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NEAR SEASONAL INTEGRATION

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This paper presents asymptotic results for the seasonal unit root test proposed by Hylleberg, Engle, Granger and Yoo (1990, *Journal of Econometrics* 44, 215–238) in a near integration context. The findings are important in that they provide the asymptotic power functions of the Hylleberg et al. statistics when the characteristic roots of a seasonal process are local to unity. These conclusions extend the available asymptotic results for this test and serve as a framework for the potential development of more powerful test procedures.

1. INTRODUCTION

The interest in unit root processes has recently been extended to cover a class of processes that specifically deal with the presence of roots close to but not equal to one. As indicated by Perron (1989), this local to unity theory has been useful in different contexts. Among others, it has been used by Evans and Savin (1981, 1984), Phillips and Perron (1988), and Nabeya and Tanaka (1990) in the analysis of the power of unit root tests under local alternatives; Stock (1991, 1994) in the derivation of confidence intervals for the largest root of an autoregressive (AR) process; and Phillips (1987) for the calculation of the power of unit root tests with a continuum of observations.

However, in contrast with the results available for conventional AR processes, greater attention has yet to be directed to the analysis of seasonal near integration. There are a large number of seasonal unit root test procedures in the literature (to name a few, see, e.g., Breitung and Franses, 1998; Dickey, Hasza, and Fuller, 1984; Hylleberg, Engle, Granger, and Yoo, 1990; Osborn, Chui, Smith, and Birchenhall, 1988; Taylor, 1999), but very few results exist for these procedures in a near nonstationary framework.

Chan (1988, 1989), Chan and Wei (1987), Perron (1992), and Tanaka (1996, pp. 253–264) offer results for a first-order near seasonally integrated autoregressive (NSIAR) process. Essentially, Chan (1988) and Chan and Wei (1987) provide the limiting distribution for a t-ratio obtained from a NSIAR model; Chan (1989) derives the asymptotic distributions for sample moments of a NSIAR

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model; Perron (1992) presents a closed-form expression for the limiting joint moment generating function of the least squares estimator of the autoregressive parameter from a NSIAR model; and Tanaka (1996) using the stochastic process and Fredholme approach generalizes Chan's and Perron's results by considering NSIAR models with deterministic components.

The objective of this paper is to extend the literature on near seasonally integrated processes by analyzing the approach of Hylleberg et al. (1990) in this context. This procedure is possibly one of the most frequently applied tests in empirical work; hence, the understanding of its behavior in a local to unity framework is of fundamental importance.

This paper is organized as follows. Section 2 analyzes the asymptotic properties of seasonally near integrated processes. Section 3 derives the asymptotic power functions for the Hylleberg et al. (1990) test statistics, and Section 4 concludes the paper. The Appendix presents the proofs of the limiting results put forward throughout the paper.

2. NEAR SEASONALLY INTEGRATED AR PROCESSES

To extend the results available for conventional near integrated AR(1) processes to the seasonal context, consider the seasonal stochastic process

$$y_t = \gamma y_{t-\mathtt{s}} + \varepsilon_t, \qquad t = 1, \ldots, T, \tag{2.1}$$

where $\mathtt{s}$ is a positive integer, $T = \mathtt{s}N$, N is the total number of complete years of observations, $y_{-\mathtt{s}+1} = \cdots = y_0 = 0$, and $\varepsilon_t \sim$ independent and identically distributed (i.i.d.) $(0,\sigma^2)$. The notion of near nonstationarity can be introduced into (2.1) by considering $\gamma = e^{(c/N)} \simeq 1 + (c/N) = 1 + (\mathtt{s}c/T)$, where c is a fixed constant (clearly, when $c = 0$ then $\gamma = 1$ and (2.1) is a seasonal random walk).

Given the assumption that the disturbances in (2.1) are independent, this data generating process (DGP) can be seen as the result of circular merging $\mathtt{s}$ independent near integrated processes (for details on the seasonal integrated case, see Kunst, 1997; Osborn and Rodrigues, 1998). Therefore, the near seasonally integrated process in (2.1) will correspond to $\mathtt{s}$ near integrated processes, each associated with one season of the year. Hence, the relevant partial sum process for observation t is that of season k of year n and is given by

$$S_{k,n} = \sum_{i=0}^{[(t-1)/\mathtt{s}]} e^{([t-1/\mathtt{s}]-i)c/N} \varepsilon_{\mathtt{s}i+k} = \sum_{j=1}^{n} e^{(n-j)c/N} \varepsilon_{k,j}, \tag{2.2}$$

where $k = 1 + (t-1) \bmod \mathtt{s}$, $n = [(t-1)/\mathtt{s}] + 1$, $[\cdot]$ indicates the greatest integer less or equal to $(t-1)/\mathtt{s}$, and $\varepsilon_{k,j}$ is the disturbance occurring in season k of year j.

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From the continuous mapping theorem, as $N \to \infty$ the normalized partial sum processes, $S_{k,n}$, converge to independent Ornstein–Uhlenbeck processes (see Phillips, 1987). That is,

$$N^{-1/2}S_{kn} \Rightarrow \sigma J_{kc}(r), \tag{2.3}$$

where $J_{kc}(r)$, $k = 1,\ldots,\mathsf{s}$ are independent standard Ornstein–Uhlenbeck processes and $\Rightarrow$ represents weak convergence. Note that $J_{kc}(r) = W_k(r) + c\int_0^r e^{(r-s)c}W_k(s)ds$ and $W_k(r)$, $k = 1,\ldots,\mathsf{s}$ are independent standard Brownian motions. Thus the following lemma, which presents the asymptotic behavior of several sample moments obtained from a process such as (2.1) with $\gamma = e^{(c/N)} \simeq 1 + (c/N)$, can be advanced.

LEMMA 1. *Assuming that the* DGP *is the near seasonally integrated process in* (2.1), *then from the continuous mapping theorem as* $T \to \infty$ *it can be established that*

(a) $T^{-1/2}y_{t-i} \Rightarrow (\sigma/\sqrt{\mathsf{s}})J_{kc}(r), \qquad k = 1,\ldots,\mathsf{s},$
(b) $T^{-3/2}\sum_{t=1}^{T} y_{t-i} \Rightarrow (\sigma/\mathsf{s}^{3/2})\sum_{k=1}^{\mathsf{s}}\int_0^1 J_{kc}(r)dr,$
(c) $T^{-2}\sum_{t=1}^{T} y_{t-i}^2 \Rightarrow (\sigma^2/\mathsf{s}^2)\sum_{k=1}^{\mathsf{s}}\int_0^1[J_{kc}(r)]^2\,dr,$
(d) $T^{-1}\sum_{t=1}^{T} y_{t-i}\varepsilon_t \Rightarrow (\sigma^2/\mathsf{s})\sum_{k=1}^{\mathsf{s}}\int_0^1(L^jJ_{kc}(r))dW_k(r),$

where $J_{kc}(r)$ *and* $W_k(r)$, $k = 1,\ldots,\mathsf{s}$ *are independent standard Ornstein–Uhlenbeck processes and independent standard Brownian motions, respectively*; $i = 1,\ldots,\mathsf{s}$, s *defines the number of seasons*, L *is a lag operator that shifts the Ornstein–Uhlenbeck processes between seasons* ($L^jJ_{kc}(r) = J_{k-j,c}(r)$, *with* $J_{k-j,c}(r) = J_{\mathsf{s}+k-j,c}(r)$ *for* $k - j \leq 0$), *and* j *indicates the rotation of the Ornstein–Uhlenbeck processes between seasons imposed by* y_{t-i}.

The proof of this lemma follows along similar lines as Lemma A4.1 for seasonally integrated processes in Osborn and Rodrigues (1998) and will therefore be omitted.

3. ASYMPTOTIC POWER FUNCTIONS FOR THE HYLLEBERG ET AL. TEST

For the purpose of analyzing the test proposed by Hylleberg et al. (1990), it is important to specify the periodicity of the data (e.g., quarterly or monthly) because it is this that determines the transformations of the regressors required in the test regression. For expositional convenience, only the quarterly case will be discussed here, although extensions to processes of higher periodicity can be considered.

Thus, if the near integrated version of the seasonal process given in (2.1) is investigated with $\gamma = 1 + (c/N)$ and $\mathsf{s} = 4$, that is,

$$y_t = \left(1 + \frac{c}{N}\right)y_{t-4} + \varepsilon_t, \tag{3.1}$$

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it can be observed that the resulting characteristic equation can be factorized as

$$\left(1-\left(\sqrt[4]{1+\frac{c}{N}}\right)L\right)\left(1+\left(\sqrt[4]{1+\frac{c}{N}}\right)L\right)\left(1+\left(\sqrt{1+\frac{c}{N}}\right)L^2\right). \tag{3.2}$$

Employing a Taylor expansion about one on each of the factors of (3.2) will in turn allow these terms to be approximated as

$$\left[1-\left(1+\frac{c}{4N}+O(N^{-2})\right)L\right]\left[1+\left(1+\frac{c}{4N}+O(N^{-2})\right)L\right]$$
$$\times\left[1+\left(1+\frac{c}{2N}+O(N^{-2})\right)L^2\right]. \tag{3.3}$$

Hence, the application of the Hylleberg et al. (1990) approach to the near nonstationary process in (2.1) gives the test regression

$$\Delta_4 y_t = \pi_1 y_{(1),t-1} + \pi_2 y_{(2),t-1} + \pi_3 y_{(3),t-2} + \pi_4 y_{(3),t-1} + \varepsilon_t, \tag{3.4}$$

where $y_{(1)t} = (1 + L + L^2 + L^3)y_t$, $y_{(2)t} = -(1 - L + L^2 - L^3)y_t$, $y_{(3)t} = -(1 - L^2)y_t$, L is the usual lag operator, and $\pi_1 = c/T$, $\pi_2 = c/T$, $\pi_3 = 2c/T$, and $\pi_4 = 0$. This can be confirmed by adequately expanding (3.3). (Note that $\pi_4 = 0$ occurs directly from the assumption on the frequencies of the roots.)

For the analysis of the distributions of the estimated coefficients and associated statistics of (3.4) in this near seasonally integrated context, it is important to refer to the results provided by Jeganathan (1991). Jeganathan shows that the asymptotic independence between unit roots at different frequencies, established by Chan and Wei (1988) for a general integrated process, also prevails in the near integrated context. Thus, with respect to the Hylleberg et al. (1990) test in (3.4), the asymptotic orthogonality between the regressors prevails, which allows an asymptotic examination of each component independently of the others. For this purpose, the following models will be adopted:

$$z_{(1)t} = \phi_1 z_{(1),t-1} + \varepsilon_t, \tag{3.5}$$

$$z_{(2)t} = -\phi_2 z_{(2),t-1} + \varepsilon_t, \tag{3.6}$$

$$z_{(3)t} = \phi_4 z_{(3),t-1} - \phi_3 z_{(3),t-2} + \varepsilon_t, \tag{3.7}$$

where $\phi_j = e^{c/T} \simeq (1 + c/T), j = 1,2$, $\phi_3 = e^{2c/T} \simeq (1 + (2c/T))$, and $\phi_4 = 0$. Assuming that the initial values are equal to zero, models (3.5)–(3.7) imply the partial sums

$$z_{(1),t} = \sum_{i=1}^{t} e^{(t-i)(c/T)}\varepsilon_i,$$

$$z_{(2),t} = (-1)^t \sum_{i=1}^{t-1} (-1)^i e^{(t-i)(c/T)}\varepsilon_i + \varepsilon_t,$$

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and

$$z_{(3),t} = \begin{cases} (-1)^n \sum_{i=0}^{n-1} (-1)^i e^{(n-i)(2c/T)} \varepsilon_{2i+1} + \varepsilon_t & t \bmod 2 = 1 \\ (-1)^n \sum_{i=0}^{n-1} (-1)^i e^{(n-i)(2c/T)} \varepsilon_{2i+2} + \varepsilon_t & t \bmod 2 = 0 \end{cases},$$

where $n = [(t+1)/2] - 1$.

3.1. Local to One Behavior of the Zero Frequency Root

For the purpose of deriving the distribution of the normalized least squares bias,

$$T(\hat{\phi}_1 - \phi_1) = \frac{\frac{1}{T}\sum_{t=1}^{T} z_{(1),t-1}\varepsilon_t}{\frac{1}{T^2}\sum_{t=1}^{T} z_{(1),t-1}^2},$$

obtained from (3.5), we can see from the continuous mapping theorem and Lemma 2 in the Appendix that

$$\frac{1}{T}\sum_{t=1}^{T} z_{(1),t-1}\varepsilon_t = \frac{1}{4N}\sum_{t=1}^{T}\left(\sum_{i=0}^{t-1} e^{(t-1-i)(c/T)}\varepsilon_i\right)\varepsilon_t \Rightarrow \frac{\sigma^2}{4}\int_0^1 J_{(1)c}(r)dW_{(1)}(r)$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T} z_{(1),t-1}^2 \Rightarrow \frac{4\sigma^2}{16}\int_0^1 [J_{(1)c}(r)]^2 dr.$$

Thus,

$$T(\hat{\phi}_1 - \phi_1) \Rightarrow \frac{\int_0^1 J_{(1)c}(r)dW_{(1)}(r)}{\int_0^1 [J_{(1)c}(r)]^2 dr}, \tag{3.8}$$

and the limit distribution for the t-statistic on $(\hat{\phi}_1 - \phi_1)$ is therefore

$$t_{(\hat{\phi}_1-\phi_1)} \Rightarrow \frac{\int_0^1 J^*_{(1)c}(r)dW^*_{(1)}(r)}{\left\{\int_0^1 [J^*_{(1)c}(r)]^2 dr\right\}^{1/2}}, \tag{3.9}$$

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where $J_{(1)c}(r) = W(r) + c(\int_0^r e^{(r-s)c}W(s)ds)$, $W(h) = \sum_{k=1}^4 W_k(h)$, $W_k(h)$ $(k = 1,\ldots,4)$ are independent standard Wiener processes, $W^*_{(1)}(h) = (1/2)\sum_{k=1}^4 \times W_k(h)$ and $J^*_{(1),c}(r) = (1/2)J_{(1)c}(r)$ (see Appendix for proof of the results stated). Notice that the distributions in (3.8) and (3.9) are equivalent to the power functions of the well known Dickey–Fuller test (see Phillips, 1987).

3.2. Local to Minus One Behavior of the Biannual Root

With respect to the least squares bias $T(\hat{\phi}_2 - \phi_2)$, obtained from (3.6), it follows from Lemma 2 (see the Appendix), that

$$\frac{1}{T}\sum_{t=1}^{T} z_{(2),t-1}\,\varepsilon_t \Rightarrow -\frac{\sigma^2}{4}\int_0^1 J_{(2)c}(r)dW_{(2)}(r)$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T} z^2_{(2),t-1} \Rightarrow \frac{4\sigma^2}{16}\int_0^1 [J_{(2)c}(r)]^2 dr,$$

respectively. Hence,

$$T(\hat{\phi}_2 - \phi_2) \Rightarrow -\frac{\displaystyle\int_0^1 J_{(2)c}(r)dW_{(2)}(r)}{\displaystyle\int_0^1 [J_{(2)c}(r)]^2 dr}, \tag{3.10}$$

and consequently, the corresponding t-statistic, $t_{(\hat{\phi}_2-\phi_2)}$, converges to

$$t_{(\hat{\phi}_2-\phi_2)} \Rightarrow -\frac{\displaystyle\int_0^1 J^*_{(2)c}(r)dW^*_{(2)}(r)}{\left\{\displaystyle\int_0^1 [J^*_{(2)c}(r)]^2 dr\right\}^{1/2}}, \tag{3.11}$$

where $J_{(2)c}(r) = W_{(2)}(r) + c(\int_0^r e^{(r-s)c}W_{(2)}(s)ds)$, $W_{(2)}(h) = \sum_{k=1}^4 (-1)^k W_k(h)$, $W^*_{(2)}(h) = (1/2)W_{(2)}(h)$, and $J^*_{(2)c}(r) = (1/2)J_{(2)c}(r)$. As expected, the asymptotic distributions in (3.10) and (3.11) are mirror images of those in (3.8) and (3.9). This mirror image property was established by Chan and Wei (1988) and Fuller (1996) in a conventional AR(1) context (see also Tanaka, 1996, pp. 230–231).

3.3. Local to One Behavior of the Annual Root (Complex Roots)

Finally, with respect to the statistics $T(\hat{\phi}_3 - \phi_3)$ and $t_{(\hat{\phi}_3-\phi_3)}$ resulting from (3.7), it also follows from Lemma 2 that

$$\frac{1}{T}\sum_{t=1}^{T} z_{(3),t-2}\,\varepsilon_t \Rightarrow -\frac{\sigma^2}{4}\left\{\int_0^1 J_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J_{(4)c}(r)dR_{(4)}(r)\right\}$$

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and

$$\frac{1}{T^2}\sum_{t=1}^{T} z_{(3),\,t-2}^2 \Rightarrow \frac{2\sigma^2}{16}\left\{\int_0^1 [J_{(3)c}(r)]^2 dr + \int_0^1 [J_{(4)c}(r)]^2 dr\right\},$$

where

$$J_{(i)c}(r) = (W_{i-2}(r) - W_i(r)) + c\left\{\int_0^r e^{(r-s)c}[W_{i-2}(s) - W_i(s)]\,ds\right\},$$

$$R_{(i)}(r) = W_{i-2}(r) - W_i(r),$$

and $i = 3,4$. Thus, it can be verified that $T(\hat{\phi}_3 - \phi_3)$ converges in distribution to

$$T(\hat{\phi}_3 - \phi_3) \Rightarrow -\frac{2\left\{\int_0^1 J_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J_{(4)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 [J_{(3)c}(r)]^2 dr + \int_0^1 [J_{(4)c}(r)]^2 dr\right\}} \tag{3.12}$$

and the respective t-statistic,

$$t_{(\hat{\phi}_3-\phi_3)} \Rightarrow -\frac{\left\{\int_0^1 J^*_{(3)c}(r)dR^*_{(3)}(r) + \int_0^1 J^*_{(4)c}(r)dR^*_{(4)}(r)\right\}}{\left\{\int_0^1 [J^*_{(3)c}(r)]^2 dr + \int_0^1 [J^*_{(4)c}(r)]^2 dr\right\}^{1/2}}, \tag{3.13}$$

where $J^*_{(i)c}(r) = (1/\sqrt{2})J_{(i)c}(r)$, $R^*_{(i)}(r) = (1/\sqrt{2})R_{(i)}(r)$, and $i = 3,4$.

It should be noted that the results in (3.12) and (3.13) correspond to symmetric versions of the distribution obtained for the Dickey et al. (1984) test when applied on a near seasonally integrated process of periodicity two (see Chan, 1988, 1989; Perron, 1992; Tanaka, 1996).

Furthermore, it can also be shown, based on Lemma 2, that the least squares bias of ϕ_4 and respective t-statistic have limit distributions

$$T\hat{\phi}_4 = \frac{\frac{1}{T}\sum_{t=1}^{T} z_{(3),\,t-1}\varepsilon_t}{\frac{1}{T^2}\sum_{t=1}^{T} z_{(3),\,t-1}^2} \Rightarrow -\frac{2\left\{\int_0^1 J_{(4)c}(r)dR_{(3)}(r) - \int_0^1 J_{(3)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 [J_{(3)c}(r)]^2 dr + \int_0^1 [J_{(4)c}(r)]^2 dr\right\}} \tag{3.14}$$

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and

$$t_{\hat{\phi}_4} \Rightarrow -\frac{\left\{\int_0^1 J^*_{(4)c}(r)dR^*_{(3)}(r) - \int_0^1 J^*_{(3)c}(r)dR^*_{(4)}(r)\right\}}{\left\{\int_0^1 [J^*_{(3)c}(r)]^2dr + \int_0^1 [J^*_{(4)c}(r)]^2dr\right\}^{1/2}}, \tag{3.15}$$

respectively. Note that ϕ_4 in (3.7) is zero.

Hence, based on the results presented, the following theorem collecting the relevant limit distributions pertaining to the Hylleberg et al. (1990) test statistics can be stated.

THEOREM 1. *If* $\{y_t\}$ *is a near seasonally integrated time series generated by* (3.1), *then as* $T \to \infty$, *the least-squares estimators of* (3.4) *are asymptotically distributed as*

$$T(\hat{\pi}_i - \pi_i) \Rightarrow \frac{\int_0^1 J_{(i)c}(r)dW_{(i)}(r)}{\int_0^1 [J_{(i)c}(r)]^2dr}, \qquad i = 1,2, \tag{3.16}$$

$$T(\hat{\pi}_3 - \pi_3) \Rightarrow \frac{2\left\{\int_0^1 J_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J_{(4)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 [J_{(3)c}(r)]^2dr + \int_0^1 [J_{(4)c}(r)]^2dr\right\}}, \tag{3.17}$$

$$T(\hat{\pi}_4 - \pi_4) \Rightarrow \frac{2\left\{\int_0^1 J_{(4)c}(r)dR_{(3)}(r) - \int_0^1 J_{(3)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 [J_{(3)c}(r)]^2dr + \int_0^1 [J_{(4)c}(r)]^2dr\right\}}, \tag{3.18}$$

and the limiting distributions for the corresponding t-statistics are

$$t_{\hat{\pi}_i} \Rightarrow \frac{\int_0^1 J^*_{(i)c}(r)dW^*_{(i)}(r)}{\left\{\int_0^1 [J^*_{(i)c}(r)]^2dr\right\}^{1/2}}, \qquad i = 1,2, \tag{3.19}$$

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$$t_{\hat{\pi}_3} \Rightarrow \frac{\left\{\int_0^1 J^*_{(3)c}(r)dR^*_{(3)}(r) + \int_0^1 J^*_{(4)c}(r)dR^*_{(4)}(r)\right\}}{\left\{\int_0^1 [J^*_{(3)c}(r)]^2dr + \int_0^1 [J^*_{(4)c}(r)]^2dr\right\}^{1/2}}, \tag{3.20}$$

$$t_{\hat{\pi}_4} \Rightarrow \frac{\left\{\int_0^1 J^*_{(4)c}(r)dR^*_{(3)}(r) - \int_0^1 J^*_{(3)c}(r)dR^*_{(4)}(r)\right\}}{\left\{\int_0^1 [J^*_{(3)c}(r)]^2dr + \int_0^1 [J^*_{(4)c}(r)]^2dr\right\}^{1/2}} \equiv \Im_4. \tag{3.21}$$

Note that the distributions in (3.16) and (3.19) for $i = 2$, (3.17), (3.18), (3.20), and (3.21) given the manner in which the Hylleberg et al. (1990) test is set up are symmetric versions of (3.10), (3.11), (3.12), (3.14), (3.13), and (3.15), respectively. Furthermore, the distributions in (3.16), (3.19), (3.17), and (3.20) are equivalent to the power functions of the least squares estimators and corresponding *t*-statistics obtained from Dickey and Fuller (1979) and Dickey et al. (1984) type test regressions (with no deterministic components), respectively.

Moreover, to emphasize the significance of the noncentrality parameter on the distributions of the statistics of the Hylleberg et al. (1990) test, (3.16) and (3.17) can be rewritten as

$$T\hat{\pi}_i \Rightarrow c + \frac{\int_0^1 J_{(i)c}(r)dW_{(i)}(r)}{\int_0^1 [J_{(i)c}(r)]^2dr}, \qquad i = 1,2,$$

$$T\hat{\pi}_3 \Rightarrow c + \frac{2\left\{\int_0^1 J_{(3)c}(r)dR_{(3)}(r) + \int_0^1 J_{(4)c}(r)dR_{(4)}(r)\right\}}{\left\{\int_0^1 [J_{(3)c}(r)]^2dr + \int_0^1 [J_{(4)c}(r)]^2dr\right\}}$$

and (3.19) and (3.20) as

$$t_{\hat{\pi}_i} \Rightarrow c\left\{\int_0^1 [J^*_{(i)c}(r)]^2dr\right\}^{1/2} + \frac{\int_0^1 J^*_{(i)c}(r)dW^*_{(i)}(r)}{\left\{\int_0^1 [J^*_{(i)c}(r)]^2dr\right\}^{1/2}} \equiv \Im_i, \qquad i = 1,2, \tag{3.22}$$

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$$t_{\hat{\pi}_3} \Rightarrow c\left\{\int_0^1 [J^*_{(3)c}(r)]^2 dr + \int_0^1 [J^*_{(4)c}(r)]^2 dr\right\}^{1/2} + \frac{\left\{\int_0^1 J^*_{(3)c}(r)\,dR^*_{(3)}(r) + \int_0^1 J^*_{(4)c}(r)\,dR^*_{(4)}(r)\right\}}{\left\{\int_0^1 [J^*_{(3)c}(r)]^2 dr + \int_0^1 [J^*_{(4)c}(r)]^2 dr\right\}^{1/2}}$$
$$\equiv \Im_3. \tag{3.23}$$

Hence, from (3.22), (3.23), and Lemma 2 of Phillips (1987), the asymptotic local behavior of these statistics can be established. Specifically, Phillips indicates that as the noncentrality parameter tends to minus infinity, $(c \to -\infty)$ then

$$(-2c)\int_0^1 [J_c(r)]^2 dr \xrightarrow{p} 1 \quad \text{and} \quad (-2c)^{1/2}\int_0^1 J_c(r)\,dW(r) \Rightarrow N(0,1).$$

Thus, by acquiring these results, it can be observed that the asymptotic local power of $t_{\hat{\pi}_j}$, $j = 1,2$, and 3 tends to unity as $c \to -\infty$, because $\Im_j$ in (3.22) and (3.23) diverges to $-\infty$.

3.4. Asymptotic Distributions of Joint Tests

Joint tests also play a fundamental role in the analysis of seasonal unit roots, and their distributions are of equal importance. From the preceding analysis these are now easily obtained. In other words, given the asymptotic diagonality of the appropriately normalized covariance matrix of the regressors of the Hylleberg et al. (1990) test procedure in a local to unity framework, the *F*-statistics on parameters of the Hylleberg et al. test regression can be represented as averages of squared *t*-ratios (as shown by Ghysels, Lee, and Noh, 1994, for seasonally integrated processes). Thus, the asymptotic distributions of the *F*-statistics that test $\pi_1 = \cdots = \pi_4 = 0$ (seasonal integration), $\pi_2 = \pi_3 = \pi_4 = 0$ (unit roots at seasonal frequencies), and $\pi_3 = \pi_4 = 0$ (complex unit roots) are easily derived from the results in (3.19)–(3.21).

4. CONCLUSION

In this paper, we derived the asymptotic power functions for the well known Hylleberg et al. (1990) procedure based on a test regression with no deterministic components. The results are, however, easily extended to cover test regressions with seasonal dummies only and with seasonal dummies and trend by substituting the independent standard Ornstein–Uhlenbeck processes with ade-

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quately demeaned or detrended versions, respectively. Of important significance is the indication that in the derivation of the limit distributions considered, the noncentrality parameter c is the same at different frequencies. This, however, is not restrictive given that the asymptotic independence of the test statistics implies that regardless of whether the noncentrality parameters considered are the same, this will have no impact on the limit distributions.

The results presented are important for several reasons. They are potentially useful in the development of more powerful seasonal unit root test procedures; for instance, locally best invariant (LBI) and unbiased (LBIU) tests and also point optimal invariant (POI) tests, as suggested by Tanaka (1996, pp. 355–362). A further interesting application is the derivation of asymptotic confidence intervals for the roots of a seasonal nonstationary process as in Rodrigues and Osborn (1999). Finally, the limit distributions presented also extend the asymptotic results in Osborn and Rodrigues (1998) and Smith and Taylor (1998).

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APPENDIX

Recall that the distributions of the Hylleberg et al. (1990) test statistics under the null hypothesis of a seasonal random walk are functions of S independent Brownian motions. Thus, to analyze the asymptotic behavior under local alternatives, similar functional forms are explored. For simplicity of notation, throughout this Appendix it is assumed that $J_{kc} = J_{kc}(r)$ and $W_k = W_k(r)$.

Consider therefore, the models of Section 3,

$$\text{(i)}\quad z_{(1)t} = \phi_1 z_{(1),t-1} + \varepsilon_t, \tag{A.1}$$

$$\text{(ii)}\quad z_{(2)t} = -\phi_2 z_{(2),t-1} + \varepsilon_t, \tag{A.2}$$

$$\text{(iii)}\quad z_{(3)t} = \phi_4 z_{(3),t-1} - \phi_3 z_{(3),t-2} + \varepsilon_t, \tag{A.3}$$

where $\phi_j = e^{c/T} \simeq (1 + (c/T)), j = 1,2$, $\phi_3 = e^{2c/T} \simeq (1 + (2c/T))$, and $\phi_4 = 0$, and assume that the initial values are equal to zero and $T = 4N$.

(i) From model (A.1), it can be observed that as $T \to \infty$ $(N \to \infty)$,

$$z_{(1)t} = \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i}$$

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can be approximated as

$$z_{(1)t} = \begin{cases} \sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{1i} + \sum_{k=2}^{4}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki} + o_p(1) & t \bmod 4 = 1 \\ \sum_{k=1}^{2}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{k=3}^{4}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki} + o_p(1) & t \bmod 4 = 2 \\ \sum_{k=1}^{3}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{4i} + o_p(1) & t \bmod 4 = 3 \\ \sum_{k=1}^{4}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + o_p(1) & t \bmod 4 = 0 \end{cases},$$

where $n = [(t-1)/4] + 1$ and ε_{ki}, $k = 1,\ldots,4$ refers to the disturbance occurring in season k of year i. Thus, from the functional central limit theorem (FCLT) it follows that

$$\frac{1}{T^{1/2}} z_{(1)t} \Rightarrow \frac{\sigma}{2}\left(\sum_{k=1}^{4} J_{kc}\right),$$

where J_{kc}, $k = 1,2,3,4$ are standard independent Ornstein–Uhlenbeck processes. Furthermore,

$$\frac{1}{T^2}\sum_{t=1}^{T} z_{(1)t}^2 \Rightarrow \frac{4\sigma^2}{16}\int_0^1 \left(\sum_{k=1}^{4} J_{kc}\right)^2 dr \qquad \textbf{(A.4)}$$

and

$$\begin{aligned} &\frac{1}{T}\sum_{t=1}^{T} z_{(1)t-1}\varepsilon_t \\ &\quad = \frac{1}{4N}\sum_{n=1}^{N}\left\{\left(\sum_{k=1}^{4}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{1n}\right. \\ &\qquad + \left(\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{1i} + \sum_{k=2}^{4}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{2n} \\ &\qquad + \left(\sum_{k=1}^{2}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{k=3}^{4}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{3n} \\ &\qquad \left. + \left(\sum_{k=1}^{3}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{4i}\right)\varepsilon_{4n} + o_p(1)\right\} \\ &\quad \Rightarrow \frac{\sigma^2}{4}\int_0^1 \left(\sum_{k=1}^{4} J_{kc}\right) d\left(\sum_{k=1}^{4} W_k\right). \end{aligned} \qquad \textbf{(A.5)}$$

(ii) From model (A.2), it can be observed that

$$z_{(2)t} = \sum_{i=1}^{t-1} (-\phi_2)^i \varepsilon_{t-i} + \varepsilon_t. \qquad \textbf{(A.6)}$$

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Thus, as $T \to \infty$ ($N \to \infty$), (A.6) can be approximated as

$$z_{(2)t} = \begin{cases} \sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{1i} + \sum_{k=2}^{4}(-1)^{k+1}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki} + o_p(1) \\ \qquad t \bmod 4 = 1 \\ \sum_{k=1}^{2}(-1)^{k}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{k=3}^{4}(-1)^{k}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki} + o_p(1) \\ \qquad t \bmod 4 = 2 \\ \sum_{k=1}^{3}(-1)^{k+1}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} - \sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{4i} + o_p(1) \\ \qquad t \bmod 4 = 3 \\ \sum_{k=1}^{4}(-1)^{k}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + o_p(1). \\ \qquad t \bmod 4 = 0 \end{cases}.$$

Using similar arguments as in (i) we can see that

$$\frac{1}{T^{1/2}} z_{(2)t} \Rightarrow \begin{cases} \frac{\sigma}{2}\sum_{k=1}^{4}(-1)^{k+1}J_{kc} & t \bmod 2 = 1 \\ \frac{\sigma}{2}\sum_{k=1}^{4}(-1)^{k}J_{kc} & t \bmod 2 = 0 \end{cases},$$

$$\frac{1}{T^2}\sum_{t=1}^{T} z_{(2)t}^2 \Rightarrow \frac{4\sigma^2}{16}\int_0^1 \left[\sum_{k=1}^{4}(-1)^k J_{kc}\right]^2 dr, \tag{A.7}$$

and

$$\begin{aligned} &\frac{1}{T}\sum_{t=1}^{T} z_{(2)t-1}\varepsilon_t \\ &= \frac{1}{4N}\sum_{n=1}^{N}\left\{\left(\sum_{k=1}^{4}(-1)^k\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{1n}\right. \\ &\qquad + \left(\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{1i} + \sum_{k=2}^{4}(-1)^{k+1}\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{2n} \\ &\qquad + \left(\sum_{k=1}^{2}(-1)^k\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} + \sum_{k=3}^{4}(-1)^k\sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{ki}\right)\varepsilon_{3n} \\ &\qquad \left. + \left(\sum_{k=1}^{3}(-1)^{k+1}\sum_{i=1}^{n} e^{(n-i)(c/N)}\varepsilon_{ki} - \sum_{i=1}^{n-1} e^{(n-1-i)(c/N)}\varepsilon_{4i}\right)\varepsilon_{4n} + o_p(1)\right\} \\ &\Rightarrow -\frac{\sigma^2}{4}\int_0^1\left[\sum_{k=1}^{4}(-1)^k J_{kc}\right] d\left[\sum_{k=1}^{4}(-1)^k W_k\right]. \end{aligned} \tag{A.8}$$

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(iii) Finally, for (A.3) we can verify that

$$z_{(3)t} = \begin{cases} (-1)^n \sum_{i=0}^{n-1} (-1)^i e^{(n-i)(2c/T)} \varepsilon_{2i+1} + \varepsilon_t & t \bmod 2 = 1 \\ (-1)^n \sum_{i=0}^{n-1} (-1)^i e^{(n-i)(2c/T)} \varepsilon_{2i+2} + \varepsilon_t & t \bmod 2 = 0 \end{cases}, \tag{A.9}$$

where $n = [(t+1)/2] - 1$. Thus, along similar arguments as in (i) and (ii), it can be observed that (A.9) can be approximated as

$$z_{(3)t} = \begin{cases} \sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{1i} - \sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)} \varepsilon_{3i} + o_p(1) \\ \qquad t \bmod 4 = 1 \\ \sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{2i} - \sum_{i=1}^{[n/2]} e^{([n/2]-i)c/N} \varepsilon_{4i} + o_p(1) \\ \qquad t \bmod 4 = 2 \\ -\sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{1i} + \sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{3i} + o_p(1) \\ \qquad t \bmod 4 = 3 \\ -\sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{2i} + \sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)} \varepsilon_{4i} + o_p(1) \\ \qquad t \bmod 4 = 0 \end{cases}. \tag{A.10}$$

Therefore, as $T \to \infty$ ($N \to \infty$) it follows that

$$\frac{1}{T^{1/2}} z_{(3)t} \Rightarrow \begin{cases} \frac{\sigma}{2}(J_{1c} - J_{3c}) & t \bmod 4 = 1 \\ \frac{\sigma}{2}(J_{2c} - J_{4c}) & t \bmod 4 = 2 \\ -\frac{\sigma}{2}(J_{1c} - J_{3c}) & t \bmod 4 = 3 \\ -\frac{\sigma}{2}(J_{2c} - J_{4c}) & t \bmod 4 = 0 \end{cases}, \tag{A.11}$$

$$\frac{1}{T^2} \sum_{t=1}^{T} z_{(3)t}^2 \Rightarrow \frac{\sigma^2}{16} \int_0^1 2[(J_{1c} - J_{3c})^2 + (J_{2c} - J_{4c})^2]\, dr, \tag{A.12}$$

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$$\frac{1}{T}\sum_{t=1}^{T} z_{(3)t-2}\varepsilon_t$$

$$= \frac{1}{4N}\sum_{n=1}^{N}\left\{\left(-\sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{1i} + \sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{3i}\right)\varepsilon_{1n}\right.$$

$$+\left(-\sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{2i} + \sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{4i}\right)\varepsilon_{2n}$$

$$+\left(\sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)}\varepsilon_{1i} - \sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{3i}\right)\varepsilon_{3n}$$

$$\left.+\left(\sum_{i=1}^{[(n/2)+1} e^{([n/2]+1-i)(c/N)}\varepsilon_{2i} - \sum_{i=1}^{[n/2]} e^{([n/2]-i)(c/N)}\varepsilon_{4i}\right)\varepsilon_{4n} + o_p(1)\right\}$$

$$\Rightarrow \frac{\sigma^2}{4}\left[-\int_0^1 (J_{1c}-J_{3c})dW_1 - \int_0^1 (J_{2c}-J_{4c})dW_2\right.$$

$$\left.+\int_0^1 (J_{1c}-J_{3c})dW_3 + \int_0^1 (J_{2c}-J_{4c})dW_4\right]$$

$$= -\frac{\sigma^2}{4}\left[\int_0^1 (J_{1c}-J_{3c})d(W_1-W_3) + \int_0^1 (J_{2c}-J_{4c})d(W_2-W_4)\right] \qquad \textbf{(A.13)}$$

and

$$\frac{1}{T}\sum_{t=1}^{T} z_{(3)t-1}\varepsilon_t \Rightarrow -\frac{\sigma^2}{4}\left[\int_0^1 (J_{2c}-J_{4c})d(W_1-W_3) - \int_0^1 (J_{1c}-J_{3c})d(W_2-W_4)\right]. \qquad \textbf{(A.14)}$$

Hence, based on the results in (A.4), (A.5), (A.7), (A.8), (A.12), (A.13), and (A.14) the following lemma can be put forward.

LEMMA 2. *If* $\{y_t\}_{t=1}^{T}$ *is generated from a near seasonally integrated process such as* (3.1) *then as* $T \to \infty$,

$$\left\{\frac{1}{T^2}\sum_{t=1}^{T} z_{(1)t}^2, \frac{1}{T^2}\sum_{t=1}^{T} z_{(2)t}^2, \frac{1}{T^2}\sum_{t=1}^{T} z_{(3)t}^2\right\}$$

$$\Rightarrow \left\{\frac{\sigma^2}{4}\int_0^1 [J_{(1)c}]^2 dr, \frac{\sigma^2}{4}\int_0^1 [J_{(2)c}]^2 dr, \frac{\sigma^2}{8}\left(\int_0^1 [J_{(3)c}]^2 dr + \int_0^1 [J_{(4)c}]^2 dr\right)\right\}$$

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and

$$\left\{\frac{1}{T}\sum_{t=1}^{T} z_{(1)t-1}\varepsilon_t, \frac{1}{T}\sum_{t=1}^{T} z_{(2)t-1}\varepsilon_t, \frac{1}{T}\sum_{t=1}^{T} z_{(3)t-2}\varepsilon_t, \frac{1}{T}\sum_{t=1}^{T} z_{(3)t-1}\varepsilon_t\right\}$$

$$\Rightarrow \left\{\frac{\sigma^2}{4}\int_0^1 J_{(1)c}\,dW_{(1)}, -\frac{\sigma^2}{4}\int_0^1 J_{(2)c}\,dW_{(2)},\right.$$

$$\left.-\frac{\sigma^2}{4}\left(\int_0^1 J_{(3)c}\,dR_{(3)} + \int_0^1 J_{(4)c}\,dR_{(4)}\right), -\frac{\sigma^2}{4}\left(\int_0^1 J_{(4)c}\,dR_{(3)} - \int_0^1 J_{(3)c}\,dR_{(4)}\right)\right\}$$

where $J_{(i)c}(r) = W_{(i)}(r) + c\int_0^1 e^{(r-s)c}W_{(i)}(s)ds$, $i = 1,2$, $J_{(l)c}(r) = R_{(l)}(r) + c\int_0^1 e^{(r-s)c}R_{(l)}(s)ds$, $l = 3,4$, $W_{(1)} = \sum_{k=1}^{4} W_k(r)$, $W_{(2)} = \sum_{k=1}^{4}(-1)^k W_k(r)$, *and* $R_{(j)} = W_{j-2}(r) - W_j(r)$, $j = 3,4$.

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