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The Gradient Superconvergence of Bilinear Finite Volume Element for Elliptic Problems

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Abstract. We study the gradient superconvergence of bilinear finite volume element (FVE) solving the elliptic problems. First, a superclose weak estimate is established for the bilinear form of the FVE method. Then, we prove that the gradient approximation of the FVE solution has the superconvergence property:

$$\max_{P\in S}|(\nabla u - \overline{\nabla} u_h)(P)| = O(h^2)|\ln h|,$$

where $\overline{\nabla} u_h(P)$ denotes the average gradient on elements containing point P and S is the set of optimal stress points composed of the mesh points, the midpoints of edges and the centers of elements.

AMS subject classifications: 65N30, 65M60

Key words: Bilinear finite volume element, elliptic problem, gradient approximation, superconvergence.

1. Introduction

The finite volume element (FVE) method has been widely used in numerically solving partial deferential equations. The main feature of FVE method is that it inherits some physical conservation laws of original problems locally, which are very desirable in practical applications. During the last decades, many research works have been presented for FVE methods solving various partial differential equations, see [1–10, 12–15, 17, 18, 20] and the references cited therein.

Superconvergence of numerical solutions has been an active research area for finite element method (FEM) since its practical importance in enhancing the accuracy of finite element approximation. But the study of superconvergence properties of FVE methods is far behind that of FEMs. For elliptic problems in two-dimensional domain,

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the early superconvergence results of linear and bilinear FVE solutions are of this form [3, 10]

$$\Big(\frac{1}{N}\sum_{z\in S}|(\nabla u-\overline{\nabla}u_h)(z)|^2\Big)^{\frac{1}{2}}\le Ch^2\|u\|_{3,\infty}, \tag{1.1}$$

where S is the set of optimal stress points of interpolation function on partition T_h, $N=O(h^{-2})$ is the total number of points in S, and $\overline{\nabla}$ denotes the average gradient on elements containing point z. Later, Lv and Li in [12] extended result (1.1) to the isoparametric bilinear FVE on quadrilateral meshes under the h^2-uniform mesh condition. Recently, Zhang and Zou in [20] also derived some superconvergence results for the bi-complete k-order FVE on rectangular meshes, and in the case of bilinear FVE ($k=1$), their result is

$$|\nabla(u-u_h)(G)|\le Ch^2|\ln h|^{\frac{1}{2}}\|u\|_{4,\infty}, \tag{1.2}$$

where G is the Gauss point of element (the midpoint of element). Moreover, by using the postprocessing technique, Chou and Ye [7] obtain the superconvergence estimate:

$$\|\nabla u-\nabla Qu_h)\|\le Ch^{\frac{3}{2}}\|u\|_3, \tag{1.3}$$

where Qu_h is the postprocessed linear FVE solution obtained by the L_2-projection method; Zhang and Sheng [18] further derive the superconvergence estimate:

$$\|\nabla u-R\nabla u_h)\|\le Ch^2\|u\|_3, \tag{1.4}$$

where $R\nabla u_h$ is the reconstructed gradient of bilinear FVE solution obtained by using the patch interpolation recovery method.

In this paper, we consider the bilinear FVE method to solve the following problem

$$\begin{cases} -div(A\nabla u)+c\,u=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega\subset R^2$ is a rectangular domain with boundary $\partial\Omega$, coefficient matrix $A=(a_{ij})_{2\times 2}$. Our main goal is to give some piecewise-point gradient superconvergence for the bilinear FVE approximation to problem (1.5). To the authors' best knowledge, in existing literatures, only the midpoints of elements are proved to be the superconvergence point of gradient approximation [20] (also see (1.2)), we here will prove that except the midpoints of elements, all interior mesh points and midpoints of interior edges are also the superconvergence points. Generally speaking, the analysis of bilinear FVE on rectangular meshes is more difficult then that of linear FVE on triangle meshes, the reason is that ∇u_h is not constant in the former case. By calculating exactly some integrals on element and its boundary, we first establish the superclose weak estimate for the bilinear form of the FVE method,

$$\begin{aligned} &|a_h(u-\Pi_h u,\Pi_h^* v)|\le Ch^2\|u\|_{3,p}\|v\|_{1,q},\\ &\qquad \forall\, v\in U_h,\ 2\le p\le\infty,\ \frac{1}{p}+\frac{1}{q}=1, \end{aligned} \tag{1.6}$$

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where Π_h is the bilinear interpolation operator, Π_h^* is the projection operator from the trial function space U_h to the test function space. It is well known that such weak estimate plays an important role in the superconvergence analysis of FEMs [11,19,21]. By means of this weak estimate and some integral estimates on element, we derive the following superconvergence result for the gradient approximation

$$\max_{P\in S}|(\nabla u - \overline{\nabla} u_h)(P)| \le Ch^2|\ln h|\|u\|_{3,\infty}, \tag{1.7}$$

where S is the optimal stress point set of bilinear interpolation function which is composed of the interior mesh points, the midpoints of interior edges and the midpoints of elements.

This paper is organized as follows. In Section 2, we introduce the FVE method and give some lemmas. In Section 3, we establish the superconvergence weak estimate. Section 4 is devoted to the piecewise-point gradient supconvergence estimate on the set S. Finally, in Section 5, numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notations $W^{m,p}(D)$ to indicate the usual Sobolev spaces on domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$, and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$. When $D = \Omega$, we omit the index D. We will use letter C to represent a generic positive constant, independent of the mesh size h.

2. Finite volume element method

Consider problem (1.5). As usual, we assume that there exist positive constants C_1 and C_2 such that

$$C_1\xi^T\xi \le \xi^T A(x,y)\xi \le C_2\xi^T\xi, \quad \forall\,\xi \in R^2,\ (x,y) \in \Omega, \tag{2.1}$$

We further assume that $A \in [W^{1,\infty}(\Omega)]^{2\times 2}$, $c \in L_\infty(\Omega)$ and $c \ge 0$.

Let $T_h = \bigcup\{K\}$ be a rectangular partition of domain Ω so that $\overline{\Omega} = \bigcup_{K\in T_h}\{\overline{K}\}$, where $h = max\, h_K$, h_K is the diameter of element K. We assume that partition T_h is regular, that is, there exists a positive constant $\gamma > 0$ such that

$$h_K/\rho_K \le \gamma, \quad \forall\, K \in T_h, \tag{2.2}$$

where ρ_K denotes the diameter of the biggest ball included in K.

Associated with partition T_h, we construct the central dual partition T_h^* by connecting the center of each element to the midpoints of edges by straight lines. Thus, for each nodal point P in T_h, there exists a rectangle K_P^* surrounding P, $K_P^* \in T_h^*$ is called the dual element or the control volume at point P, see Fig. 1.

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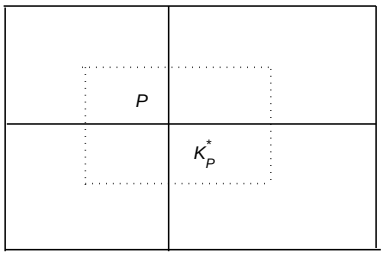


Figure 1: The dual element K_P^* surrounding point P.

To partition T_h and T_h^*, we associate the following trial function space U_h and test function space V_h, respectively,

$$U_h = \Big\{ u_h \in C^0(\overline{\Omega}) : \ u_h|_K \in Q_1(K), \quad \forall\, K \in T_h,\ u_h|_{\partial\Omega} = 0 \Big\},$$
$$V_h = \Big\{ v_h \in L_2(\Omega) : \ v_h|_{K_P^*} = constant, \quad \forall\, P \in\ N_h,\ v_h|_{K_P^*} = 0, \quad \forall\, P \in\ \partial\Omega \Big\},$$

where $Q_1(K)$ is the set of all bilinear polynomials on K and N_h is the set of all nodal points of T_h.

The conventional weak form for problem (1.5) is that find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall\, v \in H_0^1(\Omega), \tag{2.3}$$

where

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v + cu\, v, \ \ (f,v) = \int_\Omega f\, v\,. \tag{2.4}$$

Weak form (2.3) usually is adopted for the finite element method. But for the FVE method, we need a new weak form. Let u be the solution of problem (1.5), then by using Green's formula, we have

$$-\int_{\partial K_P^*} n \cdot (A\nabla u) v ds + \int_{K_P^*} cuv = \int_{K_P^*} fv, \quad K_P^* \in T_h^*,\ v \in V_h, \tag{2.5}$$

where n is the outward unit normal vector on the boundary concerned. Motivated by this weak formula, we introduce the following bilinear form

$$a_h(u,v) = \sum_{K_P^* \in T_h^*} \Big(-\int_{\partial K_P^*} n \cdot (A\nabla u) v ds + \int_{K_P^*} cuv \Big),\ u \in\ H^1(\Omega),\ v \in V_h, \tag{2.6}$$

and define the FVE approximation of problem (1.5) by finding $u_h \in U_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall\, v_h \in V_h. \tag{2.7}$$

Let $\Pi_h^* : \ U_h \to V_h$ be the interpolation operator defined by

$$\Pi_h^* v_h = \sum_{P \in N_h} v_h(P)\chi_P, \quad \forall\, v_h \in U_h,$$

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where χ_P is the characteristic function of the dual element K_P^*. Since Π_h^* is a one to one mapping from U_h onto V_h, then the equivalent form of problem (2.7) is that find $u_h \in U_h$ such that

$$a_h(u_h, \Pi_h^* v_h) = (f, \Pi_h^* v_h), \quad \forall\, v_h \in U_h. \tag{2.8}$$

This is the FVE scheme to be used in our analysis. From (2.5) we know that scheme (2.8) is consistent and the following error equation holds.

$$a_h(u - u_h, \Pi_h^* v_h) = 0, \quad \forall\, v_h \in U_h. \tag{2.9}$$

Let $\Pi_h u$ be the usual bilinear interpolation function of continuous function u. In our analysis, the following approximation property, trace inequality and inverse inequality will be used frequently. For $1 \le p, q \le \infty$, we have

$$\|u - \Pi_h u\|_{m,p,K} \le C h_K^{2-m} \|u\|_{2,p,K}, \quad 0 \le m \le 2,\ 1 < p \le \infty, \tag{2.10}$$

$$\|u\|_{0,p,\partial K} \le C h_K^{-\frac{1}{p}} \Big(\|u\|_{0,p,K} + h_K \|\nabla u\|_{0,p,K} \Big), \quad u \in W^{1,p}(K), \tag{2.11}$$

$$\|u_h\|_{m,p,K} \le C h_K^{\frac{2}{p}-\frac{2}{q}} h_K^{l-m} \|u_h\|_{l,q,K}, \quad u_h \in Q_1(K),\ 0 \le l \le m \le 1. \tag{2.12}$$

Furthermore, the following two lemmas hold [10, 18].

Lemma 2.1. *Let $K \in T_h$, $\tau \subset \partial K$ be an edge of K. Then, for $v_h \in U_h$, $1 \le q \le \infty$, we have*

$$\int_K (v_h - \Pi_h^* v_h) = 0, \quad \int_\tau (v_h - \Pi_h^* v_h) ds = 0, \tag{2.13}$$

$$\|v_h - \Pi_h^* v_h\|_{0,q,K} \le C h_K \|v_h\|_{1,q,K}, \tag{2.14}$$

$$\|v_h - \Pi_h^* v_h\|_{0,q,\partial K} \le C h_K^{1-\frac{1}{q}} \|v_h\|_{1,q,K}. \tag{2.15}$$

Lemma 2.2. *For sufficiently small h, we have*

$$a_h(v_h, \Pi_h^* v_h) \ge C \|v_h\|_1^2, \quad \forall\, v_h \in U_h.$$

Lemma 2.2 implies that the solution u_h of problem (2.8) uniquely exists.

3. Interpolation weak estimate

It is well known that the interpolation weak estimate plays an important role in the superconvergence analysis [11,19,21]. To FEM defined on regular rectangular meshes, the following interpolation weak estimate has been established [19, 21]

$$|a(u - \Pi_h u, v)| \le C h^2 \|u\|_{3,p} \|v\|_{1,q}, \quad \forall\, v \in U_h,\ 2 \le p \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1\,. \tag{3.1}$$

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In this section, we will give a similar weak estimate for the FVE method, which is very useful in our superconvergence analysis.

In order to utilize the known result (3.1), we need to give the difference between the bilinear form of FVE and that of FEM.

Lemma 3.1. *For any $w \in U_h + H^2(\Omega)$, $v_h \in U_h$, we have*

$$a_h(w, \Pi_h^* v_h) - a(w, v_h) = \sum_{K \in T_h} \int_{\partial K} n \cdot (A\nabla w)(\Pi_h^* v_h - v_h) ds + \sum_{K \in T_h} (-div(A\nabla w) + cw, \Pi_h^* v_h - v_h)_K, \quad (3.2)$$

where $U_h + H^2(\Omega) = \{w = u_h + v : u_h \in U_h, v \in H^2(\Omega)\}$ is the the algebraic sum space.

Proof. By Green's formula, we have

$$\int_K A\nabla w \cdot \nabla v_h = -\int_K div(A\nabla w) v_h + \int_{\partial K} n \cdot (A\nabla w) v_h ds,$$

and (see Fig. 1)

$$\sum_{K \in T_h} \int_K div(A\nabla w)\Pi_h^* v_h = \sum_{K \in T_h} \sum_{K_P^* \in T_h^*} \int_{K_P^* \cap K} div(A\nabla w)\Pi_h^* v_h$$
$$= \sum_{K \in T_h} \int_{\partial K} n \cdot (A\nabla w)\Pi_h^* v_h ds + \sum_{K_P^* \in T_h^*} \int_{\partial K_P^*} n \cdot (A\nabla w)\Pi_h^* v_h ds.$$

Substituting this two identities into the definitions of $a(w, v_h)$ and $a_h(w, \Pi_h^* v_h)$ (see (2.4) and (2.6)), the proof is completed. □

Let $K = \Box P_1 P_2 P_3 P_4$ be a rectangular element, $h_x = x_2 - x_1$, $h_y = y_2 - y_1$, see Fig. 2. The following lemma gives some exact calculations for integrals on ∂K.

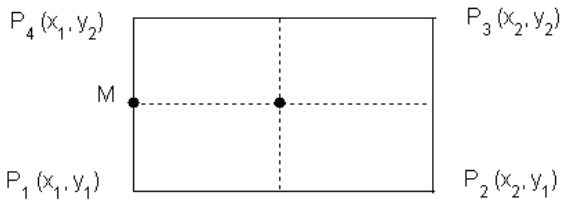


Figure 2: Rectangular element $K = \Box P_1 P_2 P_3 P_4$.

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Lemma 3.2. *For $w, v \in U_h$, we have*

$$\int_{P_1P_4} (\Pi_h^* v - v) w_x dy = \frac{h_y^3}{24} v_y(x_1, y) w_{xy}, \qquad \int_{P_1P_4} (\Pi_h^* v - v) w_y dy = 0, \tag{3.3}$$

$$\int_{P_2P_3} (\Pi_h^* v - v) w_x dy = \frac{h_y^3}{24} v_y(x_2, y) w_{xy}, \qquad \int_{P_2P_3} (\Pi_h^* v - v) w_y dy = 0, \tag{3.4}$$

$$\int_{P_1P_2} (\Pi_h^* v - v) w_y dx = \frac{h_x^3}{24} v_x(x, y_1) w_{xy}, \qquad \int_{P_1P_2} (\Pi_h^* v - v) w_x dx = 0, \tag{3.5}$$

$$\int_{P_4P_3} (\Pi_h^* v - v) w_y dx = \frac{h_x^3}{24} v_x(x, y_2) w_{xy}, \qquad \int_{P_4P_3} (\Pi_h^* v - v) w_x dx = 0. \tag{3.6}$$

Proof. We only prove (3.3), the other are similar. Noting that w_x and v are linear on line segment P_1P_4 (M is its middle point), we have from the definition of $\Pi_h^* v$ that

$$\begin{aligned}\int_{P_1P_4} \Pi_h^* v w_x dy &= \int_{P_1M} v(P_1) w_x dy + \int_{MP_4} v(P_4) w_x dy \\ &= \frac{h_y}{4} v(P_1)\Big(w_x(P_1) + w_x(M)\Big) + \frac{h_y}{4} v(P_4)\Big(w_x(M) + w_x(P_4)\Big) \\ &= \frac{h_y}{4}\Big(v(P_1)w_x(P_1) + 2v(M)w_x(M) + v(P_4)w_x(P_4)\Big).\end{aligned}$$

On the other hand, using the composite trapezoidal rule:

$$\int_a^b f(y)dy = \frac{b-a}{4}\left(f(a) + 2f(\frac{a+b}{2}) + f(b)\right) - \frac{(b-a)^3}{48} f_y''$$

we obtain by taking $f = vw_x$ that

$$\begin{aligned}\int_{P_1P_4} v w_x dy = \frac{h_y}{4}\Big(v(P_1)w_x(P_1) + 2v(M)w_x(M) + v(P_4)w_x(P_4)\Big) \\ -\frac{h_y^3}{24} v_y(x_1, y) w_{xy}.\end{aligned}$$

The difference of above two equalities implies the first equality in (3.3). The second equality comes from (2.13), noting that w_y is constant on P_1P_4. □

Let w^c denote the piecewise constant approximation of function w on T_h,

$$w^c|_K = \frac{1}{|K|}\int_K w, \quad K \in T_h; \tag{3.7a}$$

$$\|w - w^c\|_{0,p,K} \le C h_K \|w\|_{1,p,K}, \quad 1 \le p \le \infty. \tag{3.7b}$$

Theorem 3.1. *Let T_h be a regular rectangular mesh and $u \in W^{3,p}(\Omega)$. Then we have*

$$\begin{aligned}|a_h(u - \Pi_h u, \Pi_h^* v)| &\le C h^2 \|u\|_{3,p} \|v\|_{1,q}, \\ &\forall\, v \in U_h,\ 2 \le p \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1\,.\end{aligned} \tag{3.8}$$

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Proof. Let $A_M = (\overline{a}_{ij})_{2\times 2}$ be the value of A at the midpoint M of edge $\tau \subset \partial K$, see Fig. 2. From Lemma 3.1, we have

$$\begin{aligned}
&a_h(u-\Pi_h u, \Pi_h^* v) - a(u-\Pi_h u, v)\\
&= \sum_{K\in T_h}\int_{\partial K} n\cdot(A-A_M)\nabla(u-\Pi_h u)(\Pi_h^* v - v)ds\\
&\quad + \sum_{K\in T_h}\int_{\partial K} n\cdot(A_M\nabla(u-\Pi_h u))(\Pi_h^* v - v)ds\\
&\quad + \sum_{K\in T_h}(-div(A\nabla(u-\Pi_h u)), \Pi_h^* v - v)_K\\
&\quad + \sum_{K\in T_h}(c(u-\Pi_h u), \Pi_h^* v - v)_K\\
&=: E_1 + E_2 + E_3 + E_4. \qquad (3.9)
\end{aligned}$$

Using (2.10)-(2.11) and (2.14)-(2.15), we obtain

$$E_1 \le C\sum_{K\in T_h} h_K|A|_{1,\infty}\|\nabla(u-\Pi_h u)\|_{0,p,\partial K}\|v-\Pi_h^* v\|_{0,q,\partial K} \le Ch^2\|u\|_{2,p}\|v\|_{1,q}.$$

$$E_4 \le C\sum_{K\in T_h} \|u-\Pi_h u\|_{1,p,K}\|v-\Pi_h^* v\|_{0,q,K} \le Ch^2\|u\|_{2,p}\|v\|_{1,q}.$$

For E_3, set $A = (\mathbf{a_1}, \mathbf{a_2})$. Since

$$div(A\nabla w) = (div\,\mathbf{a_1}, div\,\mathbf{a_2})\cdot\nabla w + \mathbf{a_1}\cdot\nabla w_x + \mathbf{a_2}\cdot\nabla w_y, \qquad (3.10)$$

we have

$$\begin{aligned}
E_3 &= \sum_{K\in T_h}(-(div\,\mathbf{a_1}, div\,\mathbf{a_2})\cdot\nabla(u-\Pi_h u), \Pi_h^* v - v)_K\\
&\quad - \sum_{K\in T_h}(\mathbf{a_1}\cdot\nabla(u-\Pi_h u)_x + \mathbf{a_2}\cdot\nabla(u-\Pi_h u)_y, \Pi_h^* v - v)_K\\
&=: E_{31} + E_{32}.
\end{aligned}$$

Obviously, $E_{31} \le Ch^2\|u\|_{2,p}\|v\|_{1,q}$. Now, let $u_i = u(P_i)$, $\xi = (x-x_1)/h_x$, $\eta = (y-y_1)/h_y$, see Fig. 2. Since

$$\Pi_h u = u_1 + (u_2-u_1)\xi + (u_4-u_1)\eta + (u_3+u_1-u_2-u_4)\xi\eta, \quad (x,y)\in K,$$

we have

$$\begin{aligned}
(\Pi_h u)_{xy} &= (u_3+u_1-u_2-u_4)h_x^{-1}h_y^{-1}\\
&= \frac{1}{|K|}\left(\int_{x_1}^{x_2} u_x(x,y_2)dx - \int_{x_1}^{x_2} u_x(x,y_1)dx\right)\\
&= \frac{1}{|K|}\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}dxdy = \frac{1}{|K|}\int_K u_{xy}dxdy = (u_{xy})^c. \qquad (3.11)
\end{aligned}$$

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Therefore, we have from (2.13) that

$$\begin{aligned}E_{32} =& -\sum_{K\in T_h}(a_{11}u_{xx}+a_{21}(u_{xy}-(\Pi_h u)_{xy})+a_{12}\big(u_{xy}-(\Pi_h u)_{xy}\big)+a_{22}u_{yy},\Pi_h^* v-v)_K\\ =& -\sum_{K\in T_h}(a_{11}u_{xx}+a_{21}(u_{xy}-u_{xy}^c)+a_{12}(u_{xy}-u_{xy}^c)+a_{22}u_{yy},\Pi_h^* v-v)_K\\ =& -\sum_{K\in T_h}(a_{11}u_{xx}+a_{22}u_{yy}-(a_{11}u_{xx}+a_{22}u_{yy})^c,\Pi_h^* v-v)_K\\ &-\sum_{K\in T_h}(a_{21}(u_{xy}-u_{xy}^c)+a_{12}(u_{xy}-u_{xy}^c),\Pi_h^* v-v)_K.\end{aligned}$$

Using the approximation property, we obtain $E_{32}\le Ch^2\|u\|_{3,p}\|v\|_{1,q}$ so that

$$E_3=E_{31}+E_{32}\le Ch^2\|u\|_{3,p}\|v\|_{1,q}.$$

Now we need to estimate

$$E_2=\sum_{K\in T_h}\sum_{\tau\subset\partial K\setminus\partial\Omega}\int_\tau n\cdot(A_M\nabla(u-\Pi_h u))(\Pi_h^* v-v)ds. \tag{3.12}$$

Note that $(\Pi_h^* v-v)|_{\partial\Omega}=0$. Let τ be an interior edge, that is, a common edge of two adjacent element K and K'. Since $n|_{\tau\cap\partial K}=-n|_{\tau\cap\partial K'}$ and $A_M\nabla u(\Pi_h^* v-v)$ is continuous across edge τ, we have

$$\begin{aligned}E_2=&\sum_{K\in T_h}\sum_{\tau\subset\partial K\setminus\partial\Omega}\int_\tau -n\cdot(A_M\nabla\Pi_h u)(\Pi_h^* v-v)ds\\ =:&\sum_{K\in T_h}\sum_{\tau\subset\partial K\setminus\partial\Omega}F(\tau).\end{aligned} \tag{3.13}$$

Without loss of generality, let $K=\square P_1P_2P_3P_4=(x_1,x_2)\times(y_1,y_2)$ be a rectangular element and the common edge $\tau=\partial K\bigcap\partial K'=P_1P_4$, see Fig.2. From Lemma 3.2 we have

$$\begin{aligned}F(\tau)=&\int_{P_1P_4}\Big(\overline{a}_{11}(\Pi_h u)_x+\overline{a}_{12}(\Pi_h u)_y\Big)(\Pi_h^* v-v)dy\\ =&\overline{a}_{11}\frac{h_y^3}{24}v_y(x_1,y)(\Pi_h u)_{xy}.\end{aligned} \tag{3.14}$$

Then, from (3.14), (3.4) and (3.11), and noting that $n|_{\tau\cap\partial K}=-n|_{\tau\cap\partial K'}$, $\overline{a}_{11}v_y(x_1,y)$ is continuous acrose $\tau=\partial K\cap\partial K'=P_1P_4$, we obtain

$$F(\tau\cap\partial K)+F(\tau\cap\partial K')=\overline{a}_{11}\frac{h_y^3}{24}v_y(x_1,y)\Big((u_{xy})^c(K)-(u_{xy})^c(K')\Big). \tag{3.15}$$

Let $i_h w\in C^0(\Omega)$, $i_h w|_K\in Q_1(K)$, be an approximation of function w, which satisfies

$$\|w-i_h w\|_{0,p}\le Ch\|w\|_{1,p},\quad 2\le p\le\infty.$$

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For example, for $p = 2$ we may take $i_h w = P_h w$, P_h is the L_2-projection operator; For $p > 2$, we may take $i_h w = \Pi_h w$. Then, from (3.15) and the inverse inequality, we have

$$\begin{aligned}
&F(\tau \cap \partial K) + F(\tau \cap \partial K') \\
&= \overline{a}_{11} \frac{h_y^3}{24} v_y(x_1, y) \Big((u_{xy})^c(K) - i_h u_{xy}(M) + i_h u_{xy}(M) - (u_{xy})^c(K') \Big) \\
&\le C h_y^3 h_K^{-\frac{2}{q}} \|v\|_{1,q,K} \Big(h_K^{-\frac{2}{p}} \|(u_{xy})^c - i_h u_{xy}\|_{0,p,K} + h_{K'}^{-\frac{2}{p}} \|(u_{xy})^c - i_h u_{xy}\|_{0,p,K'} \Big) \\
&\le C h_y \|v\|_{1,q,K} \Big(\|(u_{xy})^c - u_{xy}\|_{0,p,K\cup K'} + \|u_{xy} - i_h u_{xy}\|_{0,p,K\cup K'} \Big), \qquad (3.16)
\end{aligned}$$

where we have used the fact that $h_y = |\partial K \bigcap \partial K'| \le \min\{h_K, h_{K'}\}$. Combining (3.13) with (3.16), we can conclude that

$$E_2 \le C h \|v\|_{1,q} \Big(\|u_{xy} - (u_{xy})^c\|_{0,p} + \|u_{xy} - i_h u_{xy}\|_{0,p} \Big) \le C h^2 \|u\|_{3,p} \|v\|_{1,q}. \qquad (3.17)$$

Substituting estimates $E_1 \sim E_4$ into (3.9), we complete the proof by using (3.1). □

From Theorem 3.1, we immediately obtain the following superclose result.

Theorem 3.2. *Let u and u_h be the solutions of problem* (1.5) *and* (2.8)*, respectively, $u \in H^3(\Omega)$. Then we have*

$$\|\Pi_h u - u_h\|_1 \le C h^2 \|u\|_3.$$

Proof. From Lemma 2.2, error equation (2.9) and weak estimate (3.8), we have

$$\begin{aligned}
C\|u_h - \Pi_h u\|_1^2 &\le a_h(u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) \\
&= a_h(u - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) \le C h^2 \|u\|_3 \|u_h - \Pi_h u\|_1.
\end{aligned}$$

This gives the conclusion. □

By using the embedding theory and weak embedding inequality of finite element space [21], a direct result from Theorem 3.2 is the following optimal L_p-error estimates,

$$\begin{aligned}
\|u - u_h\|_{0,p} &\le \|u - \Pi_h u\|_{0,p} + \|\Pi_h u - u_h\|_{0,p} \\
&\le C_p h^2 \Big(\|u\|_{2,p} + \|u\|_3 \Big), \quad 1 < p < \infty, \\
\|u - u_h\|_{0,\infty} &\le C \Big(h^2 \|u\|_{2,\infty} + |\ln\, h|^{\frac{1}{2}} \|\Pi_h u - u_h\|_1 \Big) \\
&\le C h^2 \Big(\|u\|_{2,\infty} + |\ln\, h|^{\frac{1}{2}} \|u\|_3 \Big).
\end{aligned}$$

4. Piecewise-point superconvergence for gradient approximation

Let point set S be composed of all interior mesh points, midpoints of interior edges and the midpoints of elements. In this section, we will prove that the bilinear finite volume solution u_h has the $W^{1,\infty}$-superconvergence on set S.

We first give some lemmas.

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Lemma 4.1. ([16, 21]) *Point set S is the optimal stress point set of bilinear interpolation $\Pi_h u$, that is, for each $P \in S$, it holds*

$$|(\nabla u - \overline{\nabla}\Pi_h u)(P)| \le Ch^2\|u\|_{3,\infty,E}, \tag{4.1}$$

where E is the union of elements containing point P and $\overline{\nabla}w(P)$ denotes the average value of gradient $\nabla w(P)$ in elements containing point P.

Let rectangular element $K = \square P_1P_2P_3P_4$. For $w_h \in U_h$, set $w_i = w(P_i), i = 1, \cdots, 4$. Introduce the discrete H^1-norm by

$$|w_h|_{1,h} = \Big(\sum_{K\in T_h} |w_h|^2_{1,h,K}\Big)^{\frac{1}{2}},\ w_h \in U_h, \tag{4.2}$$

where

$$|w_h|^2_{1,h,K} = (w_2 - w_1)^2 + (w_3 - w_2)^2 + (w_3 - w_4)^2 + (w_4 - w_1)^2. \tag{4.3}$$

Lemma 4.2. *The discrete norm $|w_h|_{1,h}$ is equivalent to the norm $|w_h|_1$ on space U_h, and*

$$\frac{1}{6\gamma}|w_h|^2_{1,h,K} \le |w_h|^2_{1,K} \le \frac{\gamma}{2}|w_h|^2_{1,h,K}, \quad \forall\, K \in T_h. \tag{4.4}$$

Proof. Let $K = \square P_1P_2P_3P_4 = [x_1, x_1 + h_x] \times [y_1, y_1 + h_y], \xi = (x - x_1)/h_x,\ \eta = (y - y_1)/h_y$. For $w_h \in Q_1(K)$, we may write

$$w_h(x, y) = w_1 + (w_2 - w_1)\xi + (w_4 - w_1)\eta + (w_3 + w_1 - w_2 - w_4)\xi\eta. \tag{4.5}$$

Set

$$w_{21} = w_2 - w_1, \quad w_{41} = w_4 - w_1, \quad w_{32} = w_3 - w_2, \quad w_{34} = w_3 - w_4, \tag{4.6}$$

$$w_{1234} = w_3 + w_1 - w_2 - w_4 = w_{34} - w_{21} = w_{32} - w_{41}. \tag{4.7}$$

Then, we have from (4.5)-(4.7) that

$$\begin{aligned}
|w_h|^2_{1,K} &= \int_K |\nabla w_h|^2 dxdy = \int_0^1\int_0^1 \Big((\partial_\xi w_h/h_x)^2 + (\partial_\eta w_h/h_y)^2\Big)h_xh_y d\xi d\eta \\
&= \frac{h_y}{h_x}\int_0^1\int_0^1 (w_{21} + w_{1234}\eta)^2 d\xi d\eta + \frac{h_x}{h_y}\int_0^1\int_0^1 (w_{41} + w_{1234}\xi)^2 d\xi d\eta \\
&= \frac{h_y}{h_x}\Big(w^2_{21} + w_{21}w_{1234} + \frac{1}{3}w^2_{1234}\Big) + \frac{h_x}{h_y}\Big(w^2_{41} + w_{41}w_{1234} + \frac{1}{3}w^2_{1234}\Big) \\
&= \frac{1}{3}\frac{h_y}{h_x}\Big(w^2_{21} + w_{21}w_{34} + w^2_{34}\Big) + \frac{1}{3}\frac{h_x}{h_y}\Big(w^2_{32} + w_{32}w_{41} + w^2_{41}\Big).
\end{aligned}$$

Hence, it follows from the Cauchy inequality and the regularity condition (2.2) that

$$\frac{1}{6\gamma}\Big(w^2_{21} + w^2_{34} + w^2_{32} + w^2_{41}\Big) \le |w_h|^2_{1,K} \le \frac{\gamma}{2}\Big(w^2_{21} + w^2_{34} + w^2_{32} + w^2_{41}\Big).$$

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Combining this with (4.3), the proof is completed. □

Let $A^c = (a^c_{ij})_{2\times 2}$ be the piecewise constant approximation of matrix function A.

Lemma 4.3. *The following identity holds for* $w, v \in U_h$,

$$\int_{\partial K} n\cdot(A^c\nabla w)(\Pi_h^* v - v)ds = \frac{h_y^3 h_x}{24}a^c_{11}v_{xy}w_{xy} + \frac{h_y h_x^3}{24}a^c_{22}v_{xy}w_{xy}. \tag{4.8}$$

Proof. From Lemma 3.2, we obtain

$$\begin{aligned}
&\int_{\partial K} n\cdot(A^c\nabla w)(\Pi_h^* v - v)ds\\
&= \int_{P_2P_3}(\Pi_h^* v - v)(a^c_{11}w_x + a^c_{12}w_y)dy - \int_{P_1P_4}(\Pi_h^* v - v)(a^c_{11}w_x + a^c_{12}w_y)dy\\
&\quad + \int_{P_4P_3}(\Pi_h^* v - v)(a^c_{21}w_x + a^c_{22}w_y)dx - \int_{P_1P_2}(\Pi_h^* v - v)(a^c_{21}w_x + a^c_{22}w_y)dx\\
&= \frac{h_y^3}{24}a^c_{11}v_y(x_2,y)w_{xy} - \frac{h_y^3}{24}a^c_{11}v_y(x_1,y)w_{xy} + \frac{h_x^3}{24}a^c_{22}v_x(x,y_2)w_{xy} - \frac{h_x^3}{24}a^c_{22}v_x(x,y_1)w_{xy}\\
&= \frac{h_y^3 h_x}{24}a^c_{11}v_{xy}w_{xy} + \frac{h_y h_x^3}{24}a^c_{22}v_{xy}w_{xy}.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we need to introduce the regularized Green function [19, 21]. For any given $z \in \Omega$, let $\delta_h^z \in U_h$ be the smooth δ-function which satisfies,

$$(\delta_h^z, v_h) = v_h(z),\ z\in\Omega, \quad \forall\, v_h \in U_h.$$

For any appointed direction L, define the direction derivative

$$\partial_z v(z) = \lim_{\triangle z\to 0, \triangle z//L}\big(v(z+\triangle z) - v(z)\big)/|\triangle z|.$$

Then, there exists a regularized Green function of derivative type $\partial_z G^z(x) \in H_0^1(\Omega)\bigcap H^2(\Omega)$ such that

$$a(v, \partial_z G^z) = (\partial_z\delta_h^z, v), \quad \forall\, v \in H_0^1(\Omega).$$

Let $\partial_z G_h^z \in U_h$ be the finite element approximation of $\partial_z G^z$ such that

$$a(v_h, \partial_z G^z - \partial_z G_h^z) = 0, \quad \forall\, v_h \in U_h.$$

Clearly, we have

$$a(v_h, \partial_z G_h^z) = a(v_h, \partial_z G^z) = (\partial_z\delta_h^z, v_h) = \partial_z v_h(z), \quad \forall\, v_h \in U_h. \tag{4.9}$$

Assume that partition T_h is quasi-uniform so that the inverse inequality holds on finite element space U_h. Under this condition, the following boundness estimates were given in [19, 21]

$$\|\partial_z G_h^z\|_1 \le Ch^{-1}|\ln h|^{\frac{1}{2}}, \quad \|\partial_z G_h^z\|_{1,1} \le C|\ln h|, \tag{4.10}$$

where C is a positive constant independent of $z \in \Omega$.

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Theorem 4.1. *Let partition T_h be quasi-uniform, and u and u_h be the solutions of problem* (1.5) *and* (2.8)*, respectively, $u \in W^{3,\infty}(\Omega)$. Then we have*

$$\|\Pi_h u - u_h\|_{1,\infty} \le Ch^2|\ln h|\|u\|_{3,\infty}. \tag{4.11}$$

Proof. From (4.9), error equation (2.9), Theorem 3.1 and (4.10), we obtain

$$\begin{aligned}
&\partial_z(u_h - \Pi_h u)(z) = a(u_h - \Pi_h u, \partial_z G_h^z)\\
&= a(u_h - \Pi_h u, \partial_z G_h^z) - a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z) + a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z)\\
&= a(u_h - \Pi_h u, \partial_z G_h^z) - a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z) + a_h(u - \Pi_h u, \Pi_h^* \partial_z G_h^z)\\
&\le a(u_h - \Pi_h u, \partial_z G_h^z) - a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z) + Ch^2\|u\|_{3,\infty}\|\partial_z G_h^z\|_{1,1}\\
&\le a(u_h - \Pi_h u, \partial_z G_h^z) - a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z) + Ch^2|\ln h|\|u\|_{3,\infty}.
\end{aligned} \tag{4.12}$$

Below we need to estimate (see Lemma 3.1)

$$\begin{aligned}
&a(u_h - \Pi_h u, \partial_z G_h^z) - a_h(u_h - \Pi_h u, \Pi_h^* \partial_z G_h^z)\\
&= \sum_{K\in T_h} \int_{\partial K} n \cdot (A - A^c)\nabla(u_h - \Pi_h u)(\partial_z G_h^z - \Pi_h^* \partial_z G_h^z) ds\\
&\quad + \sum_{K\in T_h} \int_{\partial K} n \cdot A^c \nabla(u_h - \Pi_h u)(\partial_z G_h^z - \Pi_h^* \partial_z G_h^z) ds\\
&\quad + \sum_{K\in T_h} (-div(A\nabla(u_h - \Pi_h u)), \partial_z G_h^z - \Pi_h^* \partial_z G_h^z)_K\\
&\quad + (c(u_h - \Pi_h u), \partial_z G_h^z - \Pi_h^* \partial_z G_h^z)_K\\
&=: S_1 + S_2 + S_3 + S_4.
\end{aligned} \tag{4.13}$$

First, using (2.11), (2.12) and (2.15), Theorem 3.2 and (4.10), we have

$$\begin{aligned}
S_1 &\le C \sum_{K\in T_h} h_K |A|_{1,\infty} \|\nabla(u_h - \Pi_h u)\|_{0,\partial K} \|\partial_z G_h^z - \Pi_h^* \partial_z G_h^z\|_{0,\partial K}\\
&\le Ch^{\frac{1}{2}} \|\nabla(u_h - \Pi_h u)\| h^{\frac{1}{2}} \|\partial_z G_h^z\|_1\\
&\le Ch^3 \|u\|_3 \|\partial_z G_h^z\|_1 \le Ch^2 |\ln h|^{\frac{1}{2}} \|u\|_3.
\end{aligned}$$

Next, it follows from Lemma 4.3 and inverse inequality (2.12) that

$$\begin{aligned}
S_2 &\le \sum_{K\in T_h} |A|_\infty \frac{h_K^4}{12} |(u_h - \Pi_h u)_{xy}|_{0,\infty,K} |(\partial_z G_h^z)_{xy}|_{0,\infty,K}\\
&\le C \sum_{K\in T_h} h_K^2 |(u_h - \Pi_h u)_{xy}|_{0,\infty,K} \|\partial_z G_h^z\|_{1,K}.
\end{aligned}$$

From (4.5) and Lemma 4.2, we have

$$\begin{aligned}
&|(u_h - \Pi_h u)_{xy}| = |(u_h - \Pi_h u)_{1234}|\\
&\le \sqrt{2}\,|u_h - \Pi_h u|_{1,K,h} \le \sqrt{12\gamma}\,|u_h - \Pi_h u|_{1,K}.
\end{aligned}$$

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Substituting this estimate into S_2 and using Theorem 3.2, we obtain

$$S_2 \le Ch^2\|u_h - \Pi_h u\|_1 \|\partial_z G_h^z\|_1 \le Ch^3|\ln h|^{\frac{1}{2}}\|u\|_3.$$

Similarly, by Lemma 4.2 and Theorem 3.2, we have (noting that $(u_h - \Pi_h u)_{xx} = (u_h - \Pi_h u)_{yy} = 0$)

$$\begin{aligned} S_3 &\le C \sum_{K\in T_h} \Big(\|\nabla(u_h - \Pi_h u)\|_{0,K} + \|(u_h - \Pi_h u)_{xy}\|_{0,K}\Big)\|\partial_z G_h^z - \Pi_h^* \partial_z G_h^z)\|_{0,K} \\ &\le C \sum_{K\in T_h} \|u_h - \Pi_h u\|_{1,K}\|\partial_z G_h^z - \Pi_h^* \partial_z G_h^z)\|_{0,K} \\ &\le Ch^3\|u\|_3\|\partial_z G_h^z\|_1 \le Ch^2|\ln h|^{\frac{1}{2}}\|u\|_3. \end{aligned}$$

For S_4, using Theorem 3.2, (2.14) and (4.10) to obtain

$$\begin{aligned} S_4 &\le C \sum_{K\in T_h} |A|_{1,\infty}\|u_h - \Pi_h u\|_{1,K}\|\partial_z G_h^z - \Pi_h^* \partial_z G_h^z\|_{0,K} \\ &\le Ch^2\|u\|_3\, h\|\partial_z G_h^z\|_1 \le Ch^2|\ln h|^{\frac{1}{2}}\|u\|_3. \end{aligned}$$

Substituting estimates $S_1 \sim S_4$ into (4.13), we complete the proof by (4.12). □

Now we can give the main superconvergence result.

Theorem 4.2. *Let partition T_h be quasi-uniform, and u and u_h be the solutions of problem* (1.5) *and* (2.8)*, respectively. If $u \in W^{3,\infty}(\Omega)$, then we have*

$$\max_{P\in S} |(\nabla u - \overline{\nabla} u_h)(P)| \le Ch^2|\ln h|\|u\|_{3,\infty}, \tag{4.14}$$

where $\overline{\nabla} w(P)$ denotes the average values of gradient $\nabla w(P)$ in elements containing point P and S is the optima stress point set composed of all interior mesh points, the midpoints of interior edges and the midpoints of elements.

Proof. From (4.1) and Theorem 4.1, we immediately obtain the conclusion by using the triangle inequality. □

5. Numerical example

In this section, we will give some numerical results to illustrate the theoretical analysis.

Let us consider problem (1.5) with the data:

$$A(x,y) = \begin{pmatrix} e^{2x} + y^3 + 1 & e^{x+y} \\ e^{x+y} & e^{2y} + x^3 + 1 \end{pmatrix}, \quad c(x,y) = 2 + x + y.$$

We take $\Omega = [0,1]^2$ and the exact solution $u(x,y) = 2\sin(2\pi x)\sin(3\pi y)$.

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Table 1: Computation error and convergence rate .

	$\|\nabla(\Pi_h u - u_h)\|$		$\max_{P\in S}\|\nabla u(P) - \overline{\nabla} u_h(P)\|$	
$mesh$	error	rate	error	rate
h=0.273	1.1244	N/A	1.6323	N/A
h/2	2.8463e-01	1.982	4.1856e-1	1.9634
h/4	7.1752e-02	1.988	1.0667e-1	1.9723
h/8	1.7988e-02	1.996	2.6886e-2	1.9882
h/16	4.4628e-03	2.011	6.7748e-3	1.9886
h/32	1.1149e-03	2.001	1.7038e-3	1.9914

We present numerical results using the quasi-uniform mesh sequence $\{T_i\}$. This mesh sequence is generated in the following way. We first make an original (non-uniform) rectangular mesh T_1 with mesh size h. Then we connect the midpoints of each edge of elements in T_i $(i \geq 1)$ to obtain the refined mesh T_{i+1} which has half mesh size of T_i. It is easy to see that such bisection rectangular meshes must be quasi-uniform.

Denote by e_h the computation error in the corresponding norm. The numerical convergence rate is computed by using the formula $r = \ln(e_h/e_{h/2})/\ln 2$. Table 1 gives the error and convergence rate with successively refined meshes. We observe that the convergence rate on error $\|\nabla(\Pi_h u - u_h)\|$ is of $O(h^2)$ order and the convergence rate on error $e_h = \max_{P\in S}|\nabla u(P) - \overline{\nabla} u_h(P)|$ is about $O(h^2)|\ln h|$ order, as the theoretical prediction.

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References

[1] Z.Q. CAI, *On the finite volume element method*, Numer. Math., 58 (1991), pp. 713–735.
[2] Z.Y. CHEN, *L^2 estimate of linear element generalized difference schemes*, Acta. Sci. Nat. Univ. Sunyatseni, 33 (1994), pp. 22–28.
[3] Z.Y. CHEN, *Superconvergence of generalized difference methods for elliptic boundary value problem*, Numer. Math. A J. of Chinese Univ.(English Ser), 3 (1994), pp. 163–171.
[4] Z.Y. CHEN, R.H. LI AND A.H. ZHOU, *A note on the optimal L^2-estimate of the finite volume element method*, Adv. Comput. Math., 16 (2002), pp. 291–303.
[5] L. CHEN, *A new class of high order finite volume element methods for second order elliptic equations*, SIAM J. Numer. Anal., 47 (2010), pp. 4021–4023.
[6] S.H. CHOU AND Q. LI, *Error estimates in L^2, H^1, L^∞ in covolume methods for elliptic and parabolic problem: a unified approach*, Math. Comp., 69 (2000), pp. 103–120.

https://doi.org/10.4208/nmtma.2016.m1515 Published online by Cambridge University Press

[7] S.H. Chou and X. Ye, *Superconvergence of finite volume methods for the second order elliptic problem*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 3706–3712.
[8] R.E. Ewing, T. Lin and Y.P. Lin, *On the accuracy of the finite volume element method based on piecewise linear polynomials* SIAM J. Numer. Anal., 39 (2002), pp. 1865–1888.
[9] R. Lazarov, I. Michev and P. Vassilevski, *Finite volume methods for convection-diffusion problems*, SIAM J. Numer. Anal., 33 (1996), pp. 31–55.
[10] R.H. Li, Z.Y. Chen and W. Wu, *Generalized difference methods for differential equations: numerical analysis of finite volume methods*, Marcel, New York, 2000.
[11] Q. Lin and Q.D. Zhu, *The preprocessing and postprocessing for the finite element methods*, Shanghai Sci & Tech Publishing, Shanghai, 1994.
[12] J.L. Lv and Y.H. Li, *L^2 error estimates and superconvergence of the finite volume element methods on quadrilateral meshes*, Adv. Comput. Math., 37 (2012), pp. 393–416.
[13] E. Süli, *Convergence of finite volume schemes for poission's equation on nonuniform meshes*, SIAM J. Numer. Anal., 28 (1991), pp. 1419–1430.
[14] H.J. Wu and R.H. Li, *Error estimate for finite volume element methods for general second order elliptic problem*, Numer. Meth. PDEs., 19 (2003), pp. 693–708.
[15] J.C. Xu and Q.S. Zou, *Analysis of linear and quadratic simplical finite volume methods for elliptic equations*, Numer. Math., 111 (2009), pp. 469–492.
[16] L. Zhang and L.K. Li, *On superconvergence of isopareametric bilinear finite elements*, Comm. Numer. Meth. Enrg., 12 (1996), pp. 849–862.
[17] T. Zhang, Y.P. Lin and R.J. Tait, *On the finite volume element version of Ritz-Volterra projection and applications to related equations*, J. Comp. Math., 20 (2002), pp. 491–504.
[18] T. Zhang and Y. Sheng, *Superconvergence and Gradient Recovery for a Finite Volume Element Method for Solving Convection-Diffusion Equations*, Numer. Meth. PDEs., 30 (2014), pp. 1152–1168.
[19] T. Zhang, *Finite element methods for partial differential-integral equations*, Science Press, Beijing, 2009.
[20] Z.M. Zhang and Q.S. Zou, *Some recent advances on vertex centered finite volume element methods for elliptic equations*, Sci. China Mathematics, 56 (2013), pp. 2507–2522.
[21] Q.D. Zhu and Q. Lin, *The superconvergence theory of finite elements*, Hunan Science and Technology Publishing House, Changsha, 1989.

https://doi.org/10.4208/nmtma.2016.m1515 Published online by Cambridge University Press