BREAKING OSCILLATIONS IN SERVO SYSTEMS.*

By THEODORE VOGEL, D.Sc.,

Centre Nationale de la Recherche Scientifique, Marseilles.

THE present paper is concerned with some aspects of stability and of steady periodical changes in dynamical systems. After a brief survey of the classical theory due to Poincaré, I shall sum up some of my recent work relating to a certain class of systems not considered in that theory. It is, of course, assumed that this work is relevant to the main subject of this meeting; in fact, in the course of writing down these notes I was fortunate enough to be able to look (although far too cursorily) through Dr. Ashby's book, *Design for a Brain*, and have been struck by what appeared to me to be a remarkable parallelism between our lines of thought. The arrangement of the subject-matter has been avowedly influenced by this assumption of relevancy; however, I shall keep to my legitimate grounds as a physicist and shall leave it to you to fill in whatever "psychological" conclusions you may think justified.

The theory of dynamical systems is mathematical in its essence, and abstract reasoning cannot be dispensed with; what I can, however, and shall, avoid are the technicalities of analysis: those interested might refer to the papers listed below for a more mathematical treatment. First let me define the special words that will be used.

CONCEPTS AND DEFINITIONS.

A "dynamical system with n degrees of freedom" is anything which can be wholly described by a set of $n+1$ variables, one of which is the time, the others having any physical meaning. (If a living organism is to be regarded in this way we must include all those variables in both organism and environment that are necessary to make the behaviour determinate. In this paper I shall, of course, be considering systems of much less complexity.) The n variables can be made to correspond, in the usual way for graphs, to an n-dimensional space in which the set of values of the n variables corresponds to one, and only one, of the points of the space. As the variables change in value so will the point change in position. Thus the state of the system at a given time t is represented by a point in the space, the point at the extremity of an n-vector $\mathbf{x}$. The locus of this point as the time progresses is a "characteristic" of the system. (The "line of behaviour" of Dr. Ashby's terminology is almost identical, though somewhat more general; Poincaré's word is, however, more suitable when discussing systems of the type studied by him.) Such a representation of behaviour by a line is well known in everyday life—many a map of the Coronation, for instance, showed by a winding line how the "chain

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of events" of the day would occur. A "characteristic" represents the events in a dynamic system similarly.

(It will be convenient from now on to use the language of plane geometry, which, of course, applies strictly only to a system with two degrees of freedom; but any proposition stated in that simple language can be easily extended to the general case.)

Any characteristic represents one solution of a certain set of first-order differential equations, corresponding to given values of the arbitrary integration constants. All the solutions of the set of equations yield a family of characteristics, and it will be considered that any curve of that family represents the behaviour of the same system, when starting from different "initial conditions" as enforced, at some given time, but not subsequently, by an external agency. If, at any stage of its free evolution, the state of a system is disturbed by an external agency enforcing for a moment new conditions and then leaving the system alone again, the representative point will be thrown off its original path to some new position, and will then go on describing the particular characteristic that passes through that position. These events correspond to what happens in a physiological or psychological experiment when a transient "stimulus" is applied. They would also correspond more generally to the experience of the psychiatric patient who has undergone some traumatic, but not persisting, disturbance.

We shall be particularly concerned with a system's equilibrium and steady (periodical) motion. (It will be readily seen that a steady (periodical) motion is akin to an equilibrium, for the system will encounter the same state, if not at all times, at least an infinite number of times at fixed intervals.)

A system is in "equilibrium" when its characteristic reduces to a point, that is, when the vector $\mathbf{x}$ does not change with time. When the characteristic is a closed curve, the changes of the system are, necessarily, periodic with respect to time.

Another important property of a system is its "stability." Suppose that a system is disturbed from its rest (or closed path): it will start on a neighbouring characteristic of the same family, as explained above. If that characteristic tends to merge into the original one in course of time, then the original state (or closed path) is "stable"; if, on the contrary, it drifts farther and farther away, the original state (or closed path) is unstable.

It is hardly necessary to point out that these two questions are of primary importance in the study of servo systems, whether mechanical, or maintaining vegetative homeostasis, or concerned in the many regulations implied by well co-ordinated and self-preserving "higher" activities; for a system's stability is a major factor in determining how it will react to a transient signal or disturbance. There are, of course, other points of view equally important for the discussion of servo systems, for instance that of selectivity; but to go into them would require a whole course of lectures, not a brief paper; so it will perhaps be better not to touch on them at all.

For systems governed by equations of the type

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

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where the f's are sufficiently well-behaved functions (*v. infra*), there is a very satisfactory theory of characteristics due to Poincaré and his school. Its main results are as follows :

Properties of the Poincaré System.

(For precision it is best referred to by this name ; for convenience it may be thought of as being approximately a continuous system.) Its most important properties are, in general, those of its " singular " points ; these occur wherever the f's are null, i.e., become zero. They all correspond to well-known and commonly occurring features of behaviour. They can be classified, completely, as follows :

Focus { stable. / unstable.
Node { stable. / unstable.
Saddle point.
Vortex.

In a Poincaré system, every singular point belongs to one, and only one, of these six classes.

A graphical illustration of their main individual properties is given in Fig. 1. The behaviour of the system in the vicinity of any of these points can be easily understood if we regard the characteristics in Fig. 1 as showing the contours of a hilly country. Thus a node is a peak or a hollow, and a ball released in its

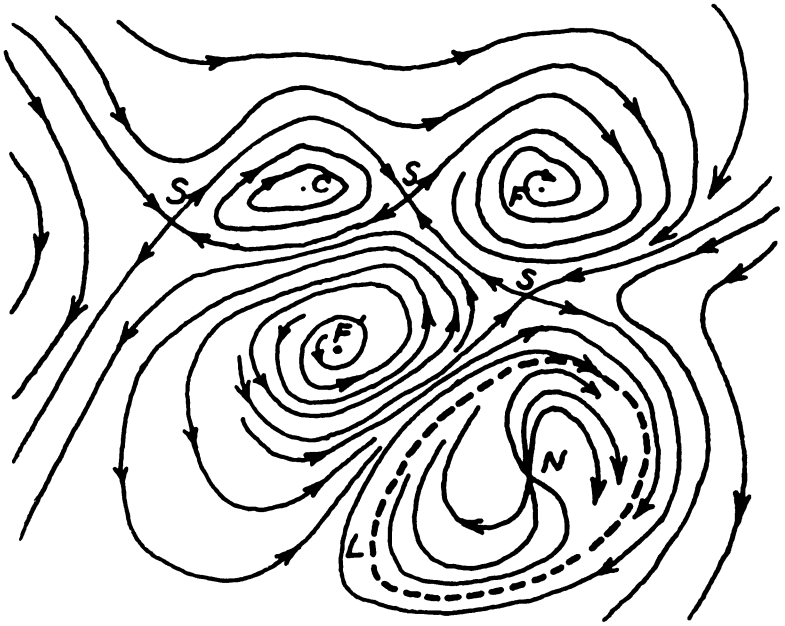


Fig. 1.—Example of a family of characteristics showing various singularities : C, vortex ; F, focus (stable) ; F', focus (unstable) ; L, limit cycle (stable) ; N, node (unstable) ; S, saddle point.

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vicinity will roll away from it if it is a peak (corresponding to instability) (*N* of Fig. 1), or towards it if it is a hollow (corresponding to stability). Foci are also peaks or hollows, but the surface is nearly or quite level around them ; so at a focus, the ball will follow a winding path, away from a peak (*F'*) or towards a hollow (*F*), each spiral closely approximating to a contour. Around a vortex (*C* in Fig. 1) it will follow a contour indefinitely (if there is no friction). Thus, foci are reached after an infinite time (if stable), or were left at time $-\infty$ (if unstable). Vortices, on the other hand, are related to a state of rest, and are surrounded by a continuum of closed paths, each path representing a periodic motion. According to our previous definition, the motions around a vortex are neither stable nor unstable, since the system, when displaced to a neighbouring path, neither drifts away from the original one nor tends to go back to it. All energy-conservative systems come within this case. However, in practice, when the system is subjected to repeated disturbances or " noise," this " neutral " stability looks more like instability ; for after each disturbance a new path is begun, and the observer sees only an aimless drifting.

The motion may also proceed from an unstable node or focus, or unwind itself from an unstable limit cycle. A limit cycle is always surrounded by an annular region where all paths tend towards the cycle or unwind themselves from it. In particular, if the cycle is stable (*L* in Fig. 1), the motion of the system tends to become ultimately periodic for all initial conditions interior to the annulus, and this limit is independent of the initial conditions. These limit cycles are *the only stable periodic motions* possible to a Poincaré system. In such a system, a limit cycle is always approached gradually, the characteristic never reaching it, strictly, in any finite time. This asymptotic approach contrasts with what I shall show later, in which an abrupt arrival is possible.

Lastly, a saddle-point (*S* in Fig. 1) is the equivalent of a pass in the hills, and is the only kind of singular point which can be crossed by the representative point.

It can be proved that a closed characteristic is bound to contain in its interior at least one singular point.

As time goes to $+\infty$, every characteristic must lead to one, and only one, of the following conditions :

(i) The motion leads to a stable node and stops there :

(ii) it leads towards a stable focus, reached after an infinite time has elapsed ;

(iii) it leads towards a closed curve, the limit cycle (e.g., *L* of Fig. 1), merging into it after an infinite time ;

(iv) it goes to infinity.

In the absence of any external agency exerting more than temporary action, a Poincaré system can therefore either proceed towards a state of rest (reached after a finite or infinite time), or proceed towards a steady periodic motion, reached after an infinite time, or, lastly, have a characteristic stretching to infinity in the phase space. Now, whatever the physical meaning of the variables **x**, it can be safely assumed that they cannot exceed certain values without a destructive change happening in the system (at best, the system will, outside a certain region of the phase space, be governed by other laws ; but

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this means that it will be another system, and the original one may be deemed destroyed). The last possibility is therefore that the system proceeds to its destruction. These remarks conclude my survey of the main properties of the Poincaré system, but the last possibility—that it may proceed to its destruction —leads into the next development.

THE SYSTEM WITH "MEMORY."

In the systems considered above, the law of change depends only on the state of the system at the time considered (on the vector **x**). This class, akin to Dr. Ashby's "absolute" system though not identical with it, is called *non-hereditary*. A system is *hereditary* (Volterra) if its behaviour at a given point of the phase space also depends on all the points previously occupied. It may also be considered as a system that has "memory," in the sense that some of its physical components have properties that depend on its past history. In the systems considered by Volterra, this "accumulated experience" of the system is expressed mathematically by a line integral taken along the path, and the governing equation is integro-differential, instead of being simply differential.

There are few physical systems where a close scrutiny does not reveal some degree of hereditary action; to take a simple example, the stiffness of a spring depends on the succession of temperatures to which it has been subjected, and even on the duration of the stress previously borne, whether the elastic limit has been reached or not.

The discussion of hereditary systems is usually a difficult problem, even in the simple case where the hereditary action is restricted to one single variable, and when it is assumed that the influence of previous states grows negligible when these recede in the distant past (Volterra's assumption of the "dissipation of hereditary action"); in more general cases, our present mathematical tools are quite inadequate. But the class of hereditary systems is of such importance that it is well worth while discussing even a particular case, for every new solution adds to our knowledge of the general behaviour of these systems. They are of particular importance in the living world, where most systems are of the type that cannot be considered adequately except in reference to their past history.

It has been pointed out that the classical theory outlined above is restricted to non-hereditary systems. More than that: of this class, it applies only to those systems that are governed by equations involving "well-behaved" functions. This vague term is used intentionally to stand for a set of precise technical conditions into which we need not go here, and whose general effect is to ensure a certain degree of smoothness in the representative curves or manifolds. The question now clearly arises, what happens when the functions are *not* well-behaved? The question is of great importance, for the common functions in the physical world are those of this second type, not those of "well-behaved" type; Nature, as Sir Isaac Newton pointed out, does not heed mathematical difficulties. Now given a discontinuous function, it is a well-known theorem that a continuous function can always be found which

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will fit the exact values of the given one, in a stated interval, with an error as small as desired. Results of experimental measurements can therefore always be expressed by a well-behaved law in the interval under study with an excellent accuracy, while the actual law may be of a different nature, and lead to unpleasant difficulties outside the interval in question. Our general theorems are valid only for the "idealized" case, and do not enable us to say that two characteristics, now infinitely near to each other, will remain so when the one is governed by a well-behaved, the other by a fractious, law.

Properties of the "Breaking" System.

It would be too ambitious to attempt a general theory valid for all fractious functions; but some particular classes can be considered. The one which will be discussed now is at the same time hereditary and discontinuous, and may thus suggest some ideas as to the behaviour of two different, and little known, categories of systems. We shall assume from now on that the hereditary action is concentrated on a curve or manifold which will be called the "boundary", and that its effect is to add a certain term, with alternate signs, whenever the representative point of the system touches or crosses the boundary. The system is thus alternately subjected to one of two well-behaved laws, corresponding to two different families of characteristics, say (R) and (S) (Fig. 2). The actual path of the representative point is made up of an arc R, followed by an arc S, then another arc R, and so on, all the changes occurring on the boundary.

In Dr. Ashby's language, the law of motion is governed by a step-function, the steps occurring on the boundary; his Fig. 7/8/2 is analogous.

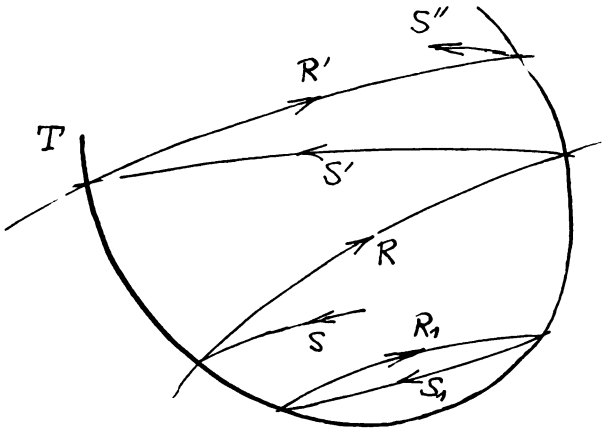


Fig. 2.—Phase plane containing paths of a breaking system.

Such a path can be closed; for instance the first arc S may cut the boundary at the same point as the first arc R, and the motion will then be $R_1S_1R_1S_1$ . . . as shown in the figure. Obviously, the period is here made up of two phases (R_1 and S_1) which may differ greatly in character; thus, in one case of great physical importance, the breaking of waves on a shore, one of the

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phases is a "jump." It is with this instance in view that I called the class of oscillations under discussion "*oscillations de déferlement*," the English for which I believe to be "breaking oscillations."

To grasp the properties of this new type of system another method of representation is advantageous. You may find it unusual, but a little practice soon makes it familiar. Let the characteristics of each of the two families be numbered, so that each point of the phase space is determined by two numbers: that of the R-characteristic which passes through it, and that of the S-characteristic which also passes through it. Take these numbers as new rectangular co-ordinates η and ξ respectively. In the η, ξ-plane, the characteristics R and S will become vertical and horizontal lines, the boundary T will be transformed into a certain curve τ, and the path of the system will be made up of vertical and horizontal segments, intersecting on τ. The closed path of Fig. 2 (R_1S_1) will obviously correspond to a *double point* of τ, where two branches of that curve intersect; one of these branches will be repulsive ($\oplus$) with respect to a given family R (or S), and the other will be attractive $\odot$. Fig. 3

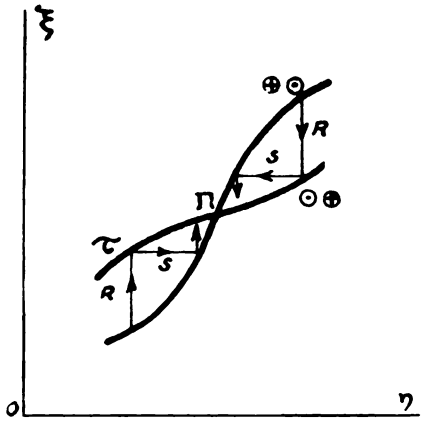


FIG. 3.—Transform of the phase plane of Fig. 2 to the plane of η, ξ. Π is the double point corresponding to a stable closed path, e.g., to R_1S_1 in Fig. 2.

shows that the necessary and sufficient condition for the periodic solution to be stable is that the repulsive branch should have the greater slope with respect to the family axis.

The oscillations represented by a closed breaking characteristic such as this one (R_1S_1) may be called "of the first kind." A "second kind" will be that where the cycle is made up of four arcs, two of each family, and so on. These higher kinds cannot occur if no two characteristics of one family intersect within the boundary, i.e., if there are no interior singular points. Similarly, if there is at most one singular interior point of each family, the periodic solutions may be either of the first or of the second kind, and so on. Fig. 4, though somewhat complex in the diagram, represents a simple system with pentagonal boundary. It is purely theoretical, but has been constructed to show some of the elementary properties with special clarity. One family of characteristics is made up of straight vertical lines (considered as degenerate

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parabolae with no interior singular point), the other of hyperbolae (having an internal saddle point at O). The behaviour of the system can be found simply by starting at some initial state and following the appropriate characteristic, changing to the other family every time the boundary is reached. Thus possible behaviours are: (**A**) If started at A (in the lower quadrant), it goes through B to C; here it falls just below the critical line, so it turns down to D (instead of going eventually up to B'); after going to D, it converges to the cycle Π_1', Π_2', Π_1', . . . (**A'**) If, however, it is started at A', it converges to the cycle Π_1, Π_2, Π_3, Π_4, Π_1, . . .

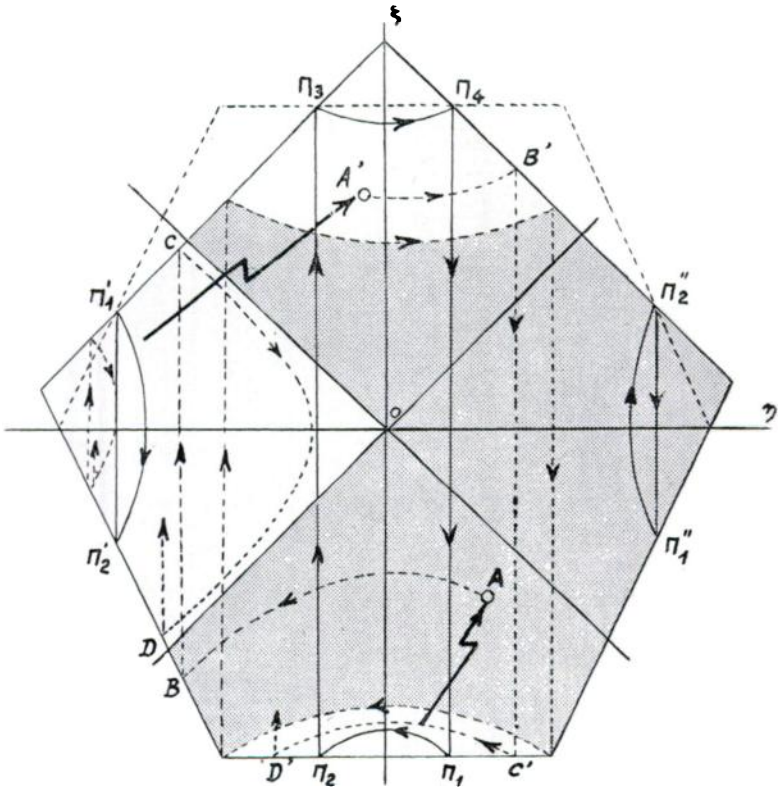


FIG. 4.—Phase plane of the breaking system that is realised in Fig. 6 (*c*). It shows closed paths of the first kind: $\Pi_1'\Pi_2'$ (stable) and $\Pi''_1\Pi''_2$ (unstable) and of the second kind: $\Pi_1\Pi_2\Pi_3\Pi_4$ (stable).

There are two steady motions of the first kind (Π_1', Π_2' and Π_1'', Π_2'') and one of the second (Π_1, Π_2, Π_3, Π_4). The rules for stability are the same as in the simple regular case, applying to each set of boundary points concerned. So much for the general description. I shall now point out very briefly some of the peculiar properties of the systems we have been considering.

PECULIARITIES OF THE "BREAKING" SYSTEM.

The occurrence of interior foci or nodes obviously excludes the possibility of any periodic motion, since they cannot be crossed an infinite number of

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times, but there may well be a node on the boundary, and in that case a periodic motion of the first kind will exist, stable if the node is attractive, as shown in Fig. 5. It is readily seen that the steady motion is reached after *one single discontinuity*, whatever the initial conditions. A breaking system with a node on the boundary reaches its steady motion almost immediately, not asymptotically as was the rule in the classical case.

Now this is rather remarkable, and worth while checking experimentally. In a previous paper I have shown that the multivibrator circuit, well known in electronics, may be considered in a large number of cases (under not very restrictive conditions) as a breaking system with both families having a node on the boundary (a complication which makes no significant difference). And in fact, oscillograms (Fig. 6(*a*)) of the building-up of the steady periodic motion

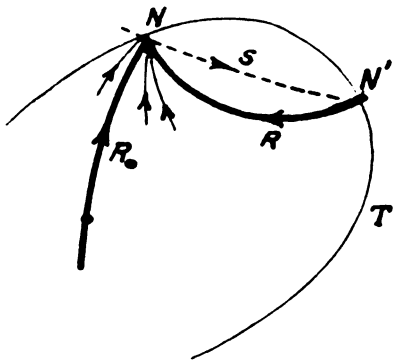


FIG. 5.—Phase plane of a breaking system that possesses a node on the boundary.

in such circuits show that the amplitude and period are reached completely at the second "flip," just as predicted by the theory.

Another noteworthy property relates to systems with saddle-points: such systems may display a number of steady states of the first and the second kinds, not separated by any unstable regions; on the occurrence of a disturbance of sufficient force, the system will jump suddenly from one way of behaving (one stable path) to another, the second way of behaving being perhaps quite different from the first. If disturbances are recurrent and invisible, the system will give the impression of behaving at random when it is really stable and quite able to continue steadily in one path once the disturbances cease. The possibility is predicted in Fig. 4. If it was moving steadily and cyclically between Π_1' and Π_2' (on the left) and a disturbance (shown by a solid zigzag line) threw the system suddenly to state A' (in the upper quadrant), then it would go to the cycle Π_1, Π_2, Π_3, Π_4, Π_1 . . . around which it would circulate permanently.

The same phenomenon can be demonstrated in an actual model. Its details need not be referred to here as we are interested only in its behaviour. Its phase space is similar to that shown in Fig. 4. Fig. 6(*b*) shows two of its cycles (photographed successively) as they appeared on the recording oscillograph. A disturbance, applied between the two exposures, has thrown it

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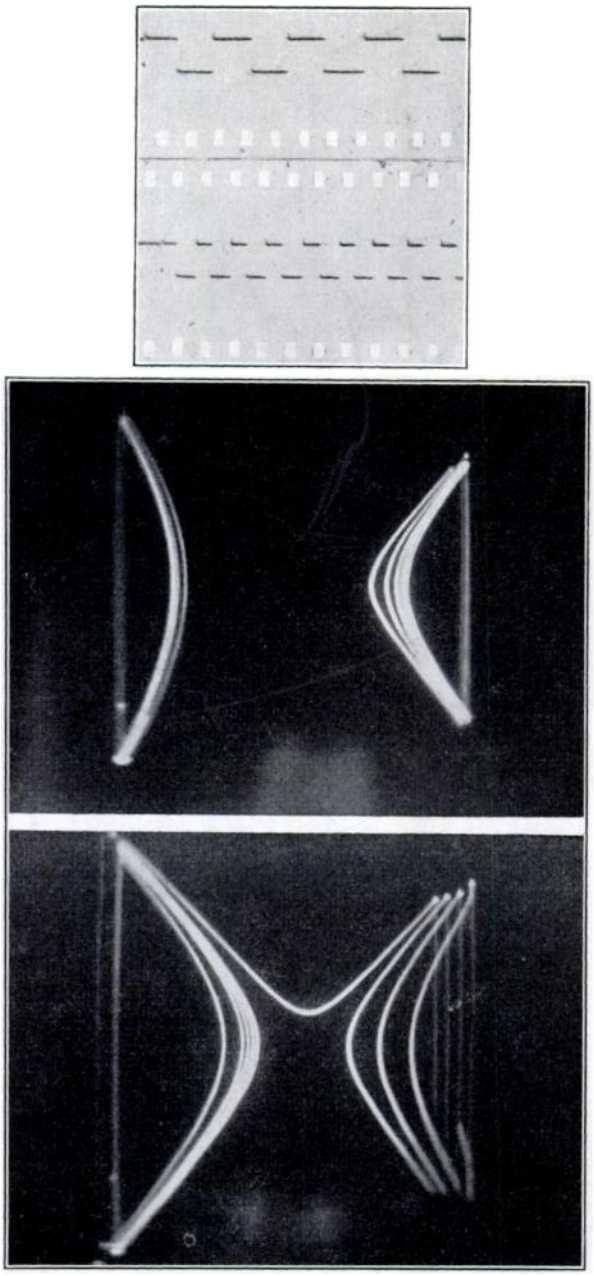


FIG. 6.—(*a*) Two oscillograms of a multivibrator as it starts. (Time goes from left to right.) Notice that the steady motion is reached at the second oscillation, there being no gradual build-up, in contrast to *L* of Fig. 1.

(*b*) Oscillogram of the phase plane for a system similar to that of Fig. 4, but with different slopes at the boundary. It shows two stable oscillations of the first kind.

(*c*) (As *b*, but the boundary is now as shown in Fig. 4). It shows two oscillations of the first kind. The system was started in that on the right side ; but the oscillation was unstable, so the point drifted away, then jumped to the other cycle, and finally settled there.

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from one cycle to the other. (The mechanism was modified for 6(*b*) to make both cycles stable.)

These "breaking" systems are much richer in properties than the basic Poincaré (continuous) systems. Fig. 6(*c*), for instance, shows a property that is impossible in the Poincaré system yet is probably common in systems with discontinuities. It is obtained from the system shown in Fig. 4. (In contrast to 6(*b*) the right-hand cycle is now unstable.) If such a system is started in the unstable cycle on the right it will for a time seem to be following a regular cycle (Fig. 6(*c*), on the right); then, *without any disturbance from the outside*, it will suddenly jump to another way of behaving, and will settle down to the steady cycle on the left. The possibility can thus both be predicted, in the theory of Fig. 4, and demonstrated, in the model whose behaviour is recorded in Fig. 6(*c*).

I do not feel that any elaborate conclusions would be in order, especially as I am a physicist encroaching on the realm of neurophysiology. The general impression I have tried to convey is that complicated systems can do quite unusual things while still being orderly and determinate. If, as often happens, a system can undergo *dis*continuous changes, then the behaviour of the system can be surprisingly "original", making the behaviour look disorderly while it is actually stable and normal. When faced with a behaviour which strongly differs from what one considers to be normal, one generally looks for an external agency; but it might be wise, sometimes, to consider whether the internal properties of the system are not really more complex than was assumed at first.

Summary.

Our understanding of the dynamics of the brain must be built up stage by stage, the elementary and simple being clearly understood before the complex is attempted.

A description is given of the main properties of the system whose variables are continuous; and a further study is given of the properties of the system that contains a few discontinuities ("breaks" or step-functions).

Systems in which "breaks" can occur show some interesting properties, three of which are described. Particularly noteworthy is the fact that such a system, even if strictly determinate and within constant external conditions, can change suddenly from one way of behaving to another, apparently spontaneously.

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