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A two-dimensional circular inclusion problem

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Abstract. The elastic fields in an elastic circular inclusion and its surrounding infinite dissimilar elastic matrix, are determined when either the matrix or inclusion is subject to a concentrated force or edge dislocation.

1. Two recent papers, (4) and (5), have been concerned with attempting to determine some aspects of the interaction between dislocations and large scale inhomogeneities, such as occur with precipitates in alloys.

An infinite region or matrix was considered, containing a dissimilar circular region or inclusion, with the assumption that matrix and inclusion were perfectly bonded together, and that a single edge dislocation was located in either the matrix or inclusion. The first part of the analysis was to find expressions for the stress functions from which the stresses and displacements within the matrix and inclusion were obtained. It is this aspect of the previous work that will be reconsidered in this paper.

There are three basic conditions that must be satisfied by the elastic fields of matrix and inclusion in this problem. They are

(i) there is no discontinuity in the stress or displacements across the boundary of the inclusion,

(ii) the solution has an edge dislocation type singularity in either the matrix or inclusion,

(iii) the stresses should vanish at infinity.

The stress functions were derived by assuming their probable structure and adjusting the relative magnitudes of the separate terms to satisfy the three conditions above.

A mathematically similar inclusion problem has been considered by the same method in (1–3) and (6). In these papers the single edge dislocation is replaced by a concentrated point force. In (1), elastic fields are given for matrix and inclusion when the matrix contains a concentrated radial force while the case of a concentrated tangential force is considered in (2). These forces act in the inclusion in (3). In (6), the case of a concentrated point force in the matrix is solved and used to solve another inclusion problem.

This method of solution is rather awkward because of the need to guess beforehand the structure of the stress functions and the large number of arbitrary constants to be determined.

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In this paper, it will be shown that the complex variable technique of Muskhelishvili (7), offers an analytic method of solution, which eliminates the problem of guessing beforehand the probable form of the solution and gives the previous published results as particular cases.

2. Consider a circular inclusion of radius a, embedded in an infinite matrix. It will be assumed that the matrix and inclusion are perfectly welded along their common boundary L, i.e. there will be no discontinuities in stress or displacement fields across L, and initially unstressed.

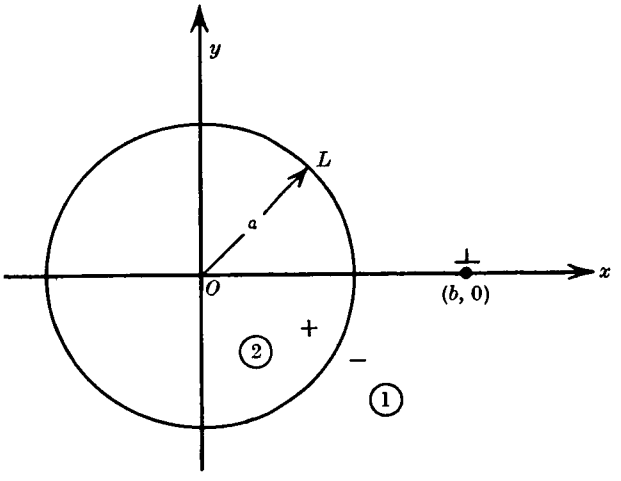


Fig. 1

As L is a circle, it is most convenient to introduce polar coordinates r, θ by

$$z = x + iy = re^{i\theta}$$

with pole at the centre of the inclusion. As shown by Muskhelishvili in (7), (most of the material in this section can be found in this reference) the stress components $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$ and displacement components v_r, v_θ can be expressed in terms of complex functions ϕ, ψ, Φ, Ψ, (the notation of (7) is used throughout this paper), which are related by $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$, as follows:

$$\widehat{rr} + \widehat{\theta\theta} = 2\{\Phi(z) + \overline{\Phi(z)}\}, \tag{2·1}$$

$$\widehat{\theta\theta} - \widehat{rr} + 2i\widehat{r\theta} = 2(z/\bar{z})\{\bar{z}\Phi'(z) + \Psi(z)\},$$

$$2\mu e^{i\theta}(v_r + iv_\theta) = \kappa\phi(z) - z\overline{\Phi(z)} - \overline{\Psi(z)} + \text{constant}. \tag{2·2}$$

On L, as the matrix and inclusion are perfectly welded,

$$\widehat{rr}^+ + i\widehat{r\theta}^+ = \widehat{rr}^- + i\widehat{r\theta}^-, \tag{2·3}$$

$$v_r^+ + iv_\theta^+ = v_r^- + iv_\theta^-, \tag{2·4}$$

where the + and − superscripts indicate the limiting values from the inclusion and matrix respectively.

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If an edge dislocation or concentrated force is located at some arbitrary point $z = z_0$, of a region S, the functions Φ and Ψ have the form

$$\left.\begin{aligned}\Phi(z) &= -\frac{C}{z-z_0}+\Phi^*(z),\\ \Psi(z) &= \frac{D}{z-z_0}-\frac{\bar{z}_0 C}{(z-z_0)^2}+\Psi^*(z),\end{aligned}\right\} \tag{2·5}$$

where Φ^* and Ψ^* are holomorphic near $z = z_0$. In particular, for an edge dislocation with Burger's vector

$$\text{(i)}\quad (b_x, 0, 0)\quad C = D = \frac{\mu b_x i}{\pi(1+\kappa)}, \tag{2·6}$$

$$\text{(ii)}\quad (0, b_y, 0)\quad C = D = \frac{\mu b_y}{\pi(1+\kappa)}, \tag{2·7}$$

while for a concentrated force with components (X, Y)

$$C = \frac{X+iY}{2\pi(1+\kappa)},\quad D = \frac{\kappa(X-iY)}{2\pi(1+\kappa)}. \tag{2·8}$$

Subtracting equations (2·1) and taking conjugates yields

$$\widehat{rr}+i\widehat{r\theta} = \Phi(z)+\overline{\Phi(z)}-\bar{z}\overline{\Phi'(z)}-(\bar{z}/z)\,\overline{\Psi(z)}. \tag{2·9}$$

In view of the expressions for the singularities (2·5) and equation (2·9), it is more convenient to replace (2·4) by

$$\frac{\partial}{\partial\theta}(v_r^+ + iv_\theta^+) = \frac{\partial}{\partial\theta}(v_r^- + iv_\theta^-) \tag{2·10}$$

which is equivalent provided the displacements are single-valued. Differentiating (2·2) with respect to θ gives

$$2\mu\frac{\partial}{\partial\theta}\{e^{i\theta}(v_r+iv_\theta)\} = iz\{\kappa\Phi(z)-\overline{\Phi(z)}+\bar{z}\overline{\Phi'(z)}+(\bar{z}/\bar{z})\,\overline{\Psi(z)}\}. \tag{2·11}$$

As $\bar{z} = a^2/z$ on L, the expressions (2·9) and (2·11) suggest writing

$$\Omega\left(\frac{a^2}{\bar{z}}\right) = \overline{\Phi(z)}-\bar{z}\overline{\Phi'(z)}-\frac{\bar{z}^2}{a^2}\overline{\Psi(z)},$$

i.e.
$$\Omega(z) = \bar{\Phi}\left(\frac{a^2}{z}\right)-\frac{a^2}{z}\bar{\Phi}'\left(\frac{a^2}{z}\right)-\frac{a^2}{z^2}\bar{\Psi}\left(\frac{a^2}{z}\right). \tag{2·12}$$

Substituting in (2·9) and (2·11)

$$\widehat{rr}+i\widehat{r\theta} = \Phi(z)+\Omega\left(\frac{a^2}{\bar{z}}\right)+\bar{z}\left(\frac{\bar{z}}{a^2}-\frac{1}{z}\right)\overline{\Psi(z)}, \tag{2·13}$$

$$2\mu\frac{\partial}{\partial\theta}\{e^{i\theta}(v_r+iv_\theta)\} = iz\left\{\kappa\Phi(z)-\Omega\left(\frac{a^2}{\bar{z}}\right)-\bar{z}\left(\frac{\bar{z}}{a^2}-\frac{1}{z}\right)\overline{\Psi(z)}\right\}. \tag{2·14}$$

As
$$\lim_{r\to a}\left\{\frac{\bar{z}}{a^2}-\frac{1}{z}\right\}\overline{\Psi(z)} = 0,$$

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the boundary condition (2·3) now becomes

$$\{\Phi(t)-\Omega(t)\}^{+}-\{\Phi(t)-\Omega(t)\}^{-}=0, \tag{2·15}$$

where $t=a\,e^{i\theta}$, and (2·10) transforms into

$$\frac{1}{\mu_2}\{\kappa_2\Phi^{+}(t)-\Omega^{-}(t)\}=\frac{1}{\mu_1}\{\kappa_1\Phi^{-}(t)-\Omega^{+}(t)\} \tag{2·16}$$

indicating the different elastic constants of matrix and inclusion by subscripts 1 and 2 respectively.

Thus the problem is reduced to solving (2·15) and (2·16) and hence obtaining expressions for Φ_1, Ψ_1 and Φ_2, Ψ_2 where the subscripts indicate the respective connections with the matrix and inclusion.

3. The solution of equation (2·15) is dependent on the nature of the singularities of $\Phi(z)-\Omega(z)$. If there are singularities of order not greater than $m_1, m_2, \ldots, m_k, m$ at the points $z_1, z_2, \ldots, z_k, \infty$ the solution of (2·15) is

$$\Phi(z)-\Omega(z)=R(z),$$

where
$$R(z)=\sum_{j=1}^{k}\left\{\frac{C_{j1}}{z-z_j}+\frac{C_{j2}}{(z-z_j)^2}+\ldots+\frac{C_{jm_j}}{(z-z_j)^{m_j}}\right\}+C_0+C_1z+\ldots+C_mz^m.$$

Consider first the case of an edge dislocation or point force located in the matrix at $z_0=b(b>a)$. As the inclusion is a circle there is no loss of generality in assuming b real. The singularities of $\Phi(z)$ are known from (2·5) while those of $\Omega(z)$ can be determined as follows. Substituting (2·5) into (2·12) gives

$$\Omega(z)=\frac{\bar{C}}{b}-\frac{\bar{D}}{z}+\frac{\bar{D}}{z-\dfrac{a^2}{b}}+\bar{C}\,\frac{a^2\left(\dfrac{b^2-a^2}{b^3}\right)}{\left(z-\dfrac{a^2}{b}\right)^2}+\Omega^{*}(z), \tag{3·1}$$

where Ω^{*} is analytic for $|z|<a$, indicating singularities at $z=0$ and $z=a^2/b$.

As Φ and Ψ are holomorphic in the region $z=0$ they have expansions of the form

$$\left.\begin{aligned}\Phi(z)&=A_0+A_1z+A_2z^2+\ldots\ |z|<a,\\ \Psi(z)&=B_0+B_1z+B_2z^2+\ldots\ |z|<a.\end{aligned}\right\} \tag{3·2}$$

Substitution in (2·12) again, shows that $\Omega(z)$ is analytic at infinity and $\Omega(\infty)=\bar{A}_0$.

Thus, the solution of (2·15) can now be written in the form

$$\Phi(z)-\Omega(z)=-\frac{C}{z-b}-\left\{-\frac{\bar{D}}{z}+\frac{\bar{D}}{z-\dfrac{a^2}{b}}+\frac{\bar{C}\,\dfrac{a^2(b^2-a^2)}{b^3}}{\left(z-\dfrac{a^2}{b}\right)^2}+\bar{A}_0\right\}=g(z). \tag{3·3}$$

This solution can now be used to simplify (2·16). Eliminating the function Ω between (3·3) and (2·16) gives

$$\Phi^{+}(t)-\frac{\mu_2\kappa_1+\mu_1}{\mu_1\kappa_2+\mu_2}\Phi^{-}(t)=\frac{\mu_2-\mu_1}{\mu_1\kappa_2+\mu_2}\,g(t). \tag{3·4}$$

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Defining a new function Φ_0 by

$$\Phi_0(z) = \begin{cases} \Phi_2(z) & (|z| < a), \\ \dfrac{\mu_2\kappa_1+\mu_1}{\mu_1\kappa_2+\mu_2}\Phi_1(z) & (|z| > a) \end{cases} \tag{3·5}$$

enables (3·4) to be written as

$$\Phi_0^+(t) - \Phi_0^-(t) = \frac{\mu_2-\mu_1}{\mu_1\kappa_2+\mu_2} g(t),$$

an equation similar to (2·15). As the only singularity of $\Phi_0(z)$ occurs at $z = b$, the solution of this equation is

$$\Phi_0(z) = \frac{1}{2\pi i}\frac{\mu_2-\mu_1}{\mu_1\kappa_2+\mu_2}\int_L \frac{g(t)\,dt}{t-z} - \frac{\mu_2\kappa_1+\mu_1}{\mu_1\kappa_2+\mu_2}\frac{C}{z-b}. \tag{3·6}$$

After substituting for $g(t)$ from (3·3), evaluating the Cauchy integrals, and using the definition of Φ_0 in (3·5), the following expressions for Φ_1 and Φ_2 are obtained.

$$\Phi_1(z) = \frac{\mu_1-\mu_2}{\mu_2\kappa_1+\mu_1}\left\{\frac{\bar{D}}{z} - \frac{\bar{D}}{z-\dfrac{a^2}{b}} - \frac{\bar{C}\dfrac{a^2(b^2-a^2)}{b^3}}{\left(z-\dfrac{a^2}{b}\right)^2}\right\} - \frac{C}{z-b}, \tag{3·7}$$

$$\Phi_2(z) = -\frac{\mu_2(1+\kappa_1)}{\mu_1\kappa_2+\mu_2}\frac{C}{z-b} + \frac{\mu_1-\mu_2}{\mu_1\kappa_2+\mu_2}\bar{A}_0. \tag{3·8}$$

An expression for $\bar{A}_0$ can be readily obtained. From (3·2) and (3·8)

$$\bar{A}_0 = \frac{\mu_2(1+\kappa_1)}{\mu_1\kappa_2+\mu_2}\frac{\bar{C}}{b} + \frac{\mu_1-\mu_2}{\mu_1\kappa_2+\mu_2}A_0. \tag{3·9}$$

Rearranging (2·12), on L

$$\Psi(t) = \frac{a^2}{t^2}\Phi(t) - \frac{a^2}{t}\Phi'(t) - \frac{a^2}{t^2}\bar{\Omega}(\bar{t}). \tag{3·10}$$

Multiplying through by $1/(t-z)$ and integrating around L, for z in the matrix gives

$$-\Psi_1(z) + \frac{D}{z-b} - \frac{bC}{(z-b)^2} = \frac{1}{2\pi i}\int_L \frac{(a^2/t^2)\,\Phi_1(t)\,dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{(a^2/t)\,\Phi_1'(t)\,dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{(a^2/t^2)\,\bar{\Omega}_2(a^2/t)\,dt}{t-z}. \tag{3·11}$$

Using the expression for $\Phi_1(z)$ from (3·7) and

$$\Omega_2(z) = \Phi_2(z) - g(z)$$

from (3·3), gives, after evaluation of the Cauchy integrals,

$$\begin{aligned}\Psi_1(z) = {} & \frac{D}{z-b} - \frac{bC}{(z-b)^2} + \frac{B\bar{C}}{z-(a^2/b)} - \frac{A}{b}\{(\bar{D}-\bar{C})b^2 + a^2\bar{C}\}\Big/\left(z-\frac{a^2}{b}\right)^2 \\ & - 2A\bar{C}\frac{a^2(b^2-a^2)}{b^2}\Big/\left(z-\frac{a^2}{b}\right)^3 - \frac{B\bar{C}}{z}, \\ & - \frac{1}{z^2}\left[\frac{A}{b}\{(\bar{C}-\bar{D})b^2 - a^2\bar{C}\} + \frac{\mu_1(1+\kappa_2)}{\mu_1\kappa_2+\mu_2}a^2A_0 - \frac{Ca^2}{b}\right] + \frac{2Aa^2\bar{D}}{z^3},\end{aligned} \tag{3·12}$$

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where
$$A = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2 \kappa_1} \quad \text{and} \quad B = \frac{\mu_1 \kappa_2 - \mu_2 \kappa_1}{\mu_2 + \mu_1 \kappa_2}.$$

Repeating this procedure with z in the inclusion gives

$$\Psi_2(z) = \frac{(1-A)D}{z-b} + \frac{C}{(z-b)^2}\left\{(B-A) - (1-A)\frac{b^2}{a^2}\right\}\frac{a^2}{b}. \tag{3·13}$$

4. When the edge dislocation or point force is located within the inclusion ($b < a$), it is necessary to reconsider the singularities of Ω. The expression (3·1) is still correct, except that now $|z| > a$, showing that Ω has a singularity at $z = a^2/b$ within the matrix.

For large $|z|$, Φ and Ψ are holomorphic and can be expanded in the form

$$\left.\begin{aligned} \Phi(z) &= \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots \quad (|z| > a), \\ \Psi(z) &= \frac{B_1}{z} + \frac{B_2}{z^2} + \ldots \quad (|z| > a). \end{aligned}\right\} \tag{4·1}$$

Substitution of (4·1) in (2·12), the definition of Ω, shows that Ω has a pole at $z = 0$ with principal part $-\bar{B}_1/z$.

As before, Ω is analytic at infinity with $\Omega(\infty) = \bar{\Phi}_2(0) = k$ (say) and for large $|z|$, no term in $1/z$. Thus $g(z)$ is replaced by

$$g^*(z) = -\frac{C}{z-b} - \left[-\frac{\bar{B}_1}{z} + \frac{\bar{D}}{z-\frac{a^2}{b}} + \frac{\bar{C}\frac{a^2(b^2-a^2)}{b^3}}{\left(z-\frac{a^2}{b}\right)^2} + k\right]. \tag{4·2}$$

Due to the change in the location of the singularities, the expression (3·6) for Φ_0 is replaced by

$$\Phi_0(z) = \frac{1}{2\pi i}\frac{\mu_2 - \mu_1}{\mu_1 \kappa_2 + \mu_2}\int_L \frac{g^*(t)\,dt}{t-z} - \frac{C}{z-b}$$

giving
$$\Phi_1(z) = -\frac{\mu_1(1+\kappa_2)}{\mu_2 \kappa_1 + \mu_1}\frac{C}{z-b} + \frac{\mu_1 - \mu_2}{\mu_2 \kappa_1 + \mu_1}\frac{\bar{B}_1}{z}, \tag{4·3}$$

$$\Phi_2(z) = \frac{\mu_1 - \mu_2}{\mu_1 \kappa_2 + \mu_2}\left\{\frac{\bar{D}}{z-\frac{a^2}{b}} + \frac{\bar{C}\frac{a^2(b^2-a^2)}{b^3}}{\left(z-\frac{a^2}{b}\right)^2} + k\right\} - \frac{C}{z-b}. \tag{4·4}$$

k can be determined from (4·4) as $\Phi_2(0) = \bar{k}$. Thus

$$\Phi_2(0) = F\frac{b}{a^2}(\bar{D} - \bar{C}) + \frac{1}{b}\{C + F\bar{C}\} - Fk = \bar{k}, \tag{4·5}$$

where
$$F = \frac{\mu_2 - \mu_1}{\mu_1 \kappa_2 + \mu_2}.$$

As Ω_1 has no term in $1/z$ for large $|z|$, an expression for B_1 can be obtained using $\Omega_1(z) = \Phi_1(z) - g^*(z)$. Substituting from (4·2) and (4·3) gives

$$\bar{B}_1 \frac{\mu_2(\kappa_1 + 1)}{\mu_2 \kappa_1 + \mu_1} = EC + \bar{D}, \tag{4·6}$$

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where
$$E = \frac{\mu_2\kappa_1 - \mu_1\kappa_2}{\mu_1 + \mu_2\kappa_1}.$$

Determining Ψ as before gives

$$\Psi_1(z) = \frac{(1-F)D}{z-b} - \left\{(1-F)b + \frac{a^2}{b}(F-E)\right\}\frac{C}{(z-b)^2} + \{(F-1)D + B_1\}\frac{1}{z} + \left\{(F-1)\bar{k}a^2 + (1-E)\frac{a^2C}{b}\right\}\frac{1}{z^2} + \frac{\mu_1-\mu_2}{\mu_2\kappa_1+\mu_1}\frac{2a^2\bar{B}_1}{z^3}, \quad (4{\cdot}7)$$

$$\Psi_2(z) = \frac{D}{z-b} - \frac{bC}{(z-b)^2} + \frac{E\bar{C}}{z-\frac{a^2}{b}} + F\left\{(\bar{C}-\bar{D})b - \bar{C}\frac{a^2}{b}\right\}\frac{1}{\left(z-\frac{a^2}{b}\right)^2} - 2F\bar{C}\frac{(b^2-a^2)a^2}{b^2}\frac{1}{\left(z-\frac{a^2}{b}\right)^3}. \quad (4{\cdot}8)$$

5. The general formulae obtained in sections 3 and 4 do simplify in special cases. As an example, consider an edge dislocation with Burger's vector $(b_x, 0, 0)$, located in the matrix at $z = b$ (b real), outside a circular inclusion of radius a, i.e. $b > a$.

From (2·6) we have
$$C = D = -\bar{C} = -\bar{D} = \frac{\mu_1 b_x i}{\pi(1+\kappa_1)}. \quad (5{\cdot}1)$$

For the matrix we have from (3·7), after substituting from (5·1), that

$$\Phi_1(z) = \frac{\mu_1 b_x i}{\pi(1+\kappa_1)}\left\{-\frac{A}{z} + \frac{A}{z-\frac{a^2}{b}} + \frac{\frac{a^2(b^3-a^2)A}{b^3}}{\left(z-\frac{a^2}{b}\right)^2} - \frac{1}{z-b}\right\}, \quad (5{\cdot}2)$$

where
$$A = \frac{\mu_1-\mu_2}{\mu_1+\mu_2\kappa_1}.$$

The expression for $\Psi_1(z)$ given in (3·12), simplifies as follows. From (3·9) we have

$$A_0 = \frac{\mu_2(1+\kappa_1)}{\mu_1(1+\kappa_2)}\frac{C}{b} \quad (5{\cdot}3)$$

and hence the coefficient of $-1/z^2$ reduces to $(a^2/b)\,C(A-B)$ where

$$B = \frac{\mu_1\kappa_2 - \mu_2\kappa_1}{\mu_2+\mu_1\kappa_2}.$$

Thus
$$\Psi_1(z) = \frac{\mu_1 b_x i}{\pi(1+\kappa_1)}\left\{\frac{1}{z-b} - \frac{b}{(z-b)^2} - \frac{B}{z-\frac{a^2}{b}} + \frac{\frac{Aa^2}{b}}{\left(z-\frac{a^2}{b}\right)^2} + \frac{2A\frac{a^2(b^2-a^2)}{b^2}}{\left(z-\frac{a^2}{b}\right)^3} + \frac{B}{z} - \frac{\frac{a^2(A-B)}{b}}{z^2} - \frac{2Aa^2}{z^3}\right\}. \quad (5{\cdot}4)$$

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For the inclusion, from (3·8) and using (5·3)

$$\Phi_2(z) = \frac{\mu_1 b_x i}{\pi(1+\kappa_1)}\left\{\frac{B-1}{z-b} + \frac{\mu_1-\mu_2}{\mu_2+\mu_1\kappa_2}\frac{\mu_2(1+\kappa_1)}{\mu_1(1+\kappa_2)}\frac{1}{b}\right\}. \tag{5·5}$$

Finally, from (3·13), again using (5·1),

$$\Psi_2(z) = \frac{\mu_1 b_x i}{\pi(1+\kappa_1)}\left[\frac{1-A}{z-b} + \frac{1}{(z-b)^2}\left\{(B-A)-(1-A)\frac{b^2}{a^2}\right\}\frac{a^2}{b}\right]. \tag{5·6}$$

The Airy stress functions U_1 and U_2 for the matrix and inclusion respectively, can now be obtained by substituting into

$$U = Rl\{\bar{z}\phi(z) + \chi(z)\}$$

where $\phi'(z) = \Phi(z)$ and $\chi''(z) = \Psi(z)$. If the further substitutions $z = re^{i\theta}$, $z-b = r_1 e^{i\theta_1}$, $z-(a^2/b) = re^{i\theta_2}$, $\beta = b/a$, $b = c$ and $\mu = G$ are made, the expression for the Airy stress functions as given in (4) are obtained.

However, it is easier to obtain the stresses and displacements by using the standard results from (7)

$$X_x + Y_y = 2[\Phi(z) + \overline{\Phi(z)}],$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)],$$

$$2\mu(u+iv) = \kappa\phi(z) - z\overline{\Phi(z)} - \overline{\psi(z)},$$

where

$$\psi'(z) = \Psi(z).$$

These expressions for the stresses and displacements are obviously rather lengthy and as they do not appear to lead to any useful discussion, will not be recorded here.

The references to this paper contain a number of these expressions, written out in full, and attention is drawn to the fact that some curves have been produced showing the variation of the stresses in various regions. For example, in (1), some special cases of the point force with components $(X, 0)$ are given.

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