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RESIDUAL STOCHASTIC PRECEDENCE ORDER

AMIT KUMAR MISRA
Department of Statistics, School of Physical & Decision Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow 226 025, India
E-mail: mishraamit31@gmail.com

VAISHALI GUPTA
Department of Statistics, School of Physical & Decision Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow 226 025, India
E-mail: vaishali.gupta3091@gmail.com

NEERAJ MISRA
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208 016, India
E-mail: neeraj@iitk.ac.in

The aim of this paper is to introduce a new stochastic order based on the residual lifetimes of two nonnegative dependent random variables and the stochastic precedence order. We develop some characterizations and preservation properties of this stochastic order. In addition, we study some of its reliability properties and its relation with other existing stochastic orders. One of the possible applications in reliability theory has also been discussed.

Keywords: joint stochastic orders, preservation, residual lifetime, univariate stochastic orders

1. INTRODUCTION

Let (X, Y) be a random vector with the Lebésgue probability density function (pdf) $f_{X,Y}(x, y)$, distribution function $F_{X,Y}(x, y)$, and survival function $\bar{F}_{X,Y}(x, y) = P(X > x, Y > y)$, $x \in \mathbb{R}$, $y \in \mathbb{R}$. For convenience of notation, we assume that the distributional support of the random vector (X, Y) is $[0, \infty) \times [0, \infty)$, although most of the results derived in this paper can be extended to situations where the support is different from $[0, \infty) \times [0, \infty)$.

In many situations, comparisons of the magnitudes of the random variables X and Y may be of interest and the goal may be to show that one random variable is larger than the other in some stochastic sense. A common approach to achieve this goal is the use of theory of stochastic orders that has been extensively studied in the literature. Stochastic orders are used to compare random variables in a more complex way when single measures, such as mean and variance, are not very informative. There are several stochastic orders defined in the literature. Some of them are the usual stochastic order, the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the stochastic precedence order (see [9,11,30,36] for the details of various stochastic orders). These orders have

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applications in many disciplines such as reliability theory, survival analysis, economics, actuarial science, operations research, epidemiology, and management science (see, for example, [3,4,11,20,24,29,30,38], and references cited therein). To make the paper comprehensive, let us recall the definitions of the stochastic orders used in the paper.

DEFINITION 1.1: *Let (X,Y) be a random vector with marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, survival functions $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$, probability density functions $f_X(\cdot)$ and $f_Y(\cdot)$, hazard rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$, and mean residual life functions $m_X(\cdot)$ and $m_Y(\cdot)$. Then X is said to be smaller than Y in the*

i. likelihood ratio order (written as $X \leq_{\mathrm{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}_+ \equiv [0,\infty)$;

ii. usual stochastic order (written as $X \leq_{\mathrm{st}} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x)$, $\forall x \in \mathbb{R}_+$;

iii. hazard rate order (written as $X \leq_{\mathrm{hr}} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $r_X(x) \geq r_Y(x), \forall x \in \mathbb{R}_+$;

iv. reversed hazard rate order (written as $X \leq_{\mathrm{rh}} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$, $\forall x \in (0,\infty)$; and

v. mean residual life order (written as $X \leq_{\mathrm{mrl}} Y$) if $\int_x^\infty \bar{F}_Y(t)\,dt / \int_x^\infty \bar{F}_X(t)\,dt$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $m_X(x) \leq m_Y(x), \forall x \in \mathbb{R}_+$.

The following relationships are well known [36]:

$$\begin{array}{ccccc} X\leq_{\mathrm{lr}}Y & \Rightarrow & X\leq_{\mathrm{hr}}Y & \Rightarrow & X\leq_{\mathrm{mrl}}Y \\ \Downarrow & & \Downarrow & & \\ X\leq_{\mathrm{rh}}Y & \Rightarrow & X\leq_{\mathrm{st}}Y & & \end{array}$$

Shanthikumar and Yao [37] pointed out that these univariate stochastic orders are basically defined with the help of marginal distributions of the random variables; therefore, any dependence between the random variables is no way reflected in the ordering of the random variables. For this purpose, many authors have introduced stochastic orders which take care of the dependence structure and are commonly known as *joint stochastic orders*. They also demonstrated applications of these orders in many areas, more specifically, analysis of random utility models, allocation of redundant components, and portfolio selection (see, for example, [1,5,7,8,10,21,22,34,37]). We recall the following definitions of some of the joint stochastic orders.

DEFINITION 1.2: *Given a random vector (X,Y), X is said to be smaller than Y in the*

i. joint likelihood ratio order (written as $X \leq_{\mathrm{lr:j}} Y$) if $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ [37];

ii. joint hazard rate order (written as $X \leq_{\mathrm{hr:j}} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x)$ is decreasing in x, whenever $0 \leq x \leq y$ [37];

iii. weak joint hazard rate order (written as $X \leq_{\mathrm{hr:wj}} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ [10];

iv. joint weak reversed hazard rate order (written as $X \leq_{\mathrm{rh:wj}} Y$) if $F_{X,Y}(x,y) - F_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ [5]; and

v. joint stochastic order (written as $X \leq_{\mathrm{st:j}} Y$) if $(X,-Y) \leq_{\mathrm{st}} (Y,-X)$, that is, $Eg(X,Y) \leq Eg(Y,X)$, $\forall g \in \{g(x,y) : g(x,y) - g(y,x) \text{ is increasing in } x, \forall x\}$ [37].

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The following relationships are well known [10]:

$$\begin{array}{ccccc} X \leq_{\text{lr:j}} Y & \Rightarrow & X \leq_{\text{hr:j}} Y & \Rightarrow & X \leq_{\text{st:j}} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq_{\text{hr:wj}} Y & \Rightarrow & X \leq_{\text{st}} Y \end{array}$$

One can also show that $X \leq_{\text{lr:j}} Y \Rightarrow X \leq_{\text{rh:wj}} Y$. It is worth mentioning that, if X and Y are independent, then the joint likelihood ratio order, the joint hazard rate (weak joint hazard rate) order, the joint weak reversed hazard rate order, and the joint stochastic order are equivalent to the likelihood ratio order, the hazard rate order, the reversed hazard rate order, and the usual stochastic order, respectively. However, for dependent X and Y, Shanthikumar and Yao [37] have shown that there is no implication between $X \leq_{\text{lr:j}} [\leq_{\text{hr:j}}] Y$ and $X \leq_{\text{lr}} [\leq_{\text{hr}}] Y$. Recently, Pellerey and Zalzadeh [33] provided conditions under which $X \leq_{\text{lr:j}} [\leq_{\text{hr:j}}] Y$ implies $X \leq_{\text{lr}} [\leq_{\text{hr}}] Y$. The authors also provided conditions under which $X \leq_{\text{st}} [\leq_{\text{hr}}, \leq_{\text{lr}}] Y$ implies $X \leq_{\text{st:j}} [\leq_{\text{hr:j}}, \leq_{\text{lr:j}}] Y$. Belzunce et al. [10] provided some conditions, on the joint density of X and Y, under which $X \leq_{\text{hr:j}} Y$ and $X \leq_{\text{hr:wj}} Y$ are equivalent. They also provided conditions under which $X \leq_{\text{hr}} [\leq_{\text{hr:wj}}] Y$ implies $X \leq_{\text{hr:wj}} [\leq_{\text{hr}}] Y$. Balakrishnan et al. [5] provided conditions under which $X \leq_{\text{rh}} [\leq_{\text{rh:wj}}] Y$ implies $X \leq_{\text{rh:wj}} [\leq_{\text{rh}}] Y$.

Note that in the above definitions of the joint stochastic orders, we have $P(X = Y) = 0$. For the case when $P(X = Y)$ may take a positive value, Boland et al. [11] gave the definition of the stochastic precedence order. According to their definition, X is said to be smaller than Y in the stochastic precedence order (written as $X \leq_{\text{sp}} Y$) if $P(X < Y) \geq P(Y < X)$. In the case of (X, Y) having a Lebésgue pdf, this definition reduces to that of Arcones et al. [3], that is, X is said to be smaller than Y in the stochastic precedence order if $P(X \leq Y) \geq \frac{1}{2}$.

It is well known in the literature that $X \leq_{\text{hr:wj}} [\leq_{\text{st:j}}] Y$ implies $X \leq_{\text{sp}} Y$ [34]. In general, there is no relationship between $X \leq_{\text{st}} Y$ and $X \leq_{\text{sp}} Y$. However, if X and Y are independent, then $X \leq_{\text{st}} Y$ implies $X \leq_{\text{sp}} Y$ [11]. It is worth mentioning that Mi [23] and Valdés et al. [39] use the term "probabilistic relation" (written as $\leq_{\text{pr}}$) instead of the stochastic precedence order.

Now consider that X and Y denote the random lifetimes of two components/systems. Then, instead of comparing X and Y, many times it may be of interest to compare the respective residual lifetimes, say $X_t = [X - t | X > t, Y > t]$ and $Y_t = [Y - t | X > t, Y > t]$, $t \geq 0$, due to the interest in comparing the lifetimes of the components/systems beyond time t [10]. Note that when X and Y are independent, $X_t = [X - t | X > t]$ and $Y_t = [Y - t | Y > t]$, $t \geq 0$, and their comparisons are of interest (see, for example, [15,16]; and references cited therein). For more details about the residual lifetime of a random variable, see Cai and Zheng [12], Guess and Proshcan [14], Nanda et al. [32], and Shaked and Shanthikumar (Chapters 1 and 2) [36].

Certain comparisons of X_t and Y_t are related to the well-defined stochastic orders. For example, when X and Y are independent, $X_t \leq_{\text{st}} Y_t$ for all $t \geq 0$ is equivalent to say that $X \leq_{\text{hr}} Y$. In fact, some joint stochastic orders between X and Y are also related to the certain comparisions between X_t and Y_t. Belzunce et al. [10] have shown that:

i. $X \leq_{\text{hr:wj}} Y \Leftrightarrow X_t \leq_{\text{st}} Y_t$, $t \geq 0$;
ii. $X \leq_{\text{hr:wj}} Y \Leftrightarrow X_t \leq_{\text{hr:wj}} Y_t$, $t \geq 0$;
iii. $X \leq_{\text{hr:j}} Y \Leftrightarrow X_t \leq_{\text{hr:j}} Y_t$, $t \geq 0$; and
iv. $X \leq_{\text{hr:j}} Y \Leftrightarrow X_t \leq_{\text{st:j}} Y_t$, $t \geq 0$.

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Motivated from this, in this paper, we define (and study) the "residual stochastic precedence order" by comparing X_t and Y_t with respect to the stochastic precedence order. The definition is as follows.

DEFINITION 1.3: *X is said to be smaller than Y in the residual stochastic precedence order (written as $X \leq_{\text{rsp}} Y$) if $P(X_t < Y_t) \geq P(Y_t < X_t)$ for all $t \geq 0$.*

It is worth mentioning that the above definition does not require (X, Y) to have a Lebésgue pdf. When (X, Y) has a Lebésgue pdf, this reduces to the definition of the residual probability order (denoted by $\leq_{\text{rpr}}$) defined by Zardasht and Asadi [40]. They studied this order for the case when X and Y are independent random variables and derived some interesting results. Kayid et al. [19] further studied the residual probability order and derived some results on characterizations and implications of the order. They, too, derived the results under the assumption that random variables X and Y are independent.

It is clear from Definition 1.3 that $X \leq_{\text{rsp}} Y$ is equivalent to say that $X_t \leq_{\text{sp}} Y_t$ for all $t \geq 0$. Since at time $t = 0$, the residual lifetimes of X and Y are the same as X and Y, respectively, it follows, on taking $t = 0$, that $X \leq_{\text{rsp}} Y$ implies $X \leq_{\text{sp}} Y$. Clearly, the converse of this implication may not always be true (see Example 2.2 in the sequel). However, we will show, in Theorem 2.2, that $X \leq_{\text{sp}} Y$ may imply $X \leq_{\text{rsp}} Y$ under certain conditions. In this paper, we discuss various properties of the residual stochastic precedence order.

The rest of the paper is organized as follows. Section 2 is devoted to some results on characterizations and implications of the residual stochastic precedence order. In Section 3, the preservation properties of this order under some reliability operations, such as monotone transformation, convolution, and mixture, have been derived. Finally, one possible application of the residual stochastic precedence order in reliability theory is given in Section 4.

Throughout this paper, we use the following notation. The terms "increasing" and "decreasing" are used in the non-strict sense. For any random variable W and an event E, we use $[W|E]$ to represent a random variable whose distribution is same as the conditional distribution of W given E. For any random variable W, we use $F_W(\cdot)$ and $\bar{F}_W(\cdot)$ to denote, respectively, the distribution function and the survival function of W. For a random variable having a Lebésgue pdf, we use $f_W(\cdot)$, $r_W(\cdot)$, $\tilde{r}_W(\cdot)$, and $m_W(\cdot)$ to denote the probability density function, the hazard rate function, the reversed hazard rate function, and the mean residual life function of W, respectively. Similarly, for a random vector (W_1, W_2) having a Lebésgue pdf, we use $f_{W_1,W_2}(\cdot,\cdot)$ to denote the joint density of (W_1, W_2). The support of all the random variables, if explicitly not mentioned, is assumed to be $\mathbb{R}_+ \equiv [0, \infty)$. While referring to the previous works, we use $\leq_{\text{rsp}}$ for $\leq_{\text{rpr}}$.

2. CHARACTERIZATIONS AND IMPLICATIONS

The aim of this section is to focus on the relationships between the residual stochastic precedence order and the other existing well-known stochastic orders. Some characterization results are also obtained. Throughout this section, we assume that (X, Y) has the Lebésgue pdf, $f_{X,Y}(\cdot,\cdot)$ in $\mathbb{R}^2$.

The following theorem describes the equivalent conditions for the residual stochastic precedence order. The proof is trivial and hence omitted.

THEOREM 2.1: *The following statements are equivalent:*

i. $X \leq_{\text{rsp}} Y$.

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ii. $P(t < X < Y) - P(t < Y < X) \geq 0$, for all $t \geq 0$.

iii. $\int_t^\infty \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy\, dx \geq 0$, *for all* $t \geq 0$.

The following corollary directly follows from the above theorem.

COROLLARY 2.1: *If* $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy \geq 0,\ \forall x \geq 0$, *then* $X \leq_{\text{rsp}} Y$.

Corollary 2.1 motivates us to introduce a new joint hazard rate order based on the joint distribution of random variables. The definition is as follows.

DEFINITION 2.1: *X is said to be smaller than Y in the joint hazard rate order (written as $X \leq_{\text{jhr}} Y$) if* $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy \geq 0,\ \forall x \geq 0$.

Belzunce et al. [10] have proved that if $X \leq_{\text{hr:wj}} Y$, then $\int_x^\infty f_{X,Y}(x,z)\, dz \geq \int_x^\infty f_{X,Y}(z,x)\, dz$ for all $x \geq 0$. Using this, it is direct to see that $X \leq_{\text{hr:wj}} Y$ implies $X \leq_{\text{jhr}} Y$. Also, using Definition 2.1 and Corollary 2.1, we get that $X \leq_{\text{jhr}} Y$ implies $X \leq_{\text{rsp}} Y$. Thus,

$$X \leq_{\text{hr:wj}} Y \Rightarrow X \leq_{\text{jhr}} Y \Rightarrow X \leq_{\text{rsp}} Y,$$

which, in turn, implies that $X_t \leq_{\text{st}} Y_t \Rightarrow X_t \leq_{\text{sp}} Y_t, \forall t \geq 0$. Note that the joint hazard rate order is weaker than the weak joint hazard rate order and is stronger than the residual stochastic precedence order. It is also important to note that $X \leq_{\text{jhr}} Y$ and $X \leq_{\text{hr}} Y$ are equivalent when X and Y are independent. We are working on the properties of the joint hazard rate order.

As we have seen that $X \leq_{\text{jhr}} Y$ implies $X \leq_{\text{rsp}} Y$, one may ask whether the converse also holds. The following counterexample gives the answer in negation.

EXAMPLE 2.1: *Let*

$$f_{X,Y}(x,y) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha) x e^{-x^2} e^{-y/2} & \text{if } x \geq 0,\ y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ *is an arbitrary fixed number. Then, for* $x \geq 0$,

$$\begin{aligned} \Delta_1(x) &= \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy \\ &= (1-\alpha) \left(x e^{-x^2} \int_x^\infty e^{-y/2}\, dy - e^{-x/2} \int_x^\infty y e^{-y^2}\, dy \right) \\ &= \frac{1}{2}(1-\alpha)(4x-1) e^{-(x^2 + (x/2))}. \end{aligned}$$

One can easily see that $\Delta_1(x) > 0$ *for* $x > \frac{1}{4}$ *and* $\Delta_1(x) < 0$ *for* $x < \frac{1}{4}$. *Thus, from Definition 2.1, we conclude that neither* $X \leq_{\text{jhr}} Y$ *nor* $Y \leq_{\text{jhr}} X$. *Now, for* $t \geq 0$, *let*

$$\begin{aligned} \Delta_2(t) &= \int_t^\infty \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy\, dx \\ &= \int_t^\infty \frac{1}{2}(1-\alpha)(4x-1) e^{-(x^2 + (x/2))}\, dx. \end{aligned}$$

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Our aim is to show that $\Delta_2(t) \geq 0,\ \forall t \geq 0$. *It is easy to see that* $\Delta_2'(t) = -\frac{1}{2}(1-\alpha)(4t-1)e^{-(1/2)(2t^2+t)},\ t \geq 0$. *Clearly,* $\Delta_2'(t) > 0$ *for* $t < \frac{1}{4}$ *and* $\Delta_2'(t) < 0$ *for* $t > \frac{1}{4}$, *which implies that* $\Delta_2(t)$ *is increasing for* $t < \frac{1}{4}$ *and is decreasing for* $t > \frac{1}{4}$. *Also,* $\lim_{t\to\infty} \Delta_2(t) = 0$. *Then, it is sufficient to show that* $\Delta_2(0) \geq 0$. *We have*

$$
\begin{aligned}
\Delta_2(0) &= (1-\alpha)\int_0^\infty \frac{1}{2}(4x-1)e^{-(1/2)(2x^2+x)}\,dx \\
&= (1-\alpha)\int_0^\infty \frac{1}{2}[(4x+1)e^{-(1/2)(2x^2+x)} - 2e^{-(1/2)(2x^2+x)}]\,dx \\
&= (1-\alpha)\left[\int_0^\infty \frac{1}{2}(4x+1)e^{-(1/2)(2x^2+x)}\,dx - \int_0^\infty e^{1/16-(x+1/4)^2}\,dx\right] \\
&= (1-\alpha)\left[1 - e^{1/16}\int_{\sqrt{2}/4}^\infty \frac{1}{\sqrt{2}}\,e^{-(1/2)v^2}\,dv\right] \\
&= (1-\alpha)[1-\sqrt{\pi}\,e^{1/16}(1-\Phi(\sqrt{2}/4))] \quad (\text{where } \Phi(\cdot) \text{ is the cdf of } N(0,1)) \\
&= 0.3147(1-\alpha) \\
&> 0.
\end{aligned}
$$

Therefore, $\Delta_2(t) \geq 0,\ \forall t \geq 0$, *and on using Theorem 2.1, we have that* $X \leq_{\text{rsp}} Y$. *Thus, we conclude that* $X \leq_{\text{rsp}} Y$ *does not imply* $X \leq_{\text{jhr}} Y$.

As an immediate consequence of Theorem 2.1, we have the following corollary which is given in Zardasht and Asadi [40] as a remark.

COROLLARY 2.2: *Let* X *and* Y *be independent random variables. Then, the following statements are equivalent:*

i. $X \leq_{\text{rsp}} Y$.
ii. $\int_t^\infty [\bar{F}_Y(x)f_X(x) - \bar{F}_X(x)f_Y(x)]\,dx \geq 0$, *for all* $t \geq 0$.
iii. $\int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)[r_X(x) - r_Y(x)]\,dx \geq 0$, *for all* $t \geq 0$.

Zardasht and Asadi [40] have proved (in Theorem 6) that if X and Y are independent random variables with $X \leq_{\text{hr}} Y$, then $X \leq_{\text{rsp}} Y$; which directly follows from part (iii) of the above corollary. Zardasht and Asadi [40] have also shown, with the help of counterexample, that the converse of this result does not hold.

As we know that $X \leq_{\text{hr}} [\leq_{\text{rh}}] Y$ implies $X \leq_{\text{st}} Y$, one may be interested in the relationship between the usual stochastic order and the residual stochastic precedence order. Moreover, one may also be interested in the relationship between the reversed hazard rate order and the residual stochastic precedence order. The following example shows, for independent random variables, that the reversed hazard rate order does not imply the residual stochastic precedence order, and hence, the usual stochastic order also does not imply the residual stochastic precedence order. Since, for independent random variables, the usual

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stochastic order implies the stochastic precedence order, this example also shows that the stochastic precedence order does not imply the residual stochastic precedence order.

EXAMPLE 2.2: *Let X and Y be two independent random variables with $F_X(x) = 1 - \bar{F}_X(x) = 1 - e^{-x}$ and $F_Y(x) = 1 - \bar{F}_Y(x) = 1 - (1 + xe^{-x})e^{-x}$, $x \geq 0$, respectively. One can verify that $F_Y(x)/F_X(x)$ is increasing in $x \in (0, \infty)$, which implies that $X \leq_{\mathrm{rh}} Y$. It is also easy to verify that $r_X(x) = 1$ and $r_Y(x) = (1 + 2xe^{-x} - e^{-x})/(1 + xe^{-x})$, $x > 0$, respectively. Now, for $t \geq 0$, let*

$$\begin{aligned}\Delta_4(t) &= \int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)[r_X(x) - r_Y(x)]\,dx \\ &= \int_t^\infty e^{-3x}(1-x)\,dx \\ &= \frac{1}{3}e^{-3t}\left(\frac{2}{3} - t\right).\end{aligned}$$

Clearly, $\Delta_4(t) \geq 0$, if $0 \leq t \leq \frac{2}{3}$ and $\Delta_4(t) < 0$, if $t > \frac{2}{3}$. Therefore, on using Corollary 2.2, we get that $X \not\leq_{\mathrm{rsp}} Y$. Thus, $X \leq_{\mathrm{rh}} Y$ does not imply that $X \leq_{\mathrm{rsp}} Y$, and hence, $X \leq_{\mathrm{st}} Y$ also does not imply that $X \leq_{\mathrm{rsp}} Y$. Since, for independent random variables, the usual stochastic order implies the stochastic precedence order, we conclude that $X \leq_{\mathrm{sp}} Y$ also does not imply that $X \leq_{\mathrm{rsp}} Y$.

Now, with the help of the following counterexample, we show that, if X and Y are dependent random variables, then the usual stochastic order and the residual stochastic precedence order may exist simultaneously, but the mean residual life order (and hence the hazard rate order) may not exist.

EXAMPLE 2.3: *Let*

$$f_{X,Y}(x,y) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha)\lambda_1\lambda_2 e^{-\lambda_1 x}e^{-\lambda_2 y} & \text{if } x \geq 0,\ y \geq 0,\ \lambda_1 > \lambda_2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ is an arbitrary fixed number. For $x \geq 0$,

$$\begin{aligned}\Delta_5(x) &= \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \\ &= (1-\alpha)\lambda_1\lambda_2\left(e^{-\lambda_1 x}\int_x^\infty e^{-\lambda_2 y}\,dy - e^{-\lambda_2 x}\int_x^\infty e^{-\lambda_1 y}\,dy\right) \\ &= (1-\alpha)\,e^{-(\lambda_1+\lambda_2)x}(\lambda_1 - \lambda_2) \\ &> 0, \quad \forall x \geq 0.\end{aligned}$$

This implies that $X \leq_{\mathrm{jhr}} Y$, which further implies that $X \leq_{\mathrm{rsp}} Y$.

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One can easily verify that the marginal probability density functions and the survival functions of X and Y are given by

$$f_X(x) = \alpha e^{-x} + (1-\alpha)\lambda_1 e^{-\lambda_1 x}, \quad x \geq 0,\ \lambda_1 > 0,$$
$$f_Y(x) = \alpha e^{-x} + (1-\alpha)\lambda_2 e^{-\lambda_2 x}, \quad x \geq 0,\ \lambda_2 > 0,$$

and

$$\bar{F}_X(x) = \alpha e^{-x} + (1-\alpha)e^{-\lambda_1 x}, \quad x \geq 0,$$
$$\bar{F}_Y(x) = \alpha e^{-x} + (1-\alpha)e^{-\lambda_2 x}, \quad x \geq 0.$$

Clearly, X and Y are not independent random variables. Since $\lambda_1 > \lambda_2$, it follows that $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \geq 0$, that is, $X \leq_{\text{st}} Y$.

Now, we show that the mean residual life order (and hence the hazard rate order) between X and Y does not exist. For $\lambda_1 = 2$, $\lambda_2 = 1.5$, $\alpha = 0.5$, we have

$$m_X(x) = \frac{0.5e^{-x} + 0.25e^{-2x}}{0.5\,(e^{-x} + e^{-2x})} \quad \text{and} \quad m_Y(x) = \frac{0.5e^{-x} + 0.33e^{-1.5x}}{0.5\,(e^{-x} + e^{-1.5x})},$$

respectively. Then, it is easy to calculate that

$$m_X(1.5) = 0.9088, \quad m_Y(1.5) = 0.8909, \quad m_X(0.5) = 0.811, \quad \text{and} \quad m_Y(0.5) = 0.851,$$

which implies that $m_X(1.5) > m_Y(1.5)$ and $m_X(0.5) < m_Y(0.5)$. Thus, neither $m_X(x) \geq m_Y(x)$ nor $m_X(x) \leq m_Y(x)$ for all $x > 0$, which implies that neither $X \leq_{\text{mrl}} Y$ nor $Y \leq_{\text{mrl}} X$.

Remark 2.1: Zardasht and Asadi [40], in Remark 4, have mentioned that if the random variables X and Y are independent and if $F_Y(\cdot)$ is concave, then $X \leq_{\text{rsp}} Y$ implies $X \leq_{\text{mrl}} Y$. It can be easily verified that in Example 2.3, $F_Y(x) = 1 - \alpha e^{-x} - (1-\alpha)e^{-\lambda_2 x}$, $x \geq 0$, is concave and $X \leq_{\text{rsp}} Y$ but $X \not\leq_{\text{mrl}} Y$, which contradicts the remark of Zardasht and Asadi [40]. This is due to the dependence between X and Y. As a consequence, Theorem 7 of Zardasht and Asadi [40], which says that $X \leq_{\text{rsp}} Y \Leftrightarrow F_Y(X) \leq_{\text{mrl}} F_Y(Y)$ does not hold when X and Y are not independent.

Recall that we have mentioned in the introduction that $X \leq_{\text{rsp}} Y$ implies $X \leq_{\text{sp}} Y$, but $X \leq_{\text{sp}} Y$ does not imply $X \leq_{\text{rsp}} Y$ in general. One may be curious to know the conditions under which the stochastic precedence order implies the residual stochastic precedence. The following theorem provides such a condition.

THEOREM 2.2: *Let $X \leq_{\text{sp}} Y$. If $\int_x^\infty f_{X,Y}(x,y)\,dy / \int_x^\infty f_{X,Y}(y,x)\,dy$ is increasing in $x > 0$, then $X \leq_{\text{rsp}} Y$.*

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PROOF: Let

$$\Delta_5(t) = \int_t^\infty \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy\, dx, \quad t \geq 0.$$

To prove that $X \leq_{\text{rsp}} Y$, using Theorem 2.1, we have to show that $\Delta_5(t) \geq 0,\ \forall t \geq 0$, which is equivalent to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx} \geq 1, \quad \forall t \geq 0.$$

Since $X \leq_{\text{sp}} Y$ implies that $P(Y > X)/P(X > Y) \geq 1$, it is sufficient to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx} \geq \frac{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}, \quad \forall t \geq 0,$$

which is equivalent to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx} \geq \frac{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}, \quad \forall t \geq 0.$$

Let Z_1 and Z_2 be two random variables having probability density functions

$$f_{Z_1}(x) = \frac{\int_x^\infty f_{X,Y}(y,x)\, dy}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}, \quad x > 0$$

and

$$f_{Z_2}(x) = \frac{\int_x^\infty f_{X,Y}(x,y)\, dy}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}, \quad x > 0,$$

respectively. Then,

$$\frac{f_{Z_2}(x)}{f_{Z_1}(x)} = \frac{\int_x^\infty f_{X,Y}(x,y)\, dy}{\int_x^\infty f_{X,Y}(y,x)\, dy} \times \frac{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}.$$

Since $\int_x^\infty f_{X,Y}(x,y)\, dy / \int_x^\infty f_{X,Y}(y,x)\, dy$ is increasing in $x > 0$, it follows that $f_{Z_2}(x)/f_{Z_1}(x)$ is increasing in $x > 0$, and therefore, $Z_1 \leq_{\text{lr}} Z_2$, and hence, $Z_1 \leq_{\text{st}} Z_2$. Therefore, $\bar{F}_{Z_2}(t) \geq \bar{F}_{Z_1}(t),\ \forall t \geq 0$, that is,

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\, dy\, dx} \geq \frac{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\, dy\, dx}, \quad \forall t \geq 0.$$

Hence, the theorem follows. ■

The following corollary is an immediate consequence of Theorem 2.2.

COROLLARY 2.3: *Let X and Y be independent random variables with $X \leq_{\text{sp}} Y$. If $r_X(x)/r_Y(x)$ is increasing in $x > 0$, then $X \leq_{\text{rsp}} Y$.*

Remark 2.2: It is worth mentioning that the assumption that $r_X(x)/r_Y(x)$ is increasing in $x > 0$, or equivalently, that $r_Y(x)/r_X(x)$ is decreasing in $x > 0$, is related to the notion of the relative aging introduced by Kalashnikov and Rachev [18] (see, for more details, [17,26,27]; Remark 2.3 of [29] and [35]).

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3. PRESERVATION PROPERTIES

Under well-defined reliability operations, the preservation properties of stochastic orders have acquired noticeable attention in the literature of reliability theory (see, for example, [2,6,28]). In this section, we derive some preservation properties of the residual stochastic precedence order.

The following theorem, which strengthens Theorem 9 of Kayid et al. [19], proves that the residual stochastic precedence order is preserved under strictly increasing transformations.

THEOREM 3.1: *Let $X \leq_{\text{rsp}} Y$. If $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function, then $\phi(X) \leq_{\text{rsp}} \phi(Y)$.*

PROOF: For $t \geq 0$, let

$$\begin{aligned}\Delta_6(t) &= P(t < \phi(X) < \phi(Y)) - P(t < \phi(Y) < \phi(X)) \\ &= P(t^* < X < Y) - P(t^* < Y < X) \quad (\text{where } t^* = \phi^{-1}(t) \geq 0) \\ &\geq 0,\end{aligned}$$

where the last inequality follows from the assumption that $X \leq_{\text{rsp}} Y$ and Theorem 2.1. Thus, using Theorem 2.1, we conclude that $\phi(X) \leq_{\text{rsp}} \phi(Y)$. ■

It is worth mentioning that the above preservation property also holds for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see Theorems 1.B.2, 1.B.43, 1.C.8, and 2.A.19 in Ref. [36]).

Next, the result shows that the residual stochastic precedence order is preserved under convolution.

THEOREM 3.2: *Let X, Y, and Z be jointly distributed nonnegative random variables and let, for each $z \geq 0$, (X_z, Y_z) be a random vector having the same joint distribution as the conditional distribution of $(X, Y|Z = z)$. If $X_z \leq_{\text{rsp}} Y_z$, $\forall z \geq 0$, then $X + Z \leq_{\text{rsp}} Y + Z$.*

PROOF: For $t \geq 0$, let

$$\begin{aligned}\Delta_7(t) &= P(t < X + Z < Y + Z) - P(t < Y + Z < X + Z) \\ &= \int_0^\infty [P(t < X + Z < Y + Z|Z = z) - P(t < Y + Z < X + Z|Z = z)]\, dF_Z(z) \\ &= \int_0^\infty [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)]\, dF_Z(z) \\ &= \int_0^t [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)]\, dF_Z(z) \\ &\quad + \int_t^\infty [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)]\, dF_Z(z)\end{aligned}$$

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$$= \int_0^t [P(t-z < X_z < Y_z) - P(t-z < Y_z < X_z)]\, dF_Z(z)$$
$$+ \int_t^\infty [P(X_z < Y_z) - P(Y_z < X_z)]\, dF_Z(z)$$
$$\geq 0, \quad \forall t \geq 0,$$

where the nonnegativity of the first integral follows from the assumption that $X_z \leq_{\text{rsp}} Y_z$ and Theorem 2.1, and that of the second integral follows from the fact that $X_z \leq_{\text{rsp}} Y_z$ implies $X_z \leq_{\text{sp}} Y_z$. Since $\Delta_7(t) \geq 0$ for all $t \geq 0$, the result follows from Theorem 2.1. ■

The following corollary directly follows from the above theorem.

COROLLARY 3.1: *Let X, Y, and Z be jointly distributed nonnegative random variables such that (X, Y) is independent of Z. If $X \leq_{\text{rsp}} Y$, then $X + Z \leq_{\text{rsp}} Y + Z$.*

Note that the results similar to the above one also holds for the hazard rate order, the reversed hazard rate order, and the mean residual life order (see Lemmas 1.B.3, 1.B.44, and 2.A.8 in Ref. [36]).

In the following theorems, we discuss the preservation of the residual stochastic precedence order under mixtures.

THEOREM 3.3: *Consider a family of absolutely continuous survival functions $\{\bar{G}_\theta, \theta \in \mathfrak{X}\}$ of nonnegative random variables $\{X(\theta), \theta \in \mathfrak{X}\}$, where $\mathfrak{X}$ is a subset of the real line $\mathbb{R}$. Let Θ_1 and Θ_2 be two random variables with support $\mathfrak{X}$, and with distribution functions H_1 and H_2, respectively. Let Y_1 and Y_2 be two independent random variables and let the survival function of Y_i be given by*

$$\bar{F}_i(x) = \int_{\mathfrak{X}} \bar{G}_\theta(x)\, dH_i(\theta), \quad x \in \mathbb{R};\ i = 1, 2.$$

If

$$X(\theta_1) \leq_{\text{rsp}} X(\theta_2), \quad \textit{whenever } \theta_1 \leq \theta_2,$$

and $\Theta_1 \leq_{\text{lr}} \Theta_2$, then

$$Y_1 \leq_{\text{rsp}} Y_2.$$

PROOF: Let $\{g_\theta, \theta \in \mathfrak{X}\}$ be the family of probability density functions of $\{X(\theta), \theta \in \mathfrak{X}\}$. Then, the pdf of Y_i is given by

$$f_i(x) = \int_{\mathfrak{X}} g_\theta(x)\, dH_i(\theta), \quad x \in \mathbb{R};\ i = 1, 2.$$

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Let, for a fixed $t \geq 0$,

$$\begin{aligned}
\Delta_8(t) &= P(t < Y_1 < Y_2) - P(t < Y_2 < Y_1) \\
&= \iint\limits_{t<y_1<y_2} f_1(y_1) f_2(y_2)\, dy_1\, dy_2 - \iint\limits_{t<y_2<y_1} f_1(y_1) f_2(y_2)\, dy_1\, dy_2 \\
&= \iint\limits_{t<y_1<y_2} \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dH_1(\theta_1)\, dH_2(\theta_2)\, dy_1\, dy_2 \\
&\quad - \iint\limits_{t<y_2<y_1} \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dH_1(\theta_1) dH_2(\theta_2)\, dy_1\, dy_2 \\
&= \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} \left[\iint\limits_{t<y_1<y_2} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dy_1\, dy_2 - \iint\limits_{t<y_2<y_1} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dy_1\, dy_2 \right] dH_1(\theta_1)\, dH_2(\theta_2) \\
&= \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} [P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))]\, dH_1(\theta_1)\, dH_2(\theta_2) \\
&= E[\phi_2(\Theta_1, \Theta_2)] - E[\phi_1(\Theta_1, \Theta_2)],
\end{aligned}$$

where, for $\theta_1, \theta_2 \in \mathfrak{X}$,

$$\phi_1(\theta_1, \theta_2) = P(t < X(\theta_2) < X(\theta_1)) \text{ and } \phi_2(\theta_1, \theta_2) = P(t < X(\theta_1) < X(\theta_2)).$$

Let $\Delta\phi_{21}(\theta_1, \theta_2) = \phi_2(\theta_1, \theta_2) - \phi_1(\theta_1, \theta_2)$. Since $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$ for all $\theta_1 \leq \theta_2$, it follows from Theorem 2.1 that $\Delta\phi_{21}(\theta_1, \theta_2) \geq 0$ whenever $\theta_1 \leq \theta_2$. It is also easy to verify that $\Delta\phi_{21}(\theta_1, \theta_2) = -\Delta\phi_{21}(\theta_2, \theta_1)$ for all θ_1, θ_2. Now, on using Theorem 1.C.22 of Shaked and Shanthikumar [36], we have $\Delta_8(t) \geq 0$. Since $t \geq 0$ is arbitrary, we have $\Delta_8(t) \geq 0,\ \forall t \geq 0$. Now, on using Theorem 2.1, we conclude that $Y_1 \leq_{\text{rsp}} Y_2$. ■

It is worth mentioning that the above theorem generalizes Theorem 5 of Kayid et al. [19]. Further, note that Theorem 3.3 also holds for the usual stochastic order, the hazard rate order, the reversed hazard rate order, and the mean residual life order (see Theorems 1.A.6, 1.B.14, 1.B.52, 1.C.17, and 2.A.15 in Ref. [36]).

One may ask whether the assumption $\Theta_1 \leq_{\text{lr}} \Theta_2$, in the previous theorem, can be relaxed to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \Theta_2$. The following counterexample shows that this may not always be true.

EXAMPLE 3.1: *Let the survival function of $X(\theta)$ be $\bar{G}_\theta(x) = e^{-x/\theta}$, $\theta \geq 0$. For $\theta_1 \neq \theta_2$, assume that $X(\theta_1)$ and $X(\theta_2)$ are independent random variables. It is easy to see that $X(\theta_1) \leq_{\text{hr}} X(\theta_2)$ for all $\theta_1 \leq \theta_2$, which implies that $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$. Now, let Θ_1 and Θ_2 be two independent gamma random variables with respective probability density functions given by*

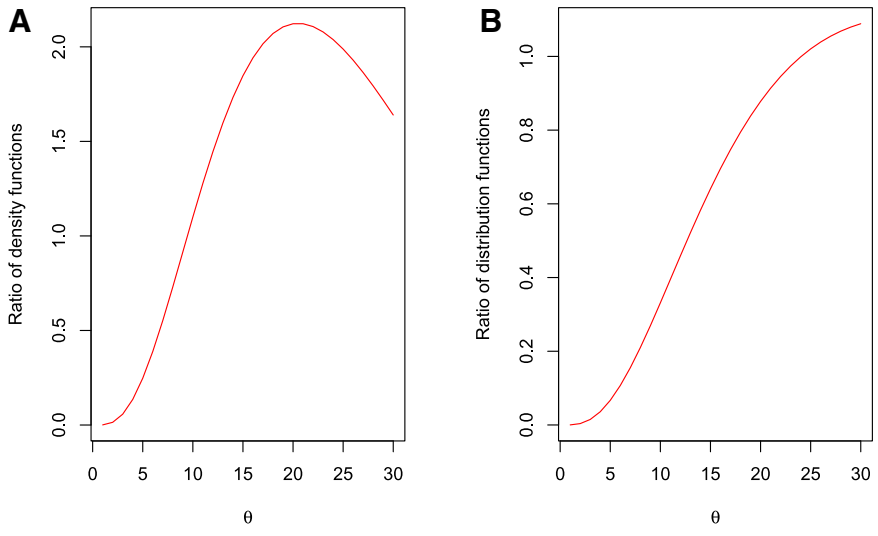
$$h_1(\theta) = \tfrac{1}{4} e^{-\theta/4}, \quad \theta > 0 \qquad \text{and} \qquad h_2(\theta) = \tfrac{1}{6}\theta^3 e^{-\theta}, \quad \theta > 0.$$

The distribution functions of Θ_1 and Θ_2 are given by the following equations:

$$H_1(\theta) = 1 - e^{-\theta/4}, \quad \theta \geq 0 \qquad \text{and} \qquad H_2(\theta) = \frac{6 - (\theta^3 + 3\theta^2 + 6\theta + 6)e^{-\theta}}{6}, \quad \theta \geq 0,$$

respectively. Then, it can be verified that $h_2(\theta)/h_1(\theta)$ is not a monotone function of $\theta \in [0, \infty)$ (see Figure 1(a)), which implies that $\Theta_1 \not\leq_{\text{lr}} \Theta_2$. It can also be verified that $H_2(\theta)/H_1(\theta)$ is increasing in $\theta \in [0, \infty)$ (see Figure 1(b)), which implies that $\Theta_1 \leq_{\text{rh}} \Theta_2$.

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Plot of $\frac{h_2(\theta)}{h_1(\theta)}$ Plot of $\frac{H_2(\theta)}{H_1(\theta)}$

FIGURE **1.** (a) Plot of $h_2(\theta)/h_1(\theta)$. (b) Plot of $H_2(\theta)/H_1(\theta)$.

Now, let Y_1 and Y_2 be two independent random variables as defined in Theorem 3.3, and let, for $t \geq 0$,

$$\begin{aligned}\Delta_9(t) &= P(t < Y_1 < Y_2) - P(t < Y_2 < Y_1) \\ &= \int_0^\infty \int_0^\infty [P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))] h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2.\end{aligned}$$

One can easily verify that

$$P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1)) = \frac{\theta_2 - \theta_1}{\theta_2 + \theta_1} e^{-(1/\theta_1 + 1/\theta_2)t}.$$

Therefore,

$$\Delta_9(t) = \frac{1}{24} \int_0^\infty \int_0^\infty \frac{\theta_2 - \theta_1}{\theta_2 + \theta_1}\, \theta_2^3\, e^{-(t/\theta_1 + \theta_1/4)} e^{-((t/\theta_2) + \theta_2)}\, d\theta_1\, d\theta_2.$$

Now, using the R-software, we obtain that $\Delta_9(0.5) = 0.01409676$ and $\Delta_9(1) = -0.009706716$, which implies that $\Delta_9(t) \not\geq 0$, $\forall t \geq 0$. Now, on using Theorem 2.1, we conclude that $Y_1 \not\leq_{\text{rsp}} Y_2$.

The following theorem provides conditions under which the assumption $\Theta_1 \leq_{\text{lr}} \Theta_2$ can be relaxed to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \Theta_2$.

THEOREM 3.4: *Consider the setup of Theorem 3.3. If, for each $\theta_1\, [\theta_2]$, $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))$ is increasing [decreasing] in $\theta_2 \in [\theta_1, \infty) \cap \mathfrak{X}$ $[\theta_1 \in (-\infty, \theta_2) \cap \mathfrak{X}]$, $\forall t \geq 0$ and $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \Theta_2$, then $Y_1 \leq_{\text{rsp}} Y_2$.*

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The proof of the above theorem can be written down using lines similar to those used in the proof of Theorem 3.3, and using Theorem 1.B.10 [1.B.48] of Shaked and Shanthikumar [36].

Remark 3.1: Observe that the assumption $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))$ is increasing [decreasing] in $\theta_2 \in [\theta_1, \infty) \cap \mathfrak{X}$ $[\theta_1 \in (-\infty, \theta_2) \cap \mathfrak{X}]$, $\forall t \geq 0$ implies that $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1)) \geq 0$, $\forall t \geq 0$, that is, $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$ for all $\theta_1 \leq \theta_2$. Thus, to relax $\Theta_1 \leq_{\text{lr}} \Theta_2$ to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \Theta_2$, an assumption stronger than $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$, is being used.

THEOREM 3.5: *Let X and Y be independent random variables having absolutely continuous survival functions $\bar{F}_X$ and $\bar{F}_Y$, respectively, and let W be a random variable, independent of X and Y, with the survival function $p\bar{F}_X + (1-p)\bar{F}_Y$ for some $p \in (0,1)$. If $X \leq_{\text{rsp}} Y$, then $X \leq_{\text{rsp}} W \leq_{\text{rsp}} Y$.*

PROOF: Clearly, the pdf of W is given by $pf_X + (1-p)f_Y$ for some $p \in (0,1)$. Now, first we show that $X \leq_{\text{rsp}} W$. For $t \geq 0$, we have

$$\begin{aligned}\Delta_{10}(t) &= \int_t^\infty [(p\bar{F}_X(x) + (1-p)\bar{F}_Y(x))f_X(x) - \bar{F}_X(x)(pf_X(x) + (1-p)f_Y(x))]\,dx \\ &= (1-p)\int_t^\infty [\bar{F}_Y(x)f_X(x) - \bar{F}_X(x)f_Y(x)]\,dx \\ &\geq 0, \quad \forall t \geq 0,\end{aligned}$$

where the last inequality follows from the assumption that $X \leq_{\text{rsp}} Y$ and Corollary 2.2. Since $\Delta_{10}(t) \geq 0$ for all $t \geq 0$, again using Corollary 2.2, it follows that $X \leq_{\text{rsp}} W$. Similarly, we can show that $W \leq_{\text{rsp}} Y$, and on combining the two results, we get that $X \leq_{\text{rsp}} W \leq_{\text{rsp}} Y$. ■

It is worth mentioning that the theorems similar to above one also holds for the hazard rate order, the likelihood ratio order, and the mean residual life order (see Theorems 1.B.22, 1.C.30, and 2.A.18 in Ref. [36]).

4. APPLICATION

In this section, we discuss an application of the residual stochastic precedence order in reliability theory.

Let X_1, X_2, and X_3 be nonnegative random variables. Consider the two series systems, each consisting of two components. Let the random lifetimes of two components of the first system be X_1 and X_2, and let that of the two components of the second system be X_1 and X_3. Then, the lifetimes of the two systems are given by $\wedge\{X_1, X_2\}$ and $\wedge\{X_1, X_3\}$, respectively, where the symbol " and" represents *minimum*. We are interested in the comparison of the lifetimes of the two systems with respect to the residual stochastic precedence order. The following theorem provides sufficient condition for the comparison.

THEOREM 4.1: *Let X_1, X_2, and X_3 be jointly distributed nonnegative random variables having a joint Lebésgue density, and let, for each $x \geq 0$, $(X_2^*(x), X_3^*(x))$ be a random vector*

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having the same joint distribution as the conditional distribution of $(X_2, X_3|X_1 = x)$. *If* $X_2^*(x) \leq_{\text{jhr}} X_3^*(x)$, $\forall x \geq 0$, *then* $\wedge\{X_1, X_2\} \leq_{\text{rsp}} \wedge\{X_1, X_3\}$.

PROOF: For $t \geq 0$, let

$$\begin{aligned}
\Delta_{11}(t) &= P(t < \wedge\{X_1, X_2\} < \wedge\{X_1, X_3\}) - P(t < \wedge\{X_1, X_3\} < \wedge\{X_1, X_2\}) \\
&= P(t < X_2 < X_1, t < X_2 < X_3) - P(t < X_3 < X_1, t < X_3 < X_2) \\
&= \int_t^\infty [P(t < X_2 < X_1, t < X_2 < X_3 | X_1 = x) \\
&\quad - P(t < X_3 < X_1, t < X_3 < X_2 | X_1 = x)] f_{X_1}(x)\, dx \\
&= \int_t^\infty [P(t < X_2^*(x) < x, t < X_2^*(x) < X_3^*(x)) \\
&\quad - P(t < X_3^*(x) < x, t < X_3^*(x) < X_2^*(x))] f_{X_1}(x)\, dx \\
&= \int_t^\infty \int_t^{x_1} \left[\int_{x_2}^\infty (f_{X_2^*(x), X_3^*(x)}(x_2, x_3) - f_{X_2^*(x), X_3^*(x)}(x_3, x_2))\, dx_3 \right] f_{X_1}(x)\, dx_2\, dx \\
&\geq 0, \quad \forall t \geq 0,
\end{aligned}$$

where the last inequality follows from the assumption that $X_2^*(x) \leq_{\text{jhr}} X_3^*(x)$, $\forall x \geq 0$. Now, on using Theorem 2.1, we conclude that $\wedge\{X_1, X_2\} \leq_{\text{rsp}} \wedge\{X_1, X_3\}$. ■

The following corollaries directly follow from the above theorem.

COROLLARY 4.1: *Let* X_1, X_2, *and* X_3 *be jointly distributed nonnegative random variables, having a joint Lebésgue density, such that* (X_2, X_3) *is independent of* X_1. *If* $X_2 \leq_{\text{jhr}} X_3$, *then* $\wedge\{X_1, X_2\} \leq_{\text{rsp}} \wedge\{X_1, X_3\}$.

COROLLARY 4.2: *Let* X_1, X_2, *and* X_3 *be nonnegative independent random variables having Lebésgue densities. If* $X_2 \leq_{\text{hr}} X_3$, *then* $\wedge\{X_1, X_2\} \leq_{\text{rsp}} \wedge\{X_1, X_3\}$.

It is worth mentioning that the similar kind of comparisons, with respect to different stochastic orders, are available in the literature (see, for example, [13,25]; and references cited therein).

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