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A NOTE ON TREE–CYCLE RAMSEY NUMBERS

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Abstract

Let $R(T_n, C_m)$ denote the Ramsey number of a tree T_n on n vertices versus a cycle C_m of length m. Burr *et al.* ['Ramsey numbers for the pair sparse graph–path or cycle', *Trans. Amer. Math. Soc.* **269**(2) (1982), 501–512] asked for the least function $f(m)$ such that $R(T_n, C_m) = 2n - 1$ for every odd $m \geq 3$ whenever $n \geq f(m)$. They proved that $f(m) \leq 756m^{10}$. This bound was later improved to $25m$ by Brennan ['Ramsey numbers of trees versus odd cycles', *Electron. J. Combin.* **23**(3) (2016), Article no. P3.2] and to $4m - 8$ by Fan and Lin ['Asymptotically optimal Ramsey goodness of sparse graphs versus odd cycles and paths', Preprint, 2025, arXiv:2507.11835v5]. In this note, we show that $f(m) \leq 2m - 4$ by using a different method and conjecture that $f(m) = \lceil (2m - 1)/3 \rceil$.

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1. Introduction

Given two graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest positive integer N such that, for every red–blue edge colouring of the complete graph K_N, there exists either a red subgraph isomorphic to G or a blue subgraph isomorphic to H. For a graph H with chromatic number $\chi(H)$, the *chromatic surplus* $s(H)$ is the smallest size of a colour class over all proper $\chi(H)$-colourings of H. For a connected graph G with $n \geq s(H)$ vertices, Burr [3] established a general lower bound for $R(G, H)$:

$$R(G, H) \geq (\chi(H) - 1)(n - 1) + s(H)\,. \tag{1.1}$$

Burr [3] defined G to be H-good if equality holds in (1.1). Let K_m, P_m and C_m denote a complete graph, a path and a cycle on m vertices, respectively, and let T_m be an arbitrary tree on m vertices. Prior to the formal definition of Ramsey goodness, Chvátal [5] had established that all n-vertex trees T_n satisfy the K_m-goodness property, as demonstrated in the following result.

THEOREM 1.1 (Chvátal [5]). *We have* $R(T_n, K_m) = (n - 1)(m - 1) + 1$.

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For the case $m = 3$, Chvátal's result implies that $R(T_n, K_3) = R(T_n, C_3) = 2n - 1$. Burr *et al.* extended this result by replacing K_3 with an odd cycle C_m as follows.

THEOREM 1.2 (Burr *et al.* [4]). *Let T_n be an arbitrary tree on n vertices. Then, for odd $m \geq 3$ and $n \geq 756m^{10}$, $R(T_n, C_m) = 2n - 1$.*

They also proposed the following problem.

PROBLEM 1.3 (Burr *et al.* [4]). What is the least function $f(m)$ such that $R(T_n, C_m) = 2n - 1$ for every odd $m \geq 3$ whenever $n \geq f(m)$?

In 2016, Brennan showed that $n \geq 25m$ will suffice.

THEOREM 1.4 (Brennan [2]). *We have $R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 25m$.*

Recently, Fan and Lin improved Brennan's bound by reducing the requirement on n from $n \geq 25m$ to $n \geq 4m - 8$.

THEOREM 1.5 (Fan and Lin [8]). *We have $R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 4m - 8$.*

In this note, by using a method different from those applied in [2, 8], we further reduce the lower bound of n to $2m - 4$.

THEOREM 1.6. *We have $R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 2m - 4$.*

For the case where T_n is a path, Faudree *et al.* [7] showed that $R(P_n, C_m) = \max\{2n - 1, m + \lfloor n/2 \rfloor - 1\}$ for $m \geq n \geq 2$ and odd m. In particular, when $n \leq 2(m - 1)/3$, the second term dominates, giving $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1 > 2n - 1$, which implies $f(m) > 2(m - 1)/3$. Combining this with the upper bound $f(m) \leq 2m - 4$ given in Theorem 1.6,

$$\lceil (2m - 1)/3 \rceil \leq f(m) \leq 2m - 4.$$

Furthermore, we conjecture that the lower bound is sharp for odd $m \geq 5$.

CONJECTURE 1.7. We have $R(T_n, C_m) = 2n - 1$ for odd $m \geq 5$ and $n \geq \lceil (2m - 1)/3 \rceil$.

2. Preliminaries

The symbols $|G|$, $e(G)$, $\delta(G)$ and $\Delta(G)$ denote the number of vertices, the number of edges, the minimum degree and the maximum degree of the graph G, respectively. By $\Delta_2(G)$, we mean the second-largest degree. That is, if $(d_1, d_2, \ldots, d_n)$ is the degree sequence of G with $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\Delta_2(G) = d_2$. Let $K_{s,t}$ be the complete bipartite graph with parts of sizes s and t. For a red–blue edge-colouring of a K_N, we let R and B denote the spanning subgraphs of K_N, whose edge sets consist of all red and all blue edges of K_N, respectively. Given a subset $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced by X. For a vertex $v \in V(G)$, $d_R(v)$ and $d_B(v)$ denote the red degree and blue degree of v, that is, the number of red and blue edges incident to v,

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respectively. We denote by $N_R(v)$ (respectively $N_B(v)$) the set of vertices adjacent to v by a red (respectively blue) edge.

Our proof also needs the following three important lemmas. The first lemma is due to Dobson and shows that when a graph contains no $K_{2,n}$, its complement graph contains a large class of trees.

LEMMA 2.1 (Dobson [6]). *Let $n \geq 2$ and let G be a graph of order $2n-1$ such that $\overline{G}$ contains no $K_{2,n}$. Then, G contains every tree T of order n such that $\Delta(T) \leq \Delta(G)$ and $\Delta_2(T) \leq \delta(G)$.*

The second lemma is due to Häggkvist and gives an upper bound for $R(T_n, P_m)$.

LEMMA 2.2 (Häggkvist [9]). *We have $R(K_{n_1,n_2}, P_m) \leq n_1 + n_2 + m - 2$.*

Since every tree is a subgraph of a complete bipartite graph, the following holds.

COROLLARY 2.3. *For $m \geq 2$ and $n \geq 2$, we have $R(T_n, P_m) \leq n + m - 2$.*

The concept of pancyclicity for graphs was introduced and studied by Bondy [1] in the early 1970s. A graph G is pancyclic if it contains cycles of every length between 3 and $|G|$. Bondy established several sufficient conditions for a graph to be pancyclic. The third lemma is one such degree condition.

LEMMA 2.4 (Bondy [1]). *Let G be a graph of order n with $\delta(G) \geq n/2$. Then, G is pancyclic or $G = K_{r,r}$ with $r = n/2$.*

3. Proof of Theorem 1.6

From (1.1), we have $R(T_n, C_m) \geq 2n-1$ for every odd integer $m \geq 3$. In what follows, we show that if $m \geq 3$ is odd and $n \geq 2m-4$, then $R(T_n, C_m) \leq 2n-1$. Let $N = 2n-1$. Assume that K_N has a red–blue edge colouring such that it contains neither a red T_n nor a blue C_m.

If $\delta(R) \geq n-1$, then clearly T_n can be embedded into R. Hence, $\delta(R) \leq n-2$, which implies $\Delta(B) = N - 1 - \delta(R) \geq n$. Let $u \in V(K_N)$ be such that $d_B(u) = \Delta(B)$. Then, there must be one edge in $B[N_B(u)]$, which together with u forms a blue C_3 and so gives a contradiction. Hence, we may assume $m \geq 5$.

We claim that $\Delta(R) \geq n-1$ and $\delta(R) \geq n/2$.

If $\delta(B) \geq n > N/2$, then by Lemma 2.4, since $N = 2n-1$ is odd, B is pancyclic and, hence, contains a blue C_m, giving a contradiction. Thus, $\delta(B) \leq n-1$ and so

$$\Delta(R) \geq N - 1 - (n-1) = n - 1. \tag{3.1}$$

Suppose next that $\delta(R) < n/2$. Let u be a vertex with $d_R(u) = \delta(R)$. Since $d_R(u)$ is an integer, $d_R(u) \leq \lceil n/2 \rceil - 1$. Therefore, $d_B(u) \geq 2n - 1 - \lceil n/2 \rceil = n - 1 + \lfloor n/2 \rfloor$. As $n \geq 2m-4$, we have $\lfloor n/2 \rfloor \geq m-2$ and, hence, $d_B(u) \geq n + m - 3$. By Corollary 2.3, $R(T_n, P_{m-1}) \leq n + (m-1) - 2 = n + m - 3$. Thus, $B[N_B(u)]$ contains a blue P_{m-1}, unless $R[N_B(u)]$ contains a red T_n. The latter is impossible, while the former together

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with u forms a blue C_m, giving a contradiction. Hence,

$$\delta(R) \geq \frac{n}{2}. \tag{3.2}$$

This proves the claim.

Since $e(T_n) = n - 1$, we have $\Delta_2(T_n) \leq n/2$ and $\Delta(T_n) \leq n - 1$ for all n-vertex trees T_n. By (3.1) and (3.2), $\Delta_2(T_n) \leq \delta(R)$ and $\Delta(T_n) \leq \Delta(R)$ for all n-vertex trees T_n. If K_N contains no blue $K_{2,n}$, then applying Lemma 2.1 with $G = R$ and $\overline{G} = B$ shows that R contains every tree T_n, that is, K_N contains a red copy of T_n, giving a contradiction. Hence, K_N contains a blue $K_{2,n}$.

Let the two partite sets of the blue $K_{2,n}$ be $\{u_1, u_2\}$ and U, where $|U| = n$. Set $V = V(K_N)$ and $W = V \setminus (U \cup \{u_1, u_2\})$. Then, $|W| = n - 3$.

Clearly, $B[U]$ contains no P_{m-3}. Indeed, if $z_1 z_2 \cdots z_{m-3}$ is a blue path in U, then since $n - (m - 3) = n - m + 3 > 0$, we may choose $y \in U \setminus \{z_1, \ldots, z_{m-3}\}$. Then, $u_1 z_1 z_2 \cdots z_{m-3} u_2 y u_1$ is a blue C_m, which is a contradiction.

Let T^1 be a subtree obtained from T_n by repeatedly deleting leaves until exactly $n - m + 5$ vertices remain. By Corollary 2.3,

$$R(T_{n-m+5}, P_{m-3}) \leq (n - m + 5) + (m - 3) - 2 = n = |U|.$$

Since $B[U]$ contains no P_{m-3}, the graph $R[U]$ contains a red copy of T^1. If $m = 5$, then $T^1 = T_n$, giving a contradiction. Thus, we may assume $m \geq 7$.

Starting from this red copy of T^1, greedily extend the embedding along a leaf-adding order from T^1 to T_n in R. Since R contains no red T_n, the procedure must fail. Hence, for the current embedded subtree H, there exists a vertex $u_3 \in V(H)$ such that u_3 is blue-adjacent to every vertex in $V(K_N) \setminus V(H)$. Moreover, $|V(H)| \leq n - 1$.

Since the initial copy of T^1 lies entirely in U, and at most $(n - 1) - (n - m + 5) = m - 6$ vertices have been added before the failure, we have $|W \cap V(H)| \leq m - 6$.

Set $A = N_B(u_3) \cap W$. Since $W \setminus V(H) \subseteq A$,

$$|A| \geq |W| - |W \cap V(H)| \geq (n - 3) - (m - 6) = n - m + 3.$$

We claim that $u_3 \notin \{u_1, u_2\}$. Otherwise, since u_3 is blue-adjacent to every vertex of U and A, we have $d_B(u_3) \geq |U| + |A| \geq n + (n - m + 3) = 2n - m + 3$. Consequently,

$$d_R(u_3) \leq 2n - 2 - (2n - m + 3) = m - 5 < n/2,$$

where the last inequality follows from $n \geq 2m - 4$. This contradicts $\delta(R) \geq n/2$. Hence, $u_3 \notin \{u_1, u_2\}$, as claimed.

Moreover, since $|V(H)| \leq n - 1$ and $|U| = n$, there exists $v_1 \in U \setminus V(H)$. By the choice of u_3, the edge $u_3 v_1$ is blue. Define

$$U_0 = U \setminus (\{v_1\} \cup (\{u_3\} \cap U)).$$

Then, $|U_0| \geq n - 2$. Since $U_0 \cap A = \emptyset$ and $n \geq 2m - 4$,

$$|U_0 \cup A| \geq (n - 2) + (n - m + 3) = 2n - m + 1 \geq n + m - 3.$$

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Again by Corollary 2.3, $R(T_n, P_{m-1}) \le n + m - 3$, which gives

$$|U_0 \cup A| \ge R(T_n, P_{m-1}).$$

Since R contains no red T_n, the graph $B[U_0 \cup A]$ contains a blue path P_{m-1}, say $P = z_1 z_2 \cdots z_{m-1}$.

We now derive a blue C_m according to the positions of the endpoints of P.

- If $z_1, z_{m-1} \in U_0$, then $u_1 z_1 z_2 \cdots z_{m-1} u_1$ is a blue C_m.
- If $z_1, z_{m-1} \in A$, then $u_3 z_1 z_2 \cdots z_{m-1} u_3$ is a blue C_m.
- If one of z_1, z_{m-1} lies in U_0 and the other in A, without loss of generality, let $z_1 \in U_0,\ z_{m-1} \in A$. Consider the position of z_{m-3}.
  - If $z_{m-3} \in U_0$, then $z_1 z_2 \cdots z_{m-3} u_1 v_1 u_2 z_1$ is a blue C_m.
  - If $z_{m-3} \in A$, then $z_1 z_2 \cdots z_{m-3} u_3 v_1 u_1 z_1$ is a blue C_m.

In all cases, we obtain a blue C_m, giving a contradiction.

The proof of Theorem 1.6 is complete.

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