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Generalised Hele-Shaw flow: A Schwarz function approach

N. R. McDONALD

Department of Mathematics, University College London, WC1E 6BT, UK
email: robb@math.ucl.ac.uk

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An equation governing the evolution of a Hele-Shaw free boundary flow in the presence of an arbitrary external potential – generalised Hele-Shaw flow – is derived in terms of the Schwarz function $g(z,t)$ of the free boundary. This generalises the well-known equation $\partial g/\partial t = 2\partial w/\partial z$, where w is the complex potential, which has been successfully employed in constructing many exact solutions in the absence of external potentials. The new equation is used to re-derive some known explicit solutions for equilibrium and time-dependent free boundary flows in the presence of external potentials, including those with singular potential fields, uniform gravity and centrifugal forces. Some new solutions are also constructed that variously describe equilibrium flows with higher order hydrodynamic singularities in the presence of electric point sources and an unsteady solution describing bubbles under the combined influence of strain and centrifugal potential.

Key words: Hele-Shaw, free boundary, Schwarz function

1 Introduction

The Hele-Shaw free boundary problem typically involves two immiscible fluids sandwiched in a narrow gap between two parallel plates in which fluid of one viscosity displaces fluid of another, with the interface between the two fluids defining the free boundary. Since the normal velocity of the interface is given by the gradient of a function satisfying Laplace's equation, the Hele-Shaw free boundary problem belongs to a class of so-called Laplacian growth problems that encompass a remarkable variety of free boundary problems arising in a wide range of applications.

In the case when one of the fluids is effectively inviscid and at constant pressure ('air'), with the free boundary being the fluid–air interface, a typical problem is to find the evolution of the free boundary driven by singularities (e.g. sources and sinks) located in the fluid. Here, such singularities are referred to as hydrodynamic singularities. In this 'one-phase' problem, the mathematical task becomes that of solving the two-dimensional Laplace equation in the fluid, subject to constant pressure and a kinematic condition that the normal velocity at the free boundary be equal to the normal pressure gradient at the free boundary. Many exact solutions are known for this problem, including the classic steady finger solution of Saffman & Taylor [19]. A variety of mathematical methods have been used to find these solutions usually, owing to their two-dimensional nature, based

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on complex variable methods. For example, the Polubarinova–Galin equation, derived by Galin [10] and Polubarinova-Kochina [14], is a partial differential equation whose solution determines the conformal map $z = z(\zeta, t)$ from the unit disk in an auxiliary parametric complex ζ-plane to the actual fluid domain Ω in the z-plane. One approach using this equation is to assume a particular form of map (e.g. a polynomial in ζ, see [5, 11]) and use the Polubarinova–Galin equation to derive ordinary differential equations governing the unknown time-varying coefficients of the polynomial map. Such ordinary differential equations being relatively simple to solve enable the evolution of free boundary $\partial\Omega$ to be found.

An alternative approach uses the Schwarz function $g(z, t)$ of the free boundary $\partial\Omega$: a function such that $g(z, t) = \bar{z}$ on $\partial\Omega$ and is analytic in its neighbourhood. The existence and uniqueness of the Schwarz function are guaranteed for an analytic curve [6]. If $\partial\Omega$ is given by the real representation $F(x, y) = 0$, then its Schwarz function can be found by substituting the relations $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ and solving for $\bar{z}$ in terms of z. For the Hele-Shaw free boundary problem in the absence of background potentials, it can be shown that where g is well defined in Ω, it satisfies (e.g. [5]) $\partial g/\partial t = 2\partial w/\partial z$, where w is the complex (hydrodynamic) potential driving the flow. There are two approaches to finding exact solutions using this equation. The first, like the Polubarinova–Galin equation method described above, is to assume a form of conformal map having unknown time-dependent coefficients. ordinary differential equations for these coefficients are then deduced by comparing the behaviour of the singular terms in the Schwarz function equation. The second method uses the Schwarz function equation directly to infer the singularity structure of $g(z, t)$ given the hydrodynamic singularities of $\partial w/\partial z$. The map $z = z(\zeta)$ is then deduced from knowledge of the singularities of $g(z, t)$. Both approaches are used in the present work. Examples of the construction of explicit solutions using these methods can be found in [1,5,11,16–18]. It is worth noting that the Schwarz function has proved a useful tool in other two-dimensional free boundary problems, such as those arising in vortex dynamics [2] and the freezing and melting of dendrites, e.g. [4].

Richardson [15] introduced the complex moments of the fluid domain Ω (see Appendix for their definition) and showed that these moments remain constant throughout the evolution except for the zeroth-order moment, which, being equivalent to the area of Ω, evolved according to the net amount of fluid being injected into the cell by the hydrodynamic forcing. The moment equations provide yet another route to finding the conformal map from the ζ-plane to the fluid domain (see, e.g. [5]), where the connection between the Schwarz function approach and moment-based method is also discussed.

In the so-called generalised Hele-Shaw problem when, in addition to hydrodynamic forcing, an external potential also acts on the fluid much less is known. Some unsteady solutions have been obtained for rotating Hele-Shaw flows in which a centrifugal potential acts everywhere using either an appropriate modifications to the Polubarinova–Galin equation [12] or the moment equations [8], or by finding the evolution of the Cauchy transform of $\Omega(t)$ [3]. Exact solutions have also been found when the gradient of the external potential is constant, this being analogous to a uniform gravitational field [8].

Recently, Entov & Etingof [7] have found exact solutions for steady free boundary shapes for a variety of background potentials, including those having singularities in the potential field that can be thought of as being due to 'electric point charges'. They use

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both moment-based methods and Cauchy transforms for the uniform gravity case and methods based on identifying the singular behaviour of the conjugate of the conformal map from the ζ-plane to the fluid domain. This latter approach is similar to that used here, except that in the present work the Schwarz function of the free boundary is explicitly used and unsteady effects accounted for.

The Schwarz function equation in the presence of an external potential is derived in Section 2 and its practical use is demonstrated in Section 3 by re-deriving some previously known exact solutions representative of various external potentials. Some new steady and unsteady solutions are derived in Section 4. In the case of the centrifugal potential, the connection of the present Schwarz function approach with previous methods based on moments and Cauchy transforms is given in the Appendix.

2 Derivation of the Schwarz function equation

Fluid in a Hele-Shaw cell occupies a region $\Omega(t)$ with boundary $\partial\Omega$ separating it from air at constant pressure. A body force with external potential $\Psi(x, y)$ acts on the fluid. Entov & Etingof [7] show that the fluid has (real) velocity potential $\phi(x, y)$ that satisfies the following non-dimensional free boundary problem:

$$\begin{aligned} \nabla^2\phi &= \sum_{j=1}^{N} Q_j \delta(x - x_j, y - y_j), \quad (x_j, y_j) \in \Omega, \\ \phi &= \Psi(x, y), \quad (x, y) \in \partial\Omega, \\ v_n &= \frac{\partial\phi}{\partial n}, \quad (x, y) \in \partial\Omega, \end{aligned} \tag{2.1}$$

where N hydrodynamic sources of strength Q_j are located at (x_j, y_j), and v_n is the normal velocity of the boundary. The task is to find the shape of the fluid domain Ω, which may be either steady or evolving in time. The modification of (2.1) to the case when higher order hydrodynamic singularities are present (e.g. dipoles and quadrupoles) is obvious. Equations (2.1) are similar to the standard Hele-Shaw free boundary problem (i.e. no external potential) except that the velocity potential ϕ does not vanish on $\partial\Omega$ but is instead given by $\Psi(x, y)$. In this work, various forms for Ψ will be considered, including the centrifugal potential $\Psi = \omega(x^2 + y^2)/2$, uniform 'gravity' $\Psi = Ux$ and the potential due to 'point electric charges' with the form $\Psi = (E/2\pi)\log(x^2 + y^2)^{1/2}$, where ω, U and E are real constants measuring the strength of these fields.

It is convenient to consider (2.1) in the complex plane $z = x + iy$ so that ϕ is the real part of the complex potential $w(z, t)$. Let $g(z, t)$ be the Schwarz function of the fluid–air interface $\partial\Omega$. That is, $\bar{z} = g(z, t)$ on $\partial\Omega$ and so upon taking the time derivative

$$\bar{v} = v\frac{\partial g}{\partial z} + \frac{\partial g}{\partial t}, \tag{2.2}$$

where v is the velocity of $\partial\Omega$. Representing $\Psi(x, y) \equiv \Psi(z, \bar{z})$ in conjugate coordinates, where $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ so that $\phi = \Psi(z, \bar{z}) = \Psi(z, g(z, t))$ on $\partial\Omega$, it follows

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that tangent to $\partial\Omega$

$$\Re\left[\bar{v}\frac{\partial z}{\partial s}\right] = \frac{\partial \Psi}{\partial s}, \tag{2.3}$$

or, equivalently,

$$\frac{1}{2}\left[\bar{v}\frac{\partial z}{\partial s} + v\frac{\partial \bar{z}}{\partial s}\right] = \frac{\partial \Psi}{\partial s}, \tag{2.4}$$

where s is an arc length parameter along $\partial\Omega$. Using (see Chapter 11 in [6])

$$\frac{\partial z}{\partial s} = \left(\sqrt{\frac{\partial g}{\partial z}}\right)^{-1}, \quad \frac{\partial \bar{z}}{\partial s} = \sqrt{\frac{\partial g}{\partial z}}, \tag{2.5}$$

(2.4) can be written

$$\bar{v} + v\frac{\partial g}{\partial z} = 2\sqrt{\frac{\partial g}{\partial z}}\frac{\partial \Psi}{\partial s}. \tag{2.6}$$

On $\partial\Omega$ $\bar{z} = g(z,t)$, and hence, (2.6) gives

$$\begin{aligned}\bar{v} + v\frac{\partial g}{\partial z} &= 2\sqrt{\frac{\partial g}{\partial z}}\left[\frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \Psi}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial s}\right], \\ &= 2\frac{\partial \Psi}{\partial z} + 2\frac{\partial g}{\partial z}\frac{\partial \Psi}{\partial \bar{z}} \\ &= 2\frac{\partial}{\partial z}\Psi(z, g(z,t)).\end{aligned} \tag{2.7}$$

Adding (2.2) and (2.7)

$$2\bar{v} = \frac{\partial g}{\partial t} + 2\frac{\partial \Psi}{\partial z}. \tag{2.8}$$

Finally, using $\bar{v} = \partial w/\partial z$ in (2.8) gives on $\partial\Omega$

$$\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t} + \frac{\partial \Psi}{\partial z}. \tag{2.9}$$

According to its derivation (2.9) holds on $\partial\Omega$. By considering $\Psi = \Psi(z, g(z,t))$, both sides of (2.9) are analytic in the neighbourhood of $\partial\Omega$ and may be analytically continued so that it is valid in the whole of Ω. In the case, there is no gradient in the background potential $\partial\Psi/\partial z = 0$ and (2.9) reduces to that previously used in investigating Hele-Shaw free boundary flows in the absence of external potential fields (see, e.g. [1, 5]).

3 Examples: Previously known solutions

3.1 Centrifugal potential

Consider a finite blob of fluid in a rotating Hele-Shaw cell subject to a centrifugal force. The external potential is $\Psi = \omega|z|^2/2 = \omega z g/2$, where $\omega > 0$ is a real constant measuring the rotation rate (see, e.g. [3, 8, 12]), and (2.9) becomes

$$2\frac{\partial w}{\partial z} = \frac{\partial g}{\partial t} + \omega\frac{\partial}{\partial z}(zg). \tag{3.1}$$

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Here, (3.1) is used to re-derive the solutions of [12], which were obtained using a modified Polubarinova–Galin equation, and that of [3], who derives and solves a PDE for the evolution of the Cauchy transform of the fluid blob.

3.1.1 *Example 1: Polynomial map*

A fluid blob evolves under the combined action of centrifugal force and a source of strength Q at the origin. Consider the polynomial map

$$z = \zeta(a_0 + a_n\zeta^n), \tag{3.2}$$

from the unit ζ-disk to the blob, where n is a positive integer. Here, $a_0(t)$ and $a_n(t)$ are time-varying real coefficients to be found. Using $\bar{\zeta} = \zeta^{-1}$ on $\partial\Omega$, as $z \to 0$ and noting $\zeta^{-1} = a_0 z^{-1} + (a_n/a_0^n)z^{n-1} + \cdots$, the Schwarz function has Laurent expansion

$$\begin{aligned} g &= \frac{a_0}{\zeta} + \frac{a_n}{\zeta^{n+1}} \\ &= \frac{a_0^{n+1}a_n}{z^{n+1}} + \frac{a_0^2 + (n+1)a_n^2}{z} + \mathrm{O}(1). \end{aligned} \tag{3.3}$$

Using (3.3) and $w = (Q/2\pi)\log z$ as $z \to 0$ in (3.1)

$$\frac{Q}{\pi z} = \frac{d}{dt}\left[\frac{a_0^{n+1}a_n}{z^{n+1}} + \frac{a_0^2 + (n+1)a_n^2}{z}\right] + \omega\frac{d}{dz}z\left[\frac{a_0^{n+1}a_n}{z^{n+1}} + \frac{a_0^2 + (n+1)a_n^2}{z}\right] + \cdots. \tag{3.4}$$

Equating powers of z^{-1} and $z^{-(n+1)}$ in (3.4)

$$\begin{aligned} \frac{d}{dt}\left(a_0^{n+1}a_n\right) &= \omega n a_0^{n+1}a_n, \\ \frac{d}{dt}\left(a_0^2 + (n+1)a_n^2\right) &= \frac{Q}{\pi}. \end{aligned} \tag{3.5}$$

The evolution equations (3.5) are precisely the same as found by [12] who used a different approach based on the Polubarinova–Galin equation modified for the presence of the centrifugal potential. Given initial values, (3.5) are easily solved for $a_0(t)$ and $a_n(t)$ (the solutions are discussed in detail in [12]). Of special interest in the behaviour of the solutions of (3.5), as [12] find and discuss, is that the centrifugal force has the ability to suppress the formation of cusp-like singularities in the case $Q < 0$, i.e. a sink. The possibility of preventing cusp formation by rotation is also noted in [8].

3.1.2 *Example 2: Rational function map*

Crowdy [3] considers the conformal map from the unit ζ-disk to $\Omega(t)$ given by

$$z = \frac{R\zeta}{\zeta^2 - a^2}, \tag{3.6}$$

where $R(t)$ and $a(t)$, $|a(t)| > 1$, are real functions to be found. He uses a method based on finding the Cauchy transform of $\Omega(t)$ to find the evolution of the blob. The

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connection between Crowdy's Cauchy transform method and the present Schwarz function formulation for the particular case of the centrifugal potential is noted in the Appendix.

To use (3.1) to find the evolution of the blob, first note that the complex potential is analytic everywhere since there are no sources or sinks. But note that, from (3.6),

$$\begin{aligned} g(z,t) &= -\frac{R\zeta/a^2}{\zeta^2 - a^{-2}} \\ &= -\frac{R}{2a^2}\left(\frac{1}{\zeta - a^{-1}} + \frac{1}{\zeta + a^{-1}}\right) \end{aligned} \tag{3.7}$$

has simple poles at $\zeta = \pm a^{-1}$. Let $z(a^{-1}) = z_0(t) = Ra/(1-a^4)$. As $\zeta \to a^{-1}$

$$\frac{1}{\zeta - a^{-1}} = \frac{z_\zeta(a^{-1})}{z - z_0} + \frac{z_{\zeta\zeta}(a^{-1})}{2z_\zeta(a^{-1})} + \mathrm{O}(z - z_0), \tag{3.8}$$

where the subscripts in z_ζ and $z_{\zeta\zeta}$ denote derivatives with respect to ζ. Considering singular terms as $z \to z_0$ and taking the time derivative of (3.7) and using (3.8)

$$\frac{\partial g}{\partial t} = \frac{R^2(1+a^4)}{2(1-a^4)^2}\frac{dz_0/dt}{(z-z_0)^2} + \frac{d}{dt}\left(\frac{R^2(1+a^4)}{2(1-a^4)^2}\right)\frac{1}{z-z_0} + \mathrm{O}(1). \tag{3.9}$$

Equations (3.7) and (3.8) are used to find the Laurent expansion of $\partial(zg)/\partial z = g + z\partial g/\partial z$ about $z = z_0$, which, after some simplification, is

$$\begin{aligned} \omega\frac{d}{dz}(zg(z,t)) &= \frac{\omega z_0 R z_\zeta(a^{-1})}{2a^2(z-z_0)^2} + \mathrm{O}(1), \\ &= -\frac{R^2(1+a^4)}{2(1-a^4)^2}\frac{\omega z_0}{(z-z_0)^2} + \mathrm{O}(1). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.1) and considering the singular terms $(z-z_0)^{-1}$ and $(z-z_0)^{-2}$ gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{R^2(1+a^4)}{(1-a^4)^2}\right) = 0, \\ &\frac{dz_0(t)}{dt} = \omega z_0(t). \end{aligned} \tag{3.11}$$

These are the same evolution equations for $a(t)$ and $R(t)$ obtained by [3], the first of which is equivalent to conservation of area of the blob. As [3] points out, a feature of this solution is that an initially circular blob ($a \to \infty$) develops cusps in finite time. Interestingly, Crowdy [3] also considers the alternative map from the exterior of the unit ζ-circle to the exterior of an elliptical blob in the z-plane and shows that an initially near-circular blob can evolve in a different way to the cusp forming solution (3.11), namely an ellipse of increasing eccentricity. He concludes that the Hele-Shaw problem with a centrifugal potential is ill-posed in the sense that blobs with starting shapes arbitrarily close to a circle can evolve in markedly different ways.

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3.2 Uniform gravitational potential

3.2.1 *Example 1: Steady source in a uniform gravitational field*

Consider the steady flow due to a source of strength Q in a uniform gravitational field given by $\Psi = Ux = U(z+g)/2$ on $\partial\Omega$ for which the governing equation for the Schwarz function (2.9) becomes

$$2\frac{\partial w}{\partial z} = U\left(1 + \frac{\partial g}{\partial z}\right). \tag{3.12}$$

In the steady state, the flow extends from near the source located at $z = 0$ to $\Re z \to \infty$, where the flow is uniform and of speed W (to be found) parallel to the $\Re z$-axis. Thus, $w \to Wz$ as $z \to \infty$. Near the source, $w \to (Q/2\pi)\log z$ as $z \to 0$. Thus, the hydrodynamic potential w has two singularities at $z = 0$ and $z \to \infty$. Let $z = z(\zeta)$ be the map from the unit ζ-disk to Ω such that $z(0) = 0$ and, without loss of generality, $z(-1) = \infty$. Integrating (3.12), it follows that, as $z \to 0$, $g = (2/U)w + \text{const}$. But $w = (Q/2\pi)\log z = (Q/2\pi)\log\zeta + \text{const}$ as $z \to 0$. Hence,

$$g = \frac{Q}{\pi U}\log\zeta + \text{const} \quad \text{as} \quad z \to 0. \tag{3.13}$$

As $z \to \infty$, recall $w \to Wz$ and therefore from (3.12) $g \to (2W - U)z/U + \text{const}$. Combining this with (3.13) determines g:

$$g = \frac{Q}{\pi U}\log\zeta + \frac{(2W-U)}{U}z + \text{const}. \tag{3.14}$$

On $\partial\Omega$, $\zeta = \exp(i\theta)$ and the choice $-\pi < \theta \leqslant \pi$ implies that the imaginary part of the constant in (3.14) vanishes in order for the the fluid domain Ω to be symmetric about the $\Re z$-axis. Also, taking the conjugate of (3.14) reveals that the real part of the constant also vanishes and $W = U$ so that the fluid velocity tends to U as $z \to \infty$. Now, with const $= 0$, from (3.14) the form of $z(\zeta)$ satisfying $z(0) = 0$ is

$$z = -\frac{Q}{\pi U}\log(1+\zeta). \tag{3.15}$$

That (3.15) satisfies (3.14) is obviously true upon taking its conjugate and using $\bar{\zeta} = \zeta^{-1}$ on $\partial\Omega$. Note that (3.15) is the same as the large time limit of the solution given in [8] (see their (3.6)). The free boundary $\partial\Omega$ is given parametrically as $x = -(Q/\pi U)\log[2\cos(\theta/2)]$ and $y = -Q\theta/2\pi U$. As $\theta \to \pm\pi$, $x \to \infty$ and the flow is confined to a channel of width Q/U and has speed U and hence transports, as it must, flux Q.

3.2.2 *Example 2: Elliptical Taylor–Saffman bubble*

Consider an air bubble propagating steadily at speed U in the positive real z-direction in an infinite fluid such that, in appropriate non-dimensional quantities, the fluid speed at infinity is unity. Taylor & Saffman [20] used a potential plane type analysis to show that in a frame moving with the bubble, the bubble shape is an ellipse with aspect ratio (i.e. y-height to x-width) $(a-b)/(a+b)$ $(a > b > 0)$ such that $b/a = (U-2)/U$.

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Their solution can be re-derived using (2.9) with $\partial g/\partial t = 0$ for steady flow in a frame moving at speed U with the bubble. The hydrodynamic singularity is at infinity such that the complex potential $w \to (1-U)z$ as $z \to \infty$. Also, a frame moving with the bubble speed U implies a background potential $\Psi = -U(z+g)/2$ and so (2.9) yields as $z \to \infty$

$$2(1-U) = -U\left(1 + \frac{\partial g}{\partial z}\right). \tag{3.16}$$

Equation (3.16) gives $g(z) \to (U-2)z/U$ as $z \to \infty$. Now, the closed curve whose Schwarz function behaves like Cz at infinity, where C is a constant, is necessarily an ellipse [13]. Given this, consider the map from the unit ζ-disk to the exterior of an ellipse

$$z = \frac{a}{\zeta} + b\zeta, \tag{3.17}$$

where $a > b > 0$ are real constants. The aspect ratio of the ellipse is $(a-b)/(a+b)$. The ellipse in the z-plane has, from (3.17), Schwarz function

$$g(z,t) = \frac{b}{a}z + \frac{a^2 - b^2}{a}\zeta. \tag{3.18}$$

Equating (3.18) with the behaviour $g(z) \to (U-2)z/U$ as $z \to \infty$ found above gives $b/a = (U-2)/U$. This being the same geometry–speed relation for the bubble was found by Taylor & Saffman [20].

3.3 Solution with an electric source

Entov & Etingof [7] derive a steady solution describing a blob of finite area with an enclosed source Q at $z = a$, sink $-Q$ at $z = b$ (a, b both real and $0 < a < b$) with an external point source potential E at $z = 0$ outside the fluid blob. That is, an 'electric point charge' such that $\Psi = (E/2\pi)\log|z| = (E/4\pi)\log(zg)$ on $\partial\Omega$. Steady solutions to (2.9) require that

$$\frac{Q}{2\pi}\log\left(\frac{z-a}{z-b}\right) = \frac{E}{4\pi}\log g + \text{const}, \quad z \to a, b. \tag{3.19}$$

Thus,

$$g = c\left(\frac{z-a}{z-b}\right)^{1/\lambda}, \tag{3.20}$$

where $\lambda = E/2Q$ and c is a constant. Let $z = z(\zeta)$ be the conformal map from the unit ζ-disk to the blob in the z-plane and suppose $\zeta = \pm\alpha$ maps to $z = a$ and $z = b$, respectively. Then,

$$g = d\left(\frac{\zeta - \alpha}{\zeta + \alpha}\right)^{1/\lambda}, \tag{3.21}$$

where d is a constant. Hence, upon taking the conjugate

$$z = d\left(\frac{1 - \alpha\zeta}{1 + \alpha\zeta}\right)^{1/\lambda}, \tag{3.22}$$

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where $d = \sqrt{ab}$ and

$$\alpha = \sqrt{\frac{1-(a/b)^{\lambda/2}}{1+(a/b)^{\lambda/2}}}, \tag{3.23}$$

as also obtained by [7].

Entov & Etingof [7] then proceed to find the equilibrium solution when a hydrodynamic dipole is located at $z = a$ and an electric point source at $z = 0$ by considering the limit of (3.22) as $b \to a$ and $Q \to \infty$ in such a way that the strength of the dipole is finite. Alternatively, their solution can be derived directly using the present Schwarz function formulation. First, note as $z \to a$, $w \to (\mu/2\pi)(z-a)^{-1}$, where μ is the strength of the dipole. Thus, from the steady version of (2.9) as $z \to a$

$$\frac{\mu}{2\pi(z-a)} = \frac{E}{4\pi}\log g + \text{const.} \tag{3.24}$$

Let $z(\zeta)$ be the map from the unit ζ-disk to the fluid domain such that $z(0) = a$. As $\zeta \to 0$, (3.24) can be written as

$$\log g = \frac{2\mu}{Ez_\zeta(0)\zeta} + \text{const}, \tag{3.25}$$

where $z_\zeta(0)$ is the derivative of $z(\zeta)$ evaluated at $\zeta = 0$. By taking the conjugate of (3.25) and using $z(0) = a$ determines $z(\zeta)$ as

$$z = a\exp\left[\frac{2\mu\zeta}{Ez_\zeta(0)}\right]. \tag{3.26}$$

Taylor expanding the right-hand side of (3.26) about $\zeta = 0$ and then differentiating with respect to ζ and putting $\zeta = 0$ determines $z_\zeta(0)$ and finally $z(\zeta)$:

$$z = a\exp\left[\sqrt{\frac{2\mu}{aE}}\zeta\right]. \tag{3.27}$$

Equation (3.27) is the same as found by [7] using the limiting argument applied directly to (3.22).

4 New examples

4.1 Elliptical bubble in a strain field with centrifugal potential

Consider the evolution of an elliptical bubble in an infinite fluid with a hydrodynamic strain field and a centrifugal external potential. The appropriate conformal map and Schwarz function for the ellipse are given by (3.17) and (3.18) with, now, $a = a(t)$ and $b = b(t)$ and $a > b$ (but not necessarily positive in this example permitting collapse to the imaginary axis). The external potential is $\Psi = \omega zg/2$ on $\partial\Omega$ and the presence of the strain field implies $w \to -Mz^2/2\pi$ as $z \to \infty$. The governing equation (2.9) is

$$2\frac{\partial w}{\partial z} = \frac{\partial g}{\partial t} + \omega\frac{\partial}{\partial z}zg. \tag{4.1}$$

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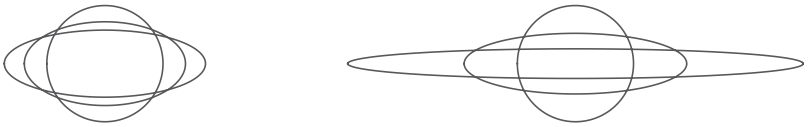


FIGURE 1. Collapse of an elliptical bubble in a centrifugal potential field with $\omega = 1$ and strain field of strength M. The bubble is initially circular with unit radius. On the left $M = -\pi/2$ and the times shown are $t = 0$, 0.5 and ∞. On the right $M = -3\pi/2$ and the bubble is shown for times $t = 0$, 0.24 and 0.44. In this case, the bubble collapses in finite time $t \approx 0.55$.

In the limit $z \to \infty$, the Schwarz function for the ellipse behaves like $g(z,t) \to (b/a)z + (a^2 - b^2)z^{-1} \cdots$. Considering terms of O(z) in (4.1) in the limit $z \to \infty$ gives the ordinary differential equation

$$\frac{d}{dt}\left(\frac{b}{a}\right) + 2\omega\frac{b}{a} = -\frac{2M}{\pi}, \tag{4.2}$$

which has solution

$$\frac{b}{a} = -\frac{M}{\pi\omega} + \left(\frac{M}{\pi\omega} + \frac{b(0)}{a(0)}\right)\exp(-2\omega t). \tag{4.3}$$

Since there are no sources or sinks, there is no O(z^{-1}) term in the large z expansion of $\partial w/\partial z$. Hence, (4.1) implies the time derivative of $a^2 - b^2$ vanishes. This is equivalent to conservation of area of the bubble, which if, say, is equal to π implies $a^2 - b^2 = 1$. This area constraint together with (4.3) determines the evolution of the ellipse. If M and ω are such that $|M|/\pi\omega > 1$ an initially circular bubble (i.e. $a(0) = 1$) collapses to the real axis in finite time. This occurs when $|b/a| = 1$ at time $t = t^*$, where

$$t^* = -\frac{1}{2\omega}\log\left(1 - \frac{\pi\omega}{|M|}\right). \tag{4.4}$$

If $|M|/\pi\omega < 1$, an equilibrium bubble shape is reached as $t \to \infty$ such that $b/a = -M/\omega\pi$. This finite time collapse or smooth approach to equilibrium as $t \to \infty$ is illustrated in Figure 1.

Note that in the special case of vanishing centrifugal force $\omega = 0$, and putting $a(0) = 1$ the solution derived by [9] for the finite-time collapse of an initially circular bubble in a strain field is recovered from (4.3):

$$\begin{aligned} a(t) &= \frac{1}{\sqrt{1 - (2Mt/\pi)^2}}, \\ b(t) &= \frac{-2Mt/\pi}{\sqrt{1 - (2Mt/\pi)^2}}. \end{aligned} \tag{4.5}$$

The other limit in which $M = 0$, (4.3) shows that $b \to 0$ as $t \to \infty$, i.e. an elliptical bubble tends a circle for large time, a result also found by [8].

4.2 Hydrodynamic dipole with two electric point sources

Consider an equilibrium Hele-Shaw free boundary flow with a hydrodynamic dipole of strength μ at $z = 0$ with point charges outside the fluid domain with strengths E at $z = a$

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($a \in \Re$, $a > 0$) and $-E$ at $z = -a$. The external potential is

$$\Psi = \frac{E}{4\pi} \log\left(\frac{(z-a)(g-a)}{(z+a)(g+a)}\right). \tag{4.6}$$

Using (4.6) and considering (2.9) in the limit $z \to 0$ gives

$$\frac{E}{4\pi} \log\left(\frac{g-a}{g+a}\right) = \frac{\mu}{2\pi z} + \text{const.} \tag{4.7}$$

Let $z(\zeta)$ be the map from the unit ζ-disk to the fluid domain such that $z(0) = 0$. Thus, as $z \to 0$, $z = z_\zeta(0)\zeta + \mathrm{O}(\zeta^2)$ and

$$\frac{g-a}{g+a} = k \exp\left(\frac{2\mu}{E z_\zeta(0)\zeta}\right), \tag{4.8}$$

where k is a constant. Taking the complex conjugate of (4.8) and using $\bar{z} = g$ and $\bar{\zeta} = \zeta^{-1}$ on $\partial\Omega$ gives $k = -1$ and

$$z = -a\tanh\left(\frac{\mu}{E z_\zeta(0)}\zeta\right). \tag{4.9}$$

Taylor expanding (4.9) about $\zeta = 0$ and then taking the derivative with respect to ζ yields $z_\zeta(0) = \sqrt{-a\mu/E}$, leading to the condition that $\mathrm{sgn}(\mu E) < 0$. Physically, this means that the flow induced by the dipole locally opposes the gradient of the external potential near each point charge. Finally, substituting for $z_\zeta(0)$ in (4.9), $z(\zeta)$ is found:

$$z = -a\tanh\left(\sqrt{\frac{-\mu}{aE}}\zeta\right). \tag{4.10}$$

For given μ and E (4.10) the shape of the blob depends on the location of the electric point charges $z = a$. As $a \to \infty$, the blob becomes a circle of radius $\sqrt{-\mu a/E}$. Figure 2 shows some examples of Ω for $\mu/E = -1$.

4.3 Hydrodynamic quadrupole with four electric point sources

Consider an equilibrium Hele-Shaw free boundary flow with a hydrodynamic quadrupole of strength δ at $z = 0$ with point charges outside the fluid domain having strengths $-E$ at $z = \pm a$ and E at $z = \pm ib$ ($a, b \in \Re$ and $a, b > 0$). The external potential is

$$\Psi = -\frac{E}{4\pi} \log\frac{(z^2-a^2)(g^2-a^2)}{(z^2+b^2)(g^2+b^2)}. \tag{4.11}$$

For a quadrupole, $w \to \delta/2\pi z^2$ as $z \to 0$, where δ is a real constant. Proceeding as in Section 4.2

$$z = a\sqrt{\frac{1-\exp\left(-\dfrac{2\delta\zeta^2}{E z_\zeta(0)^2}\right)}{1+\dfrac{a^2}{b^2}\exp\left(-\dfrac{2\delta\zeta^2}{E z_\zeta(0)^2}\right)}}. \tag{4.12}$$

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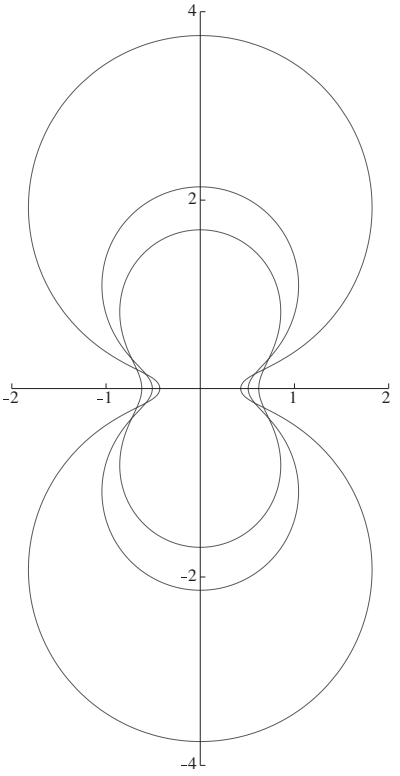


FIGURE 2. Free boundary shapes given by (4.9) for a hydrodynamic dipole of strength μ at $z = 0$ and electric point sources of strength $\pm E$ at $z = \pm a$, for $\mu/E = -1$ and $a = 0.48$ (largest), 0.59 and 0.76 (smallest).

As $\zeta \to 0$

$$z \to \frac{ab}{\sqrt{a^2+b^2}}\sqrt{\frac{2\delta}{E}}\frac{\zeta}{z_\zeta(0)}, \tag{4.13}$$

and so $z_\zeta(0)^2 = (ab/\sqrt{a^2+b^2})\sqrt{2\delta/E}$. Substituting for $z_\zeta(0)^2$ in (4.12) gives

$$z = a\sqrt{\frac{1-\exp\left(-\sqrt{\frac{2\delta}{E}}\frac{\sqrt{a^2+b^2}}{ab}\zeta^2\right)}{1+\frac{a^2}{b^2}\exp\left(-\sqrt{\frac{2\delta}{E}}\frac{\sqrt{a^2+b^2}}{ab}\zeta^2\right)}}. \tag{4.14}$$

In the limit $b \to \infty$, the solution reduces to that of [7] for a quadrupole with two same-signed electric sources outside the blob on the real axis (noting their parameter $\beta = 2\delta$). Figure 3 shows two examples of the blob shape, one of which corresponds to the $b \to \infty$ case.

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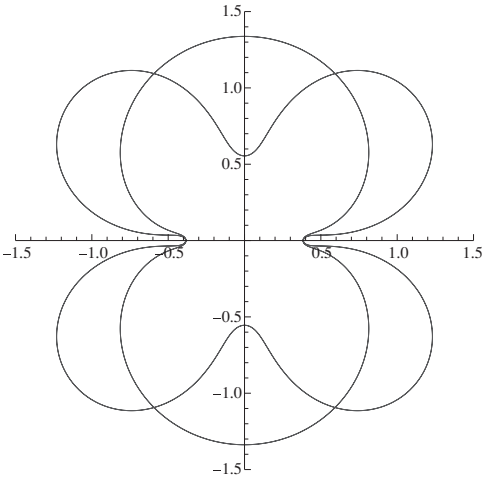


FIGURE 3. Free boundary shapes given by (4.14) for a hydrodynamic quadrupole of strength δ at $z = 0$ and electric point sources of strength $-E$ at $z = \pm a$ and E at $z = \pm ib$, for $2\delta/E = 1$ and $a = 0.4$ and $b = 0.6$ (the four 'petal' structure) and $b = \infty$.

4.4 Hydrodynamic N-pole with a superimposed electric point source

This example generalises that of Example 2 in [7]. Consider the map from the unit ζ-disk to the z-plane:

$$z = A\zeta \exp(B\zeta^n), \tag{4.15}$$

where A and B are real constants and n is positive integer. A hydrodynamic N-pole of strength μ lies at $z = 0$, i.e. $w \to \mu/2\pi z^N$ as $z \to 0$ ($N = 1, 2, \ldots$, where $N = 1$ is a dipole, $N = 2$ a quadrupole etc.) For steady flow with an electric source of strength E at the origin satisfies $\partial w/\partial z = \partial \Psi/\partial z$, where

$$\Psi = \frac{E}{4\pi} \log(zg), \tag{4.16}$$

as $z \to 0$. Note that $zg = A^2 \exp(B(\zeta^n + \zeta^{-n}))$ and therefore $\log(zg) \to B\zeta^{-n}$ as $z \to 0$. Hence,

$$\frac{\mu}{2\pi A^N \zeta^N} = \frac{EB}{4\pi \zeta^n}. \tag{4.17}$$

It follows from (4.17) $n = N$ and $B = 2\mu/EA^N$, and hence, (4.15) gives a one parameter (the parameter being A) family of solutions. Figure 4 gives examples for $N = 2$ and $N = 3$.

5 Remarks

Equation (2.9) for the Schwarz function of the free boundary governing the evolution of the fluid domain in Hele-Shaw cell under the combined action of hydrodynamic forcing and an arbitrary external potential field has been derived. The equation provides another route to tackling such free boundary problems, complementing other approaches based on for example, the moment evolution equations and the evolution equation for the Cauchy transform of the fluid domain.

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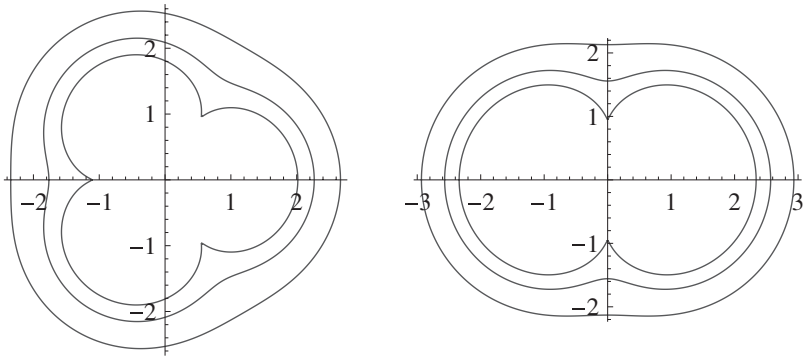


FIGURE 4. Free boundary shapes given by (4.15) for a hydrodynamic N-pole of strength μ at $z = 0$ and electric point source of strength E at $z = 0$. Here, $E/\mu = 2$ and $a = 1.5$ (smallest blob), 2 and 2.5 (largest blob) in both figures. $N = 3$ on the left and $N = 2$ on the right.

Practical use of (2.9) is demonstrated by re-deriving some known steady and un-steady exact solutions for external potentials, which are both continuous and singular. For example, the fact that a bubble propagating steadily in an unbounded Hele-Shaw cell is necessarily an ellipse with aspect ratio quantitatively related to its speed (the 'Taylor–Saffman' bubble) is particularly simple to derive using the Schwarz function equation.

Some new solutions are also derived that variously describe equilibrium flows with higher order hydrodynamic singularities in the presence of electric point sources and an un-steady solution describing bubbles under the combined influence of strain and centrifugal potential.

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Appendix Relation to moment and Cauchy transform methods

Here, it is demonstrated that the Schwarz function equation (2.9) can be used to derive the moment evolution equations of [8] for both centrifugal and gravitational potentials. Also, it is shown that for the specific case of the centrifugal potential, (2.9) is equivalent to the evolution of the Cauchy transform of $\Omega(t)$ as used in [3].

For finite Ω define the set of moments M_k, $k = 0, 1, 2, \ldots$, by

$$\begin{aligned} M_k &= \iint_{\Omega} z^k dA, \\ &= \frac{1}{2i} \oint_{\partial\Omega} z^k g(z,t) dz, \end{aligned} \tag{A 1}$$

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where the complex form of Green's theorem has been used and the fact that $\bar{z} = g(z,t)$ on $\partial\Omega$. Differentiating (A 1) with respect to time and using (2.9) and (2.1)

$$\begin{aligned}\frac{dM_k}{dt} &= \frac{1}{2i}\oint_{\partial\Omega} z^k \frac{\partial g}{\partial t} dz,\\ &= \frac{1}{i}\oint_{\partial\Omega} z^k \left(\frac{\partial w}{\partial z} - \frac{\partial \Psi}{\partial z}\right) dz,\\ &= \sum_{j=1}^{N} \frac{Q_j}{2\pi i}\oint_{\partial\Omega} \frac{z^k}{z - z_j} dz - \frac{1}{i}\oint_{\partial\Omega} z^k \frac{\partial \Psi}{\partial z} dz,\\ &= \sum_{j=1}^{N} Q_j z_j^k + \frac{k}{i}\oint_{\partial\Omega} z^{k-1}\Psi\, dz. \end{aligned} \tag{A 2}$$

For the centrifugal potential, $\Psi = \omega z g/2$ on $\partial\Omega$ and (A 2) becomes

$$\begin{aligned}\frac{dM_k}{dt} &= \sum_{j=1}^{N} Q_j z_j^k + \frac{\omega k}{2i}\oint_{\partial\Omega} z^k g\, dz,\\ &= \sum_{j=1}^{N} Q_j z_j^k + \omega k M_k, \end{aligned} \tag{A 3}$$

which is derived in [8] directly from (2.1). A similar derivation, with an appropriate re-definition of the moments, applies for bubbles, i.e. when the fluid domain Ω is infinite. For a uniform gravitational field, $\Psi = U(z + g)/2$ on $\partial\Omega$, and (A 2) becomes

$$\frac{dM_k}{dt} = \sum_{j=1}^{N} Q_j z_j^k + UkM_{k-1}, \tag{A 4}$$

as also given in [8].

The Cauchy transform of $\Omega(t)$ is

$$C(z,t) = \frac{1}{\pi}\iint_{\Omega(t)} \frac{dx'dy'}{z' - z}, \tag{A 5}$$

where z is outside $\Omega(t)$. It is known (e.g. [16]) that

$$g(z,t) = C_i(z,t) - C(z,t), \tag{A 6}$$

where $C_i(z,t)$ is an analytic function in $\Omega(t)$. It follows that in $\Omega(t)$, $g(z,t)$ has the same singularity structure as $C(z,t)$. In the case of a centrifugal potential with no hydrodynamic singularities [3] shows that $C(z,t)$ satisfies the evolution equation (see (18) in [3])

$$\frac{\partial C(z,t)}{\partial t} + \omega \frac{\partial z C(z,t)}{\partial t} = 0. \tag{A 7}$$

Comparing (A 7) with (3.1) with vanishing complex potential shows that the singularities of $g(z,t)$ and $C(z,t)$ evolve in the same way, giving the same blob evolution.

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