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A general compactness theorem

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1. *Introduction.* It is the purpose of this paper to prove a general compactness theorem and deduce from it, as corollaries, the Tychonoff theorem on the product of compact spaces and Ascoli-type theorems giving sufficient conditions for function spaces with the topology of uniform convergence on compact sets to be compact.

In (1) product and quotient topological spaces with respect to a nucleus (generalizations of the usual product and quotient spaces) were introduced and it was shown that, when certain conditions were satisfied, the operations of forming them were inverse to one another. In the present paper we shall only be interested in a slight generalization of the upper product in the case when a certain set of sets is self-indexed. With this in mind it is easy to fit the results of the present paper into the context of (1), a task we shall leave to the reader.

2. *A general compactness theorem.* Let $\mathscr{V}$ be a sub-basis for a topology $\mathscr{T}$ on a set X. Then $\{\mathscr{P}(V)\colon V \in \mathscr{V}\}$ is a sub-basis for a topology on the set $\mathscr{P}(X)$, where $\mathscr{P}(A)$ denotes the set of all subsets of a set A. The topology induced on a subset $\mathscr{M}$ of $\mathscr{P}(X)$ is called the *upper product topology* on $\mathscr{M}$ induced by $\mathscr{V}$. It depends on $\mathscr{V}$ and not just on $\mathscr{T}$.

Two subsets $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{P}(X)$ are said to be *transverse* iff each element of $\mathscr{M}$ meets each element of $\mathscr{N}$ in a singleton.

Throughout the rest of this section $\mathscr{M}$ and $\mathscr{N}$ will denote a pair of transverse subsets of $\mathscr{P}(X)$ and, for each N in $\mathscr{N}$, ϕ_N will denote the mapping $\phi_N\colon \mathscr{M} \to N$ determined by assigning to each M in $\mathscr{M}$ the unique element of $M \cap N$. We shall also assume that $\mathscr{N}$ is such that each ϕ_N is surjective.

Let L be a subset of X and $(x_N)_{N\in\mathscr{N}}$ a family such that, for each N in $\mathscr{N}$, $x_N \in L \cap N$. We say that $\mathscr{M}$ *is equicontinuous at* L (with respect to $(x_N)_{N\in\mathscr{N}}$) iff given V in $\mathscr{V}$ containing L, there exists a finite sequence $(N_i)_{1\leqslant i\leqslant n}$ of elements of $\mathscr{N}$ and, for $i = 1, 2, \ldots, n$, an open neighbourhood V_i of x_{N_i} in N_i such that

$$\bigcap_{i=1}^{n} St_{\mathscr{M}}(V_i) \subset V,$$

where $St_{\mathscr{M}}(W)$ denotes the union of those elements of $\mathscr{M}$ that meet a subset W of X.

We shall call $\mathscr{M}$ *equicontinuous with respect to* $\mathscr{N}$ iff $\mathscr{M}$ is equicontinuous at each M in $\mathscr{M}$ with respect to $(\phi_N(M))_{N\in\mathscr{N}}$.

Lemma 1. *Let* $\mathscr{K}$ *be an ultrafilter on* $\mathscr{M}$ *such that, for each* N *in* $\mathscr{N}$, $\mathscr{K}_N \equiv \phi_N(\mathscr{K})$ *converges in* N *to* x_N. *Suppose that* $\mathscr{M}$ *is equicontinuous at* L *with respect to* $(x_N)_{N\in\mathscr{N}}$. *Then* $\mathscr{K}$ *converges to the element* L *of* $\mathscr{P}(X)$ *in the upper product topology on* $\mathscr{P}(X)$.

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Proof. Suppose that V is an element of $\mathscr{V}$ containing L. Then, since $\mathscr{M}$ is equicontinuous at L with respect to $(x_N)_{N\in\mathscr{N}}$, there exists a finite sequence $(N_i)_{1\leqslant i\leqslant n}$ of elements of $\mathscr{N}$ and, for $i = 1, 2, \ldots, n$, an open neighbourhood V_i of x_{N_i} in N_i such that

$$\bigcap_{i=1}^{n} St_{\mathscr{M}}(V_i) \subset V.$$

For $i = 1, 2, \ldots, n$, the fact that $\mathscr{K}_{N_i}$ converges to x_{N_i} implies that there exists $K_i \in \mathscr{K}$ such that $\phi_{N_i}(K_i) \subset V_i$. Therefore

$$\bigcap_{i=1}^{n} St_{\mathscr{M}}(\phi_{N_i}(K_i)) \subset V,$$

and hence $\bigcap_{i=1}^{n} K_i$ is a subset of $\mathscr{P}(V)$. Since a finite intersection of elements of a filter belongs to the filter, we have shown that, for each sub-basic open set $\mathscr{P}(V)$ containing the element L of $\mathscr{P}(X)$, there exists an element K of $\mathscr{K}$ such that $K \subset \mathscr{P}(V)$. For the identical reason, the same is true for arbitrary neighbourhoods of L in $\mathscr{P}(X)$, and we have proved that $\mathscr{K}$ converges in $\mathscr{P}(X)$ to L.

Immediate is

COROLLARY. *If $\mathscr{M}$ is equicontinuous with respect to $\mathscr{N}$, then on $\mathscr{M}$ the upper product topology is coarser than the initial topology with respect to $(\phi_N)_{N\in\mathscr{N}}$ (the coarsest topology making each ϕ_N continuous).*

Remark. This corollary is also easy to prove directly using sub-basic open sets instead of ultrafilters.

We say that $\mathscr{M}$ is *$\mathscr{N}$-closed* iff given an ultrafilter $\mathscr{K}$ on $\mathscr{M}$ such that, for each N in $\mathscr{N}$, $\mathscr{K}_N \equiv \phi_N(\mathscr{K})$ converges in N, there exists an element M of $\mathscr{M}$ such that, for each N in $\mathscr{N}$, $\mathscr{K}_N$ converges in N to $\phi_N(M)$.

PROPOSITION. *If each element of $\mathscr{N}$ is compact and $\mathscr{M}$ is $\mathscr{N}$-closed, then $\mathscr{M}$ is compact in the initial topology with respect to $(\phi_N)_{N\in\mathscr{N}}$.*

Proof. Suppose that $\mathscr{K}$ is an ultrafilter on $\mathscr{M}$. Since each element of $\mathscr{N}$ is compact and each ϕ_N is surjective, $\mathscr{K}_N \equiv \phi_N(\mathscr{K})$ converges in N for each N in $\mathscr{N}$. And since $\mathscr{M}$ is $\mathscr{N}$-closed, there exists M in $\mathscr{M}$ such that each $\mathscr{K}_N$ converges to $\phi_N(M)$. Since the topology on $\mathscr{M}$ is initial with respect to $(\phi_N)_{N\in\mathscr{N}}$, $\mathscr{K}$ converges to M. Therefore $\mathscr{M}$ is compact.

Remark. There is an obvious generalization of this to arbitrary initial topologies.

Our compactness theorem now follows from the Proposition and the Corollary to Lemma 1.

THEOREM. *If $\mathscr{M}$ is equicontinuous with respect to $\mathscr{N}$, $\mathscr{M}$ is $\mathscr{N}$-closed, and each element of $\mathscr{N}$ is compact, then $\mathscr{M}$ is compact in the upper product topology.*

3. *Tychonoff's theorem.* Given a family $(X_\lambda)_{\lambda\in\Lambda}$ of sets, let

$$X = \{(\lambda, x_\lambda) : \lambda \in \Lambda \quad \text{and} \quad x_\lambda \in X_\lambda\}.$$

The subset $\mathscr{M}$ of $\mathscr{P}(X)$ consisting of those subsets of X which are functions with domain Λ is the product of the family of sets.

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When each X is a topological space, we give X the coarsest topology $\mathscr{V}$ making the natural injections $j_\lambda\colon X_\lambda \to X$, defined by

$$j_\lambda(x_\lambda) = (\lambda, x_\lambda),$$

closed mappings. (These mappings are then homeomorphisms into X.) The corresponding upper product topology on $\mathscr{M}$ is the 'usual' product topology.

Defining

$$\mathscr{N} = \{j_\lambda(X_\lambda)\colon \lambda \in \Lambda\},$$

it follows from the fact that $\mathscr{M}$ is the set of *all functions* with domain Λ which are subsets of X that $\mathscr{M}$ and $\mathscr{N}$ are transverse and that $\mathscr{M}$ is $\mathscr{N}$-closed. That $\mathscr{M}$ is equicontinuous with respect to $\mathscr{N}$ follows from the fact that a typical element V of $\mathscr{V}$ is of the form

$$V = \bigcap_{i=1}^{n} (j_{\lambda_i}(U_{\lambda_i}) \cup \bigcup_{\lambda \neq \lambda_i} j_\lambda(X_\lambda)),$$

where U_{λ_i} is open in X_{λ_i}, and that we may choose $N_i = j_{\lambda_i}(X_{\lambda_i})$ and $V_i = j_{\lambda_i}(U_{\lambda_i})$ for any M in $\mathscr{M}$ contained in V.

We are now in a position to deduce at once a corollary to our Theorem of section 2.

TYCHONOFF'S THEOREM. *The 'usual' topological product of a family of compact topological spaces is compact.*

4. *Ascoli's theorem.* A set $\mathscr{M}$ of functions on a topological space X to a topological space Y is a subset of $\mathscr{P}(X \times Y)$. We let $\mathscr{V}$ denote the sub-basis for a topology on $X \times Y$ consisting of all sets V of the form

$$V = (X \times U) \cup ((X - C) \times Y),$$

where C is a compact subset of X and U is an open subset of Y. (Then $y \to (x, y)$ is a homeomorphism of Y with the subspace $\{x\} \times Y$.) The corresponding upper product topology on $\mathscr{M}$ is the compact-open topology.

Let $\mathscr{F}$ be the set of all functions F on X to Y such that

$$F \subset \bigcup\{M\colon M \in \mathscr{M}\} \equiv \bigcup \mathscr{M}.$$

And let

$$\mathscr{N} = \{(\{x\} \times Y) \cap (\bigcup \mathscr{M})\colon x \in X\}.$$

Clearly $\mathscr{M}$ and $\mathscr{N}$ are transverse.

Kelley and Morse (see (2)) call a set of functions $\mathscr{M}$ on a topological space X to a topological space Y *evenly continuous* iff given x in X, y in Y and a neighbourhood U of y in Y, there exist neighbourhoods T of x in X and U' of y in Y such that $F(T) \subset U$ whenever F is an element of $\mathscr{M}$ satisfying $F(x) \in U'$.

LEMMA 2. *Suppose that $\mathscr{M}$ is an evenly continuous set of functions on X to Y. Then $\mathscr{M}$ is equicontinuous at F (with respect to the only family of points) for each F in $\mathscr{F}$.*

Proof. Suppose that F is contained in the element

$$V = (X \times U) \cup ((X - C) \times Y)$$

of $\mathscr{V}$, where C is a compact subset of X and U an open subset of Y. Let x be an arbitrary element of C. Then $F(x) \in U$ and the even continuity of $\mathscr{M}$ implies that there exist

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neighbourhoods T_x of x open in X and U'_x of $F(x)$ open in Y such that $M(T_x) \subset U$ whenever M in $\mathscr{M}$ satisfies $M(x) \in U'_x$. $\{T_x : x \in C\}$ forms an open covering of the compact set C and has, therefore, a finite subcovering $(T_{x_i})_{1 \leqslant i \leqslant n}$. The equicontinuity of M at F can now be seen by choosing, for $1 \leqslant i \leqslant n$,

$$N_i = (\{x_i\} \times Y) \cap (\cup \mathscr{M})$$

and

$$V_i = (\{x_i\} \times U'_{x_i}) \cap N_i$$

and noting that

$$St_{\mathscr{M}}(V_i) = \cup\{M \in \mathscr{M} : M(x_i) \in U'_{x_i}\}.$$

For the rest of the paper we give $\mathscr{F}$ (also) the compact–open topology. Using Lemma 1 of section 2, we can deduce immediately a corollary to Lemma 2.

COROLLARY. *If $\mathscr{M}$ is an evenly continuous set of functions on X to Y closed in $\mathscr{F}$, then $\mathscr{M}$ is $\mathscr{N}$-closed.*

A version of Ascoli's theorem, which may be compared with a result of Kelley and Mores (see (2), p. 236), now follows at once as a corollary to our Theorem of section 2. For x in X, let

$$\mathscr{M}[x] = \{M(x) : M \in \mathscr{M}\}.$$

ASCOLI'S THEOREM (*'topological version'*). *If $\mathscr{M}$ is an evenly continuous set of functions on a topological space X to a topological space Y, closed in $\mathscr{F}$ and such that $\mathscr{M}[x]$ is compact for each x in X, then $\mathscr{M}$ is compact.*

We recall that a set of functions $\mathscr{M}$ on a topological space X to a *uniform space* Y is said to be *equicontinuous* iff given x in X and an entourage U of Y, there exists a neighbourhood T of x in X such that, for each x' in T and each F in $\mathscr{M}$, $(F(x), F(x')) \in U$.

Our final corollary to our Theorem of section 2 may then be proved in a manner similar to the above (or deduced from the above). It may be compared with a result in Kelley ((2), pp. 233–234).

ASCOLI'S THEOREM (*'uniform version'*). *If $\mathscr{M}$ is an equicontinuous set of functions on a topological space X to a uniform space Y, closed in $\mathscr{F}$ and such that $\mathscr{M}[x]$ is compact for each x in X, then $\mathscr{M}$ is compact.*

Remark. Our versions of Ascoli's theorem may easily be generalized to the case when each $\mathscr{M}[x]$ is only relatively compact.

We conclude by noting that the methods of this paper may be used to deduce a similar (but more easily proved) compactness theorem for function spaces with the topology of pointwise convergence.

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