let D_n be the open set in $\mathcal{D}$ of maps f which have a periodic saddle orbit containing a point in Δ_n. Let $\tilde{D}_n := \mathcal{D} \setminus \partial D_n$, and set $G = \cap \tilde{D}_n$. We will show that if $f \in G$ then $\Gamma(f) = \Omega(f)$. Suppose $f \in G$ and $p \in \Delta_n \cap \Omega(f)$. We will show that $\Gamma(f) \cap \Delta_n \neq \emptyset$ which will imply that $\Gamma(f) = \Omega(f)$.

Necessarily $f \in \tilde{D}_n$. If $f \in D_n$ we are done. But by Remarks 4.2 or 5.2 f cannot be in $\mathcal{D} \setminus \overline{D_n}$, so we are done. □

7. *Closing lemma for Hamiltonian vector fields in $\mathbb{C}^2$*

Let E be the space of entire functions in $\mathbb{C}^2$ endowed with the topology of $u.c.c.$ It is a Frechet space. To $H \in E$, we associate the holomorphic vector field $X_H = (-\partial H/\partial w, \partial H/\partial z)$. Observe that X_H satisfies the relation $\Omega(X_H, .) = -dH$, where $\Omega(z,w) = dz \wedge dw$.

Fix $H \in E$, non constant. For $(z_0, w_0) \in \mathbb{C}^2$ the orbit $\gamma(t) := (z(t), w(t))$ for X_H through (z_0, w_0) is defined for $0 \leq t < T_0$, T_0 being possibly infinite; we choose T_0 as large as possible. We define

$$K_H = \{(z_0, w_0); \overline{\lim}_{t \mapsto T_0} \|(z(t), w(t))\| < \infty\}.$$

The following result is proved in [**FS2**].

THEOREM 7.1. *Let $H \in E$, non constant. Then for almost all θ, $K_{e^{i\theta}H}$ has Lebesgue measure 0 in $\mathbb{C}^2$. In particular it has empty interior. Moreover, there is a G_δ dense set $\mathcal{G} \subset E$ such that for $H \in \mathcal{G}$, K_H has empty interior.*

Here we consider the following problem. Let $H \in E$ and assume an orbit $\gamma(t)$ for X_H is recurrent, i.e. $\underline{\lim}_{t \mapsto T_0} \|\gamma(t) - \gamma(0)\| = 0$. Let V be a neighborhood of H in E.

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Fix $0 < T < T_0$. Does there exist $H_1 \in V$, such that H_1 has a closed orbit γ_1 such that $\gamma_1(t)$ lies within a given distance of $\gamma(t)$ for $t \in [0, T]$. If the answer is affirmative, we say that the closing lemma holds.

THEOREM 7.2. *The closing lemma holds for holomorphic Hamiltonians in* $\mathbb{C}^2$.

Remark 7.3. The theorem is valid without the hypothesis that $\gamma(t)$ is recurrent because we prove that all orbits are approximable by recurrent ones.

The proof of the theorem is a minor modification of a technique introduced by Fornæss and Grellier in [**FG**].

Proof. We first prove that we can approximate an orbit, unbounded in positive and negative time, by closed orbits of nearby Hamiltonians. Assume $\gamma(t)$ is an unbounded orbit in $H^{-1}(0)$. Fix $R > 0$. We want to construct $H_1 \in E$, where $H_1 \in V$, $V := \{F; |F - H|_{B(0,R)} < \epsilon\}$, a given neighborhood of H, and a periodic orbit γ_1 of X_{H_1}, $\gamma_1 \subset H_1^{-1}(0)$, and γ_1 is close to γ on $[0, T]$. We can assume that $\gamma(t)$ is transverse to $\partial B(0, R_1)$ at p, q, where $p = \gamma(t_1)$, $t_1 < 0$, $q = \gamma(t_2)$, $t_2 > T$ with $R_1 \gg R$ such that $\gamma([0, T]) \subset B(0, R_1/2)$. Suppose that γ enters the ball at t_1 and exits at t_2 and that $\gamma(t_1, t_2) \subset B(0, R_1)$. We want to connect p, q with a curve in $H^{-1}(0)$ which does not enter again in $B(0, R_1)$. This is not always possible because the Riemann surface $H^{-1}(0)$ might have more than one end. So we slightly modify H and hence also p and q. We can assume that H is a polynomial of the form $H(z, w) = \Pi_{j=1}^m (w - a_j(z))$. There exists $R_2 > R_1$ so that the $\{a_j(z)\}_{j=1}^m$ are pairwise distinct whenever $|z| \geq R_2$. We choose an entire function $h(z)$, such that h vanishes to first order only at some z_0, $|z_0| > R_2$, and h is close to 1 on $B(0, R_2)$. We define

$$H' := \Pi_{j=1}^m (w - ha_j) + \delta$$

for a suitable small constant δ. Then $(H' = 0)$ is close to $(H = 0)$ on $B(0, R)$ and $\nabla H' \neq 0$ on $(H')^{-1}(0)$. All local branches meet near $(z_0, 0)$, so we can go from one end to another by passing close to $(z_0, 0)$. Hence we are able to connect p, q in $(H' = 0)$ with a curve γ_1 staying outside $B(0, R_1)$ and passing close to $(z_0, 0)$. We denote again H' by H. Let γ be the orbit of X_H from p to q inside $B(0, R_1)$. We can modify γ_1 so that $\overline{\gamma} := \gamma_1 \cup \gamma$ is a simple closed curve of class $\mathcal{C}^k$, real analytic out of $\partial B(0, R_1)$, and $\overline{\gamma} \subset (H = 0)$. We can even assume that γ_1 has a real analytic continuation past the endpoints.

We next want to show that using an arbitrarily small perturbation of H, we can assume that $\overline{B(0, R_1)} \cup \overline{\gamma_1} =: Y$ is polynomially convex. The hull of Y is always contained in $\overline{B(0, R_1)} \cup (H = 0)$. Indeed, let $M := \max_{B(0,R_1)} |H|$. If $z_0 \notin Y$, $H(z_0) \neq 0$, then we can choose a polynomial P such that $|P|_{B(0,R_1)} \leq 1/M$ and $|P(z_0)| > 2/|H(z_0)|$. The function PH shows that, in fact, $\hat{Y} \setminus Y$ consists of the bounded connected components W of $(H = 0) \setminus Y$. There can be at most one of these since $\gamma_1 \subset \partial W$.

If W is non-empty, let p_0 be a point in W, and let L be an affine complex function so that $L \neq 0$ on $\bar{\gamma}$, $L(p_0) = 0$, and $\Re L < 0$ on $\bar{B}(0, R_1)$. Replace H by $(1 - e^{CL})H + \delta$ for a suitable large C and small δ. Then we obtain a new Y which is polynomially

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convex. Indeed, $((1-e^{CL})H+\delta=0)\setminus Y$ has no bounded component near p_0, we have created an end instead.

We next want to construct a new Hamiltonian $H_0 = H(1+f)$ with f small on $B(0,R)$, so that H_0 is in V and X_{H_0} will have a closed orbit near $\overline{\gamma}$.

We consider first the vector field Z defined on $\overline{B(0,R_1)}\cup\overline{\gamma_1}$ as follows. $Z = X_H$ on $\overline{B(0,R_1)}$ and Z is tangential to γ_1 on γ_1. Since γ_1 is real analytic, we can assume that Z is holomorphic on a neighborhood U of $\overline{\gamma_1}$ in $(H=0)$, and $\overline{U}$ touches $\partial B(0,R_1)$ only at the points p,q of γ; by shrinking U we can assume that Z is $\mathcal{C}^k$, for a given k, on $\overline{U}\cup\overline{B(0,R_1)}$. Moreover, we can suppose that U contains sectors around γ_1 with corners at p and q.

Clearly $\overline{\gamma}$ is a closed orbit for Z. We want to approximate Z by X_{H_1}, where H_1 is a Hamiltonian, such that X_{H_1} still has a closed orbit near $\overline{\gamma}$.

Let g be the function such that $Z = X_{H'}$, where $H' = H(1+g)$. Observe that H' is defined on $\overline{B(0,R_1)}\cup\overline{U}$ and holomorphic on U. Moreover, $g\equiv 0$ on $B(0,R_1)$. Let ξ be a $\mathcal{C}^2$-function, $\xi = 0$ on $\overline{B(0,R_1)}$, $\Re\xi > 0$ on U_1, an angle in U, and ξ is holomorphic on U. More precisely, if z is a holomorphic coordinate centered at p (or q), $\xi(z) = z^3+\mathcal{O}(z^4)$ on U. We define $H_\theta = e^{i\theta\xi}H(1+g)$, where θ is a real parameter near 0. Let Z_θ be the corresponding vector field.

Let l be a transverse curve to $\bar{\gamma}$ inside $(H=0)$ at a point of $\bar{\gamma}$ of maximal norm. Let γ_θ^p be the orbit of Z_θ starting at p in backwards time. By continuity in θ, the curve enters in U_1 and stays there if θ is small enough. This curve cuts l at $p_1(\theta)$. Similarly, define γ_θ^q as the orbit starting at q in forward time. It cuts l at $q_1(\theta)$. In terms of a parametrization of the curve l, we define

$$h(\theta) := p_1(\theta) - q_1(\theta).$$

Then h is a smooth function which vanishes to first order at $\theta = 0$ because $\Re\xi > 0$, which forces $p_1(\theta)$ and $q_1(\theta)$ to be on opposite sides of γ_1 when $\theta\neq 0$.

We now construct a holomorphic function ξ' on $\overline{B(0,R_1)}$ such that $\xi'(p)=\xi'(q)=0$, $\Re\xi' > 0$ on γ and vanishing in γ to second order only at p,q. We will extend ξ' as 0 on $\bar{U}$.

We first consider a real analytic function ξ_1 on γ satisfying the requirements, extend it holomorphically to a neighborhood of γ in $(H=0)$. In fact, since $(H=0)$ is an open Riemann surface we can assume that ξ_1 is defined on all of $(H=0)$. Since $(H=0)$ is a smooth complex hypersurface there is a holomorphic retraction from a neighborhood of $(H=0)$ in $\mathbb{C}^2$ to $(H=0)$. So we can consider that we have a holomorphic function ξ_1 defined on a neighborhood W of $(H=0)$ and satisfying the requirements in $\overline{B(0,R_1)}\cap W$. Let χ be a smooth function with support in W such that $\chi = 1$ in a neighborhood of $(H=0)$, the $(0,1)$ form $\alpha := (\bar{\partial}\chi)\xi_1/H$ is closed in $\mathbb{C}^2$. Let u satisfy $\bar{\partial}u = \alpha$, with $u(p)=u(q)=0$ (see [**Ho**, Chapter 4]); then $\xi' := \chi\xi_1 - Hu$ satisfies the required conditions.

Let $Z_{\theta'}$ be the vector field associated to $H'_{\theta'} := e^{i\theta'\xi'}H$. Let l' be a transverse curve to γ in $(H=0)$ at a point of minimal norm. As above we consider the orbits $\gamma_{\theta'}^p$ and $\gamma_{\theta'}^q$ starting respectively at p and q and entering in $B(0,R_1)$. For θ' small enough we define $k(\theta') = p_1'(\theta') - q_1'(\theta')$, where $p_1(\theta')$, $q_1(\theta')$ are parameters on l' corresponding

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to the points of intersections of the orbits with l'.

The function $k(\theta')$ also has a zero of order 1 at $\theta' = 0$.

Observe that $\gamma^p_{\theta'} \cup \gamma^p_\theta$ is an orbit of the Hamiltonian vector field corresponding to $H(1+g)e^{i\theta\xi+i\theta'\xi'}$. Similarly for $\gamma^q_{\theta'} \cup \gamma^q_\theta$.

The proof of the polynomial convexity of Y also shows that $\overline{B(0,R_1)} \cup \bar{U}_1$ is polynomially convex. Using Hormander's $\bar\partial$ estimates in a standard way (for example, as in Hormander and Wermer [**HoW**] or in [**FG**, Lemma 2.2]) we can approximate g, ξ, ξ' by *entire* functions on $\overline{B(0,R_1)} \cup \bar{U}_1$ in the $\mathcal{C}^1$-norm. It is in order to approximate by *entire* functions that polynomial convexity is needed. Let $H_{\theta,\theta'} := e^{i\theta\xi_1+i\theta'\xi_1'}H(1+g_1)$, where ξ_1, ξ_1', g_1 are close enough to ξ, ξ', g.

We thus obtain a Hamiltonian vector field $Z_{\theta,\theta'}$. Let $h(\theta,\theta') = p(\theta,\theta') - q(\theta,\theta')$, where $p(\theta,\theta')$, $q(\theta,\theta')$ denote the parameters corresponding to the intersection of the orbits with l. Similarly we define $k(\theta,\theta') = p'(\theta,\theta') - q'(\theta,\theta')$.

The set $(h(\theta,\theta') = 0)$ is a graph $\theta = \theta(\theta')$, where $\theta(\theta')$ is arbitrarily close to the zero-function. The set $(k(\theta,\theta') = 0)$ is also a graph, $\theta' = \theta'(\theta)$, where $\theta'(\theta)$ is arbitrarily close to the zero-function. The two graphs intersect at a point (θ_0, θ_0') which corresponds to a closed orbit.

Now let H_0 be a Hamiltonian and assume $\gamma_0(t)$ is an orbit for X_{H_0}. Fix

$$V = \{H;\, H \in E, |H - H_0|_{B(0,R)} < \epsilon\}.$$

Fix $T \gg 1$. Fix $R_1 \gg R$ such that $\gamma_0(t) \in B(0, R_1/2)$ for $t \in [0,T]$. Using Theorem 7.1, we can find a H in V and an orbit γ of X_H arbitrarily close to $\gamma_0(t)$ for $t \in [0,T]$ and such that γ is unbounded in positive and negative time. We now apply the first part of the proof in order to approximate γ by a closed orbit of a Hamiltonian in V. □

8. *The quasi-ergodic hypothesis for Hamiltonian vector fields in* $\mathbb{C}^2$

Let H be a holomorphic function in $\mathbb{C}^2$. Let X_H be the Hamiltonian vector field associated to H. The quasi-ergodic hypothesis in this context means that for a generic Hamiltonian H in $\mathbb{C}^2$ and for a generic value c, the vector field X_H has a dense orbit on $H^{-1}(c)$.

THEOREM 8.1. *There is a dense G_δ set $\mathcal{E} \subset E$ with the following property. For every $H \in \mathcal{E}$ there is a dense G_δ set $C \subset \mathbb{C}$ such that on every component of $H^{-1}(c)$, $c \in C$, there is a dense orbit.*

Proof. Fix a countable dense set $\{c_j\}_{j\geq 0} \in \mathbb{C}$.

For any integer $m \geq 1$ let W_m denote the set of non-constant $H \in E$ such that $\nabla H \neq 0$ for every point in $\overline{B(0,m)}$ at which H takes one of the values $c_1, \ldots, c_m$. Clearly, this set is open and dense in E for each m.

The set $\tilde{\mathcal{E}} := \cap W_m$ is then a dense G_δ in E and for every $H \in \tilde{\mathcal{E}}$ the sets $H^{-1}(c_j)$ are countable unions of pairwise disjoint Riemann surfaces each of which is a closed submanifold of $\mathbb{C}^2$, at most finitely many of which intersect any given compact set.

Fix an integer N. For each $H \in \tilde{\mathcal{E}}$ and $1 \leq j \leq N$ let $\{\Sigma^H_{j,k}\}_{k=1}^{n(j,H)}$ denote the components of $H^{-1}(c_j)$ which intersect the closed ball $\overline{B(0,N)}$.

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Note that if $H \in \tilde{\mathcal{E}}$, then some $\Sigma_{j,k}^H$ might touch the ball $\overline{B(0,N)}$ only at the boundary and so if we perturb H a little the component might disappear. Similarly, two distinct components of $\Sigma_{j,k}^H$ might be connected up if we perturb H a little. In either case the number of components is reduced. We say that H is N-stable if the number of components $\Sigma_{j,k}^H$ is locally minimal in $\tilde{\mathcal{E}}$. Let $\tilde{\mathcal{E}}_N$ consist of the N-stable $H \in \tilde{\mathcal{E}}$. Then $\tilde{\mathcal{E}}_N$ is a dense open set in $\tilde{\mathcal{E}}$.

Let V_N consist of those $H \in \tilde{\mathcal{E}}_N$ so that every $\Sigma_{j,k}^H$ contains an integral curve $\gamma_{j,k}^H$ of the Hamiltonian vector field of H such that every point in $\Sigma_{j,k}^H \cap \overline{B(0,N)}$ has distance strictly less than $1/N$ from $\gamma_{j,k}^H$.

LEMMA 8.2. *The set V_N is open in $\tilde{\mathcal{E}}_N$.*

Proof of Lemma 8.2 This is obvious: we just replace $\gamma_{j,k}^H$ by an integral curve in $\Sigma_{j,k}^{H'}$ if $H' \in \tilde{\mathcal{E}}_N$ is close to H. □

LEMMA 8.3. *The set V_N is dense in $\tilde{\mathcal{E}}_N$.*

Proof of Lemma 8.3. Fix an $H_0 \in \tilde{\mathcal{E}}_N$ and choose numbers $R > N$, $\epsilon > 0$. We will prove that there is an $H_1 \in V_N$ with $|H_1 - H_0| < \epsilon$ on $B(0,R)$. Pick finitely many points $\{p_{j,k,l}^{H_0}\}_l \subset \Sigma_{j,k}^{H_0}$ so that each point in $\Sigma_{j,k}^{H_0}$ has distance at most $1/2N$ from some $p_{j,k,l}^{H_0}$.

We next perturb H_0 so that the forward and backward orbit of each $p_{j,k,l}^{H_0}$ escape $\overline{B(0,R)}$. This is possible by Theorem 7.1. However, we sketch the argument.

We proceed by induction on j. Assume this is true for all points $p_{i,k,l}^{H_0}$, $i < j$. Then this remains true for all $p_{i,k,l}^{H_0}$, $i < j$, if H_0 is perturbed little enough.

Suppose some orbits of points $p_{j,k,l}^{H_0}$ remain bounded in forward or backward time. Then by the Schwartz [**Sc**] version of the Poincaré–Bendixson theorem the limit curves are closed orbits. So by the identity theorem all orbits in the level set of H_0 are closed. But then replacing H_0 by

$$e^{i\theta} H_0 + c_j - e^{i\theta} c_j$$

for arbitrarily small θ makes all these orbits escape $\overline{B(0,R)}$. Next replace this function by an arbitrarily close element of $\tilde{\mathcal{E}}_N$. Hence we may assume that the orbits of all points $p_{j,k,l}^{H_0}$ escape $\overline{B(0,R)}$ both in forward and backward time.

Let $\gamma_{j,k,l}^{H_0}$ denote the maximal orbit through $p_{j,k,l}^{H_0}$ contained in $\overline{B(0,R)}$, entering the ball at $P_{j,k,l}^{H_0}$ and exiting at $Q_{j,k,l}^{H_0}$. We can assume that the curves intersect $\partial B(0,R)$ transversally.

Next we construct a closed $\mathcal{C}^2$ loop inside $\Sigma_{j,k}^{H_0}$ containing all the pieces $\gamma_{j,k,l}^{H_0}$ by adding curves outside the ball $\overline{B(0,R)}$. So we wish to connect $Q_{j,k,l}^{H_0}$ to $P_{j,k,l+1}^{H_0}$ except the last, $Q_{j,k,l}^{H_0}$, which is connected to the first $P_{j,k,l}^{H_0}$.

We have seen how to do this: approximate, inductively on j, at the first H_0 by a polynomial of the form

$$\Pi(w - a_i(z)) + c_j$$

and then connect the ends by a function

$$\Pi(w - ha_i(z)) + \delta + c_j$$

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which we finally replace by an approximation in $\tilde{\mathcal{E}}_N$. So we connect in this way first the points in $\Sigma_{1,k}^{H_0}$ and then in $\Sigma_{2,k}^{H_0}$ (much further out) etc.

After this, we may assume that there is a $\mathcal{C}^2$ closed curve in $\Sigma_{j,k}^{H_0}$ which is real analytic except at the points $P_{j,k,l}^{H_0}$, $Q_{j,k,l}^{H_0}$.

There is however one problem, the curves might have finitely many self-crossings in the exterior of $\overline{B(0,R)}$. Now to get rid of the crossings we are going to add handles. Notice that all self-crossings in $\Sigma_{j,k}$ can be made to happen in a small disc which we are free to move where we wish on $\Sigma_{j,k}$. We add handles inductively on j. Let $(L_j = 0)$ be a surface of high genus with punctures at infinity, crossing $(H_0 = c_j)$. We can assume that L_j is close to 1 on any given ball. We replace H_0 by $(H_0 - c_j)L_j + c_j + \delta$.

Then we can connect the points $P_{j,k,l}^{H_0}$, $Q_{j,k,l}^{H_0}$ without self-crossings. Hence we obtain curves $\gamma_{j,k}^{H_0} \subset \Sigma_{j,k}^{H_0}$ passing closer than $1/N$ to any point in $\Sigma_{j,k}^{H_0} \cap \overline{B(0,N)}$. Moreover, the curves are integral curves of H_0 inside $\overline{B(0,R)}$ and real analytic, except only $\mathcal{C}^2$ on $\partial B(0,R)$. In order to replace $\gamma_{j,k}^{H_0}$ by a Hamiltonian vector field of a close-by H_0, we do an approximation argument as in the proof of Theorem 7.2. □

The end of the proof of Theorem 8.1. We define $\mathcal{E} := \cap\tilde{\mathcal{E}}_N$. Then $\mathcal{E}$ is a dense G_δ of Hamiltonians.

Fix $H \in \mathcal{E}$. We will show that there is a dense G_δ, $C_H \subset \mathbb{C}$, such that if $c \in C_H$, then $H^{-1}(c)$ is a countable union of disjoint closed submanifolds of $\mathbb{C}^2$, and on each such submanifold Σ, there is a dense orbit. Let $C_H^1 \subset \mathbb{C}$ be the dense G_δ of non-critical values of H. By construction, $\{c_j\} \subset C_H^1$.

Consider $V_N := \{c \in C_H^1$; on each component Σ of $H^{-1}(c)$ which intersect $\overline{B(0,N)}$ there exists an orbit γ of X_H such that every point of $\Sigma \cap B(0,N)$ is at distance $< 1/N$ from $\gamma\}$. Then V_N is open and we know that V_N is dense. So for c in a dense G_δ set $H^{-1}(c)$ has dense orbits. □

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