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Rigidity times for a weakly mixing dynamical system which are not rigidity times for any irrational rotation

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Abstract. We construct an increasing sequence of natural numbers $(m_n)_{n=1}^{+\infty}$ with the property that $(m_n\theta[1])_{n\geqslant 1}$ is dense in $\mathbb{T}$ for any $\theta \in \mathbb{R}\setminus\mathbb{Q}$, and a continuous measure on the circle μ such that $\lim_{n\to+\infty}\int_{\mathbb{T}} \|m_n\theta\|\, d\mu(\theta) = 0$. Moreover, for every fixed $k \in \mathbb{N}$, the set $\{n \in \mathbb{N} : k \nmid m_n\}$ is infinite. This is a sufficient condition for the existence of a rigid, weakly mixing dynamical system whose rigidity time is not a rigidity time for any system with a discrete part in its spectrum.

1. *Introduction*

Let $\mathbb{T}$ denote the circle group with addition mod 1. For $\eta \in \mathbb{R}$ we denote by $\eta[1]$ the fractional part of η and $\|\eta\|$ its distance to integers. It follows that $\|\eta\| = \min(\eta[1], (1-\eta)[1])$. Therefore, for any $\eta \in \mathbb{R}$, $\|\eta\| \leqslant \frac{1}{2}$.

In this note, we prove the following two results.

THEOREM 1. *Fix rationally independent numbers*† $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$. *There exists an increasing sequence* $(m_n)_{n=1}^{+\infty}$ *such that* $(m_n\theta[1])_{n\geqslant 1}$ *is dense in* $\mathbb{T}$ *for every irrational* θ, *and for every* $\epsilon > 0$ *and* $k \in \mathbb{N}$ *there exists* $N_0 \in \mathbb{N}$ *such that for every* $n \geqslant N_0$ *we have* $\|m_n\alpha_i\| < \epsilon$ *for at least* $k-1$ *choices of* $i \in \{1, \ldots, k\}$. *Moreover, for every* $k \in \mathbb{N}$ *the set* $\{n \in \mathbb{N} : k \nmid m_n\}$ *is infinite.*

† By this we mean that every finite collection is rationally independent.

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THEOREM 2. *Fix rationally independent numbers* $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$ *and let* $(m_n)_{n\geqslant 1}$ *be the corresponding sequence from Theorem 1. There exists a continuous probability measure* μ *on* $\mathbb{T}$ *such that*

$$\lim_{n\to+\infty} \int_{\mathbb{T}} \|m_n\theta\| \, d\mu(\theta) = 0.$$

Theorem 1 gives us an increasing sequence of natural numbers $(m_n)_{n=1}^{+\infty}$ which is not a rigidity time for any system with a discrete part in its spectrum. Indeed, if the system has an irrational eigenvalue, then it has the irrational rotation as a factor. If it has a rational eigenvalue, then it has a shift on a finite group as a factor. But a rigidity time for a dynamical system is also a rigidity time for its factors, and a sequence as in Theorem 1 cannot be a rigidity sequence for any rational or irrational rotation.

From Theorem 2, by the Gaussian measure space construction (see [**3**]), we deduce that there exists a weakly mixing dynamical system whose rigidity times contain the constructed sequence $(m_n)_{n=1}^{+\infty}$. This gives a full answer to the question stated in [**2**] of whether a rigidity times sequence of a system with discrete spectrum is a rigidity time for some weakly mixing and conversely whether a rigidity times sequence of a system with continuous spectrum is a rigidity times sequence for some discrete spectrum system. The first direction was established in [**1**] and later in [**4**], namely, any rigidity time of a system with discrete spectrum is also a rigidity time for some weakly mixing dynamical system.

Our approach is inspired by the completely spectral approach adopted in [**4**]. First we prove the existence of a sequence m_n which is not a rigidity time for any circle rotation, but still satisfies that $\|m_n\alpha_i\|$ is small for most of the indices i of a family of rationally independent numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$ (see precise statement in Theorem 1).

This allows us to construct a continuous probability measure on $\mathbb{T}$, that is a weak limit of discrete measures each supported on some finite set connected with the numbers $\alpha_1, \alpha_2, \ldots$, with a Fourier transform converging to one along this sequence.

2. *Proof of Theorem 1*

Let there be given a family of rationally independent numbers $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$. We will first state a lemma, which is a generalization of Lemma 1 in [**4**].

Definition 3. [**4**] For an interval $I \subset \mathbb{T}$ and fixed $\epsilon > 0$ one says that $\theta \in \mathcal{A}(N_1, N_2, \epsilon, I, k)$ if for every $m \in [N_1, N_2]$ such that $\|m\alpha_i\| < \epsilon$ for $i = 1, \ldots, k$, we have $m\theta[1] \notin I$.

LEMMA 4. *For every* $l \geqslant 2$ *there exists* $L(l) \in \mathbb{N}$ *such that for every* $0 < \epsilon < 1/2l^2$*, for every* $v > 0$*, for every* k*, there exist* $K(\epsilon) \in \mathbb{N}$ *and* $N' = N'(l, \epsilon, v, N, k) \in \mathbb{N}$ *such that* $\theta \in \mathcal{A}(N_1, N', I, \epsilon, k)$ *for some interval* I *of size* $1/l$*, implies that* $\|\sum_{i=1}^{k} r_i\alpha_i - l'\theta\| < v$ *for some* $|r_1|, \ldots, |r_k| < K(k, \epsilon)$ *and some* $|l'| < L(l)$.

The proof is a repetition of the proof of Lemma 1 of [**4**]. Instead of considering $\phi_\epsilon : \mathbb{T} \to \mathbb{R}$ one needs to consider $\phi_\epsilon^k : \mathbb{T}^k \to \mathbb{R}$. It follows by the proof that the number $L(l, \{\alpha_i\}_{i=1}^k)$ does not depend on the numbers $\{\alpha_i\}_{i=1}^k$ and that is why we just had $L(l)$ in the statement. Indeed, similarly to Lemma 1 in [**4**], one considers a polynomial $\varphi_l : \mathbb{T} \to \mathbb{R}$, $\varphi_l(y) := \sum_{0<|k|<L(l)} \hat{\varphi}_k e^{i2\pi ky}$, where $L(l)$ is such that:

- $\varphi_l(y) > 1$ for every $y \notin [0, 1/l]$;

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- $|\varphi_l(y)| < l^2$ for every $y \in \mathbb{T}$;

and $\phi_\epsilon(x) := \sum_{0<|r|<K} \hat{\phi}_k e^{i2\pi rx}$, where $K \in \mathbb{N}$ is such that:

- $\phi_\epsilon(x) > 1$ for every $x \in \mathbb{T}^k$ which satisfies $x_i \in [0, \epsilon] \cup [1-\epsilon, 1]$, $i = 1, \ldots, k$;
- $|\phi_\epsilon(x)| > \epsilon^3$ for every $x \in \mathbb{T}^k$.

Fix $I = [y_0, y_0 + 1/l]$. Then define $\psi_{\epsilon,l}(x, y) = \phi_\epsilon(x)\varphi_l(y - y_0)$. For $\Psi : \mathbb{T}^k \times \mathbb{T} \to \mathbb{R}$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $\theta \in \mathbb{R}$ and $m \in \mathbb{N}$ define

$$S_m^{\bar{\alpha},\theta}\psi(x, y) := \sum_{0<r<m} \Psi(x + r\bar{\alpha}, y + i\theta).$$

To conclude, note that if $N' \in \mathbb{N}$ is sufficiently big, we get by ergodicity that $S_{N'}^{\bar{\alpha},\theta}\psi_{\epsilon,l}(0, 0) > \frac{1}{2}\epsilon^2 N'$. If we assume that for any $|r_1|, \ldots, |r_k| < K(k, \epsilon)$ and $|l'| < L(l)$ we have $\| \sum_{i=1}^k r_i\alpha_i - l'\theta\| \geqslant v$, then $S_{N'}^{\bar{\alpha},\theta}\psi_{\epsilon,l}(0, 0)$ is bounded independently of N'.

Remark 5. Consider an ergodic rotation $T : T^j \to T^j$, $T(x_1, \ldots, x_j) = (x_1 + \gamma_1, \ldots, x_j + \gamma_j)$, for $\gamma_1, \ldots, \gamma_j \in \mathbb{T}$. It follows that for every $k \in \mathbb{N}$ and every $\epsilon > 0$, there exist (infinitely many) $m \in \mathbb{N}$ such that $\|m\gamma_i\| < \epsilon$ for $i = 1, \ldots, j$ and $k \nmid m$. Indeed, for every fixed $k \in \mathbb{N}$ there exist a sequence $(r_n)_{n\geqslant 1}$ such that $T^{r_n}(0) \to (1/k)T(0)$.

PROPOSITION 6. *Fix rationally independent numbers* $\{\alpha_i\}_{i\in\mathbb{N}} \in \mathbb{T}$. *There exists a sequence* (s_n) *such that* $\lim_{n\to+\infty} \|s_n\alpha_i\| = 0$ *for* $i = 1, \ldots$ *and* $(s_n\theta[1])_{n\geqslant 1}$ *is dense in* $\mathbb{T}$ *if and only if*† $\theta \notin \mathbb{Q} + \mathbb{Q}\alpha_1 + \cdots$.

Proof. We will use Lemma 4 for $k = 1, 2, \ldots$. Define for $n \geqslant 1$ the sequence $l_n = n + 1$. Let $\epsilon_n = 1/2(n+1)^2$ and $K_n := K(n, \epsilon_n)$. Define $v_n = (1/n) \inf_{0\leqslant|k_1|,\ldots,|k_{n+1}|\leqslant K_{n+1}} \|\sum_{i=1}^{n+1} k_i\alpha_i\|$. Take $N_0 = 0$ and apply Lemma 4 with $k = 1$, $l = l_1$, $\epsilon = \epsilon_1$, $N = N_0$, $v = v_1$. Denote $N_1 = N'(l_1, \epsilon_1, v_1, N_0, 1)$. We apply Lemma 4 inductively for $k = n$, $l = l_n$, $\epsilon = \epsilon_n$, $N = N_n$, $v = v_n$ and choose $N_{n+1} > N'(l_n, \epsilon_n, v_n, N_n, n)$ sufficiently large. Then we define an increasing sequence $(s_n)_{n=1}^{+\infty}$ by taking, for every $i \in \mathbb{N}$, all integers $s \in [N_i, N_{i+1}]$ such that $\|s\alpha_t\| < \epsilon_i$ for every $t = 1, \ldots, i$ (we can choose N_{i+1} so that such $s \in [N_i, N_{i+1}]$ exists). Moreover, by Remark 5 we can choose N_{i+1} so that for every $r = 1, \ldots, i$ there exists $s_r \in [N_i, N_{i+1}]$ with $r \nmid s_r$.

Note first that for every $r \in \mathbb{N}$, $\lim_{n\to+\infty} \|s_n\alpha_r\| = 0$. Indeed, for every $j > r$ and every $t \in \mathbb{N}$ such that $s_t \in [N_j, N_{j+1}]$ we have $\|s_t\alpha_r\| < \epsilon_j$.

Now, let $\theta \in \mathbb{T}$ be such that $s_n\theta[1]$ is not dense in $\mathbb{T}$. Then there exists $I \subset \mathbb{T}$, $|I| = 1/l_s$ for some s such that $s_n\theta[1] \notin I$. By the definition of the sequence $(s_n)_{n=1}^{+\infty}$ it follows that there exists n_0 such that $\theta \in \mathcal{A}(N_n, N_{n+1}, I, \epsilon, n)$ for every $n \geqslant n_0$. Therefore, by Lemma 4 it follows that there are integers $k_1^n, \ldots, k_n^n$ with $|k_i^n| < K_n$ for every $i = 1, \ldots, n$ such that $\|\sum_{i=1}^n k_i^n\alpha_i - l'\theta\| < v_n$ for some $|l'| < L = L(l_s)$. Therefore, $\|\sum_{i=1}^n h_i^n\alpha_i - L!\theta\| < L!v_n$, for some numbers $h_1^n, \ldots, h_n^n \in \mathbb{N}$ with $|h_i^n| < L!K_n$. It follows by triangle inequality that $\|\sum_{i=1}^{n+1} h_i^{n+1}\alpha_i - \sum_{i=1}^n h_i^n\alpha_i\| < L!v_n + L!v_{n+1} < 2L!v_n$. By the definition of v_n, we get that there exists $n_1 \in \mathbb{N}$ such that for $n \geqslant n_1$, these two combinations are equal. Therefore, $\|\sum_{i=1}^{n_1} h_i^{n_1}\alpha_i - L!\theta\| < L!v_n$ for every $n \geqslant n_1$. But $v_n \leqslant 1/n \to 0$ and consequently $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_{n_1}$.

On the other hand, it follows by construction of $(s_n)_{n\geqslant 1}$ that $(s_n\theta[1])_{n\geqslant 1}$ is not dense in $\mathbb{T}$ if $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + \cdots$. □

† By this we mean that there does not exist n_0 such that $\theta \in \mathbb{Q} + \mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_{n_0}$.

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Proof of Theorem 1. For every $i \in \mathbb{N}$, let $\{s_n^{(i)}\}_{n\in\mathbb{N}}$ be a sequence as in Proposition 6 applied to the family of rationally independent numbers $\{\alpha_j\}_{j\in\mathbb{N}, j\neq i} \in \mathbb{T}$. Let $(N_s(i))_{s\geqslant 1}$ be the corresponding sequence of natural numbers given in the proof of Proposition 6, that is $\|s_t^{(i)}\alpha_r\| < 1/2(j+1)^2$ for every $t \geqslant N_j(i)$ (this implies that $s_t > N_j(i)$) and every $r < j$. Then define the sequence $\tilde{s}_n^{(i)} := s_{n+N_i(i)}^{(i)}$.

Then define m_n to be the sequence $\tilde{s}_1^{(1)}, \tilde{s}_2^{(1)}, \tilde{s}_1^{(2)}, \tilde{s}_3^{(1)}, \tilde{s}_2^{(2)}, \tilde{s}_1^{(3)}, \tilde{s}_4^{(1)}, \tilde{s}_3^{(2)}, \tilde{s}_2^{(3)}, \tilde{s}_1^{(4)}, \ldots$. The sequence m_n satisfies the conditions of Theorem 1. Indeed, first note that for any irrational θ there exists i such that $\theta \notin \bigcup_{i=1}^{+\infty}(\mathbb{Q} + \cdots + \mathbb{Q}\alpha_{i-1} + \mathbb{Q}\alpha_{i+1} + \cdots)$, hence $(m_n\theta[1])_{n\geqslant 1}$ is dense by just considering the subsequence $\tilde{s}_l^{(i)}$. Second, fix $\epsilon > 0$ and $k \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $1/2(r+1)^2 < \epsilon$. Define $N_0 := (\max\{N_r(1), \ldots, N_r(r)\})^2$. Then, by definition of the sequence $(m_n)_{n\geqslant 1}$, $\|m_n\alpha_i\| < 1/2(r+1)^2 < \epsilon$, for every $n > N_0$ and every $i \in \{1, \ldots, k\}$ except for at most one i that satisfies $m_n = \tilde{s}_{l_n}^{(i)}$. □

Remark 7. It follows that for every $\epsilon > 0$, $i \in \mathbb{N}$ there exist $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $\sum_{s=1}^{i} \|m_n\alpha_s\| < \frac{1}{2} + \epsilon$.

3. *Proof of Theorem 2*

Fix rationally independent numbers $(\alpha_i)_{i\geqslant 1} \in \mathbb{T}$ and let $(m_n)_{n=1}^{+\infty}$ be the corresponding sequence given by Theorem 1.

For the construction of the measure μ we will proceed similarly to [**4**] (and we borrow notation from there). For a probability measure ν on $\mathbb{T}$ we denote by $\nu^n = |\int_{\mathbb{T}} \|m_n\theta\| \, d\mu(\theta)|$. We will define inductively a sequence $(k_n)_{n\geqslant 1}$ so that the measure μ will be a weak limit of discrete measures $\mu_p := (1/2^p)\sum_{i=1}^{2^p} \delta_{k_i\alpha_i}$ for some numbers $k_i \in \mathbb{N}$ such that there exists a sequence $(N_p)_{p=1}^{+\infty}$ for which:

(i) for every $p \geqslant 1$, for every $j \in [1, p-1]$, for every $n \in [N_j, N_{j+1}]$, $\mu_p^n < 1/2^{j-1}$ (for $j = 0$ $\mu_p^n < 1$);

(ii) for $p_0 \in \mathbb{N}$ denote by $\eta_{p_0} = \frac{1}{4}\inf_{1\leqslant i<j\leqslant 2^{p_0}} \|k_i\alpha_i - k_j\alpha_j\|$, then for every $l \in \mathbb{N}$ and every $r \in [1, 2^{p_0}]$, $\|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_r\alpha_r\| < \eta_{p_0}$.

In fact, similarly to [**4**], we get that any weak limit μ of a sequence μ_p as above, satisfies the conclusion of Theorem 2. Indeed, by (i) $\mu^n \to 0$. By (ii) it follows that, for each p_0, the intervals $I_r = [-\eta_{p_0} + k_r\alpha_r, \eta_{p_0} + k_r\alpha_r]$, $r = 1, \ldots, 2^{p_0}$ are disjoint and $\mu_p(I_r) = 1/2^{p_0}$ for every $p \geqslant p_0$ and therefore the limit measure μ is continuous.

Therefore, we just have to construct the measures μ_p as in (i) and (ii). We will carry out an inductive construction, in which we will additionally require that, for every p,

$$\mu_p^n < \frac{1}{2^{p+1}} + \frac{1}{2^{p+3}} \quad \text{for } n \geqslant N_p. \tag{1}$$

For $p = 0$ let $k_1 = 1$, then μ is the Dirac measure at α_1. Let $N_0 = 0$. For $p = 1$, $k_2 = 1$, then μ_1 is the average of Dirac measures at α_1 and α_2. We choose $N_1 = 1$. This satisfies (i) and (1) for $p = 1$.

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Assume that we have constructed k_i for $i = 1, \ldots, 2^p$, N_l for $1 \leqslant l \leqslant 2^p$ such that (i) and (1) is satisfied up to p and (ii) is satisfied for every $p_0 \leqslant p$ and $0 \leqslant l \leqslant 2^{p-p_0} - 1$. We now choose k_{2^p+1} so that $k_{2^p+1}\alpha_{2^p+1}$ is sufficiently close to $k_1\alpha_1$ so that

$$\nu_{p,1} = \frac{1}{2^p}\sum_{i=1}^{2^p} \delta_{k_i\alpha_i} + \frac{1}{2^{p+1}}(\delta_{k_{2^p+1}\alpha_{2^p+1}} - \delta_{k_1\alpha_1})$$

satisfies $\nu_{p,1}^n < 1/2^{j-1}$ for $n \in [N_j, N_{j+1}]$ and $j \in [0, p-1]$ ($\nu_{p,1}^n = \mu_p^n + (1/2^{p+1})$ ($\|m_n k_{2^p+1}\alpha_{2^p+1} - m_n k_1\alpha_1\|$)). Moreover, it follows that for $n \geqslant N_p$ we have $\nu_{p,1}^n < \mu_p^n + 1/2^{p+1} < 1/2^{p+1} + 1/2^{p+3} + 1/2^{p+1} < 1/2^{p-1}$. Let $N_{p,1} > N_p$ be sufficiently large so that $\nu_{p,1}^n < 1/2^p$ for $n \geqslant N_{p,1}$ ($\nu_{p,1}^n < \mu_p^n + 1/2^{p+1}$ and μ_p^n can be arbitrary close to $1/2^{p+1}$ by Remark 7).

Now construct inductively for $s = 1, \ldots, 2^p$ the numbers k_{2^p+s}, $N_{p,s} \in \mathbb{N}$ for the measures $\nu_{p,s}$ given by $\nu_{p,s} = \mu_p + (1/2^{p+1})(\sum_{i=1}^{s}(\delta_{k_{2^p+i}\alpha_{2^p+i}} - \delta_{k_i\alpha_i}))$. It follows that by choosing k_{2^p+s} so that $k_{2^p+s}\alpha_{2^p+s}$ is sufficiently close to $k_s\alpha_s$ and $N_{p,s}$ large enough, we can insure that:

A. $\nu_{p,s}^n < 1/2^{j-1}$ for every $n \in [N_j, N_{j+1}]$, and $j \leqslant p-1$;
B. $\nu_{p,s}^n < 1/2^{p-1}$ for $n \geqslant N_p$;
C. $\nu_{p,s}^n < 1/2^p$ for $n \geqslant N_{p,s}$.

Indeed, for $s = 1$ the above conditions are satisfied, assume that for some $s \geqslant 1$, they hold. We will prove that they hold for $s+1$. First note that $\nu_{p,s} - \nu_{p,s-1} = (1/2^{p+1})(\delta_{k_{2^p+s}\alpha_{2^p+s}} - \delta_{k_s\alpha_s})$. Therefore by choosing k_{2^p+s} so that $k_{2^p+s}\alpha_{2^p+s}$ is sufficiently close to $k_s\alpha_s$ and by the induction hypothesis, we get that $\nu_{p,s}^n < 1/2^{j-1}$ for every $n \in [N_j, N_{j+1}]$ with $j \leqslant p-1$. The same arguments give us $\nu_{p,s}^n < 1/2^{p-1}$ for $N_{p,s-1} \geqslant n \geqslant N_p$. For $n > N_{p,s-1}$ we use the fact that $\nu_{p,s-1}^n < 1/2^p$ to get $\nu_{p,s}^n < \nu_{p,s-1}^n + 1/2^{p+1} < 1/2^p + 1/2^p = 1/2^{p-1}$. For the third point we use the fact that for n sufficiently large, $\|m_n\alpha_i\|$ is arbitrary small for all but one $i \in \{1, \ldots, 2^p + s\}$ (compare with Remark 7), to get that for $N_{p,s}$ large enough, $\nu_{p,s}^n < 1/2^{p+1} + 1/2^{p+2} + 1/2^{p+2} = 1/2^p$, for $n \geqslant N_{p,s}$.

Finally we define $\mu_{p+1} = \nu_{p,2^p}$ and observe that μ_{p+1} satisfies (i). Moreover, by definition $\mu_{p+1} = (1/2^{p+1})\sum_{i=1}^{2^{p+1}} \delta_{k_i\alpha_i}$ and using the properties of the sequence $(m_n)_{n\geqslant 1}$ ($\|m_n\alpha_i\|$ is arbitrary small for all but one $i = 1, \ldots, 2^{p+1}$, see also Remark 7) we get that if N_{p+1} is sufficiently large, then (1) is satisfied for μ_{p+1}.

Moreover, for $l = 2^{p-p_0} + l' - 1$ we have $\|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_r\alpha_r\| \leqslant \|k_{l2^{p_0}+r}\alpha_{l2^{p_0}+r} - k_{l'2^{p_0}+r}\alpha_{l'2^{p_0}+r}\| + \|k_{l'2^{p_0}+r}\alpha_{l'2^{p_0}+r} - k_r\alpha_r\| < \eta_{p_0}$ by the induction hypothesis and the choice of $k_{l2^{p_0}+r}$. Therefore, (ii) is satisfied for $p+1$ and every $l \leqslant 2^{p+1}$. This finishes the proof. □

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