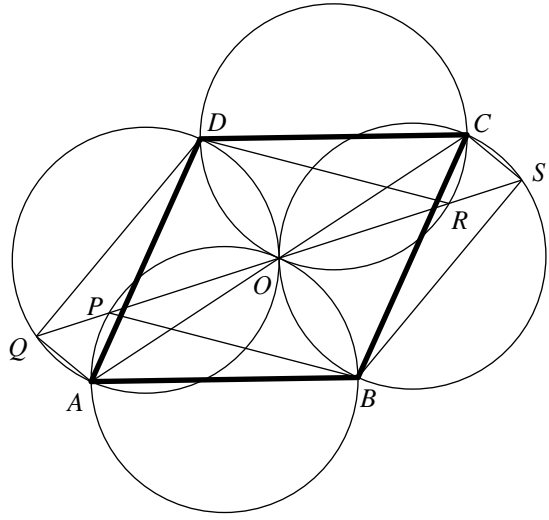
Feedback

On 102.11: Wil Ransome writes: This is an extension of a problem which has probably been around for a while. In the simple form with the square it appeared in *Mathematical Pie* 194 (Spring 2015). In his Pie Notes, Erick Gooding gave a solution that is very similar to the one given to the extension of the problem by Andrew Jobbings. This is a different proof of the result for a rhombus.



If you draw four circles on the sides of the rhombus they will cross at the centre O of the rhombus. Let $QPRS$ be any line through O meeting the circles as shown in the diagram. Note that the triangles ΔABP, ΔCBS, ΔCDR and ΔADQ are all right-angled, by angles in a semicircle.

By angles in the same segment and vertically opposite angles, we have

$$\angle CBS = \angle COS = \angle POA = \angle PBA$$

and so the triangles ΔCBS and ΔPBA are similar. Since $CB = BA$ they are also congruent, and it follows that all four triangles are congruent, as required.

On 102.12: Paul Stephenson writes: Yasuo Matsuda shows how, by setting up a particular parametric equation for the ellipse and specifying the radius of one circle within it, we can obtain the radii of the sequence of circles which complete the packing. By way of illustration he cites the particular case in which $n = 2k + 1$. The circles pack so that a diameter of the central circle coincides with the minor axis and the smallest circles touch the ends of the major axis with the greatest possible radius, namely b^2/a in the standard notation.

https://doi.org/10.1017/mag.2018.139 Published online by Cambridge University Press

Where $n = 2k$, so that two equal circles touch in the ellipse centre, we have, numbering radii r_i out from the centre:

$$a = 2\left(\sum_{i=1}^{k-1} r_i + \frac{b^2}{a}\right).$$

This is a quadratic in a, and, solving it, we obtain

$$a = \sum_{i=1}^{k-1} r_i \pm \sqrt{\left(\sum_{i=1}^{k-1} r_i\right)^2 + 2b^2}.$$

Only in the case $k = 1$, where the summation terms vanish, does a depend only on b, and we have $a = \sqrt{2}b$. But this special case is nice because of a dual relation between the ellipse and the circle. The radius at the ends of the minor axis is

$$\frac{a^2}{b} = \frac{(\sqrt{2}b)^2}{b} = 2b.$$

Two such ellipses therefore fit alongside in a circle of that radius. So the two questions below receive the same answer, namely $\sqrt{2}$:

1. What is the greatest $a : b$ ratio of the ellipse which accommodates two equal circles?
2. If a circle accommodates two congruent ellipses, what is the greatest $a : b$ ratio they can have?

Readers can confirm that the area ratio of the ellipse to the contained circles, and the circle to the contained ellipses, is also $\sqrt{2}$. As a result, the shaded and unshaded areas in both Figure 1(a) and Figure 1(b) are equal. This is consistent with the fact that we can transform the former into the latter by a one-way stretch parallel to the top of the page.

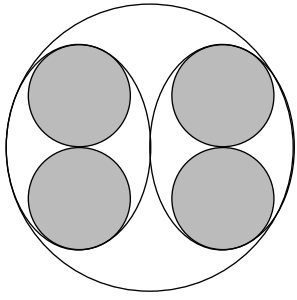


FIGURE 1(a)

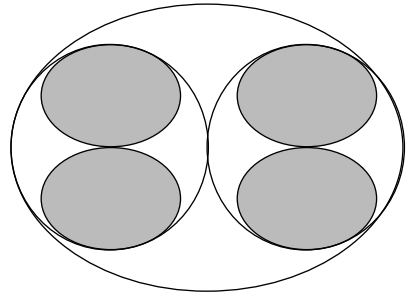


FIGURE 1(b)

This is of purely aesthetic interest, but readers will readily think of situations where one must solve the general problem of packing a fixed number n of circles in a given closed curve so that some measure is maximised.

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On 102.27: Graham Jameson writes: For two numbers a, b, the harmonic mean H is

$$\frac{2ab}{a+b} = \frac{G^2}{A}, \tag{1}$$

where $G = \sqrt{ab}$ and $A = \frac{1}{2}(a+b)$ are the geometric and arithmetic means, respectively. From (1), we get $AH = G^2$. The inequality $A + H \geqslant 2G$ is an instant consequence of this identity:

$$A(A + H - 2G) = A^2 + G^2 - 2AG = (A - G)^2 \geqslant 0.$$

(Clearly, $AH \geqslant G^2$ would be sufficient.) The same principle gives a very neat proof for the geometric progression $\left(1, x, x^2, \ldots, x^{n-1}\right)$. For this, we have $G = x^{(n-1)/2}$ and

$$\frac{n}{H} = 1 + \frac{1}{x} + \ldots + \frac{1}{x^{n-1}} = \frac{x^{n-1} + x^{n-2} + \ldots + 1}{x^{n-1}} = \frac{nA}{x^{n-1}}.$$

so that $AH = x^{n-1} = G^2$. No need even to sum the geometric series!

However, it is quite possible to have $A + H > 2G$ while $AH < G^2$. This occurs, for example, for $(1,\ 8,\ 8)$ (in which case $A = \frac{17}{3}$, $G = 4$ and $H = \frac{12}{5}$).

In view of the results in the Note, and this example, one might wonder whether there actually are any triples (a, b, c) for which $A + H < 2G$. Yes, $(1,\ 2,\ 2)$ is such a triple: then $A = \frac{5}{3}$ and $H = \frac{3}{2}$, so $A + H = \frac{19}{6}$, while $2G = 2 \times 4^{1/3} \approx 3.1748$.

The author also considers arithmetic progressions. The most natural one, $(1,\ 2,\ \ldots,\ n)$, lends itself to a pleasantly simple solution. For it, we have $A = \frac{1}{2}(n+1)$ and $H = n/H_n$, where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. Meanwhile, $G = (n!)^{1/n}$, which is greater than n/e (instant proof: from the exponential series, $e^n > n^n/n!$). So $A + H < 2G$ for all large enough n, more exactly for n satisfying $\frac{1}{2} + \frac{1}{H_n} + \frac{1}{2n} < \frac{2}{e}$. We only need to know that $H_n \to \infty$ as $n \to \infty$. In fact, one can easily check that $A + H < 2G$ already occurs for $n = 4$, as implied in the Note.

10.1017/mag.2018.139

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