6

Lovász Extensions

In this chapter, we introduce the theory of Lovász extensions as a powerful tool for studying eigenvalue problems and, in particular, Cheeger-type inequalities.

As a fundamental tool in discrete mathematics, the Lovász extension has been deeply connected to submodular analysis [59, 189] and has been applied in many areas such as combinatorial optimization, game theory, matroid theory, stochastic processes, electrical networks, computer vision, and machine learning [101]. There are many generalizations, such as the disjoint-pair Lovász extension and the Lovász extension on distributive lattices [101, 221]. Recent developments include quasi-Lovász extension on some algebraic structures and fuzzy mathematics, applications of Lovász extensions to graph cut problems and computer science, as well as Lovász-softmax loss in deep learning.

In this chapter, we mainly focus on the spectral theory regarding a pair of Lovász extensions, which offers a new perspective on the eigenvalue problems for discrete structures. The contents are largely taken from [159] and [160], which also provide convenient tools for Chapter 8 on Cheeger inequalities. However, here we also provide new examples and different formulations.

For convenience, in this chapter, we shall use slightly different notations than in the rest of the book. For instance, V will be used to denote a general set, which is not necessarily the vertex set of a graph, and its elements will be denoted by $1,\dots,N$. Also, the notation for oriented hypergraphs will be different.

6.1 Extensions and Eigenvalue Problems

Before embarking on that, however, let us describe the strategy in abstract terms. We had obtained eigenvalues and eigenfunctions as critical values and critical points of Rayleigh quotients, for instance, in Section 2.6.1,

$$\frac{(Au,u)}{(u,u)}, \tag{6.1.1}$$

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where u is in some Hilbert space with a scalar product $(\cdot,\cdot)$. In general, when we seek critical values λ and points x_0 of some quotient

$$\frac{F(x)}{G(x)}, \tag{6.1.2}$$

then, when F and G are differentiable, we obtain the equation

$$\nabla F(x_0) - \lambda \nabla G(x_0) = 0, \tag{6.1.3}$$

where

$$\lambda = \frac{F(x_0)}{G(x_0)} \tag{6.1.4}$$

is the critical value. In the case of (6.1.1), this leads to the eigenvalue equation

$$Au_0 = \lambda u_0. \tag{6.1.5}$$

Now, in the Cheeger-type inequalities, we had compared these eigenvalues, that is, the critical values of the Rayleigh quotient (6.1.1), with the extremal values of other quotients $f(S)/g(S)$ that were defined combinatorially, for $S \in \mathcal{P}(V)$, where V is the set of vertices of some geometric structure. The idea that we shall pursue in this chapter is to extend f and g also to functions on some Euclidean space and translate the combinatorial optimization problem into one on that Euclidean space that we can approach with the tools of calculus. If V has N elements $v_1,\dots,v_N$, we can identify $S = \{v_{i_1},\dots,v_{i_k}\} \in \mathcal{P}(V)$ with the vector in $\mathbb{R}^N$ whose j-th component is 1 if $j \in \{i_1,\dots,i_k\}$ and 0 else, that is, a vertex of the unit cube in $\mathbb{R}^N$. Thus, the functions f and g are naturally defined on those vertices of the unit cube, and we can then extend them to all of $\mathbb{R}^N$. Of course, many extensions are possible, but we should take one that, on one hand, translates qualitative properties of the discrete functions f, g into suitable qualitative properties of their extensions and, on the other hand, is amenable to continuous optimization tools. That will be achieved by the Lovász extension. It will extend f and g to certain piecewise linear functions. These functions in general will not be smooth, but only Lipschitz, and so, optimization will require some extension of the usual differentiable calculus; see, for instance, [88, 239]. The required concepts have already been introduced at the end of Section 2.6.2. We equip $\mathbb{R}^N$ with its Euclidean product $\langle\cdot,\cdot\rangle$. Recalling Definition 2.6.4, a function $F\colon \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is *subdifferentiable* at $\mathbf{x}$ if there exists some affine linear

$$L(\mathbf{y}) = \langle \mathbf{y} - \mathbf{x}, \mathbf{x}^* \rangle + F(\mathbf{x}) \tag{6.1.6}$$

$$\text{with } L(\mathbf{z}) \le F(\mathbf{z}) \text{ for all } \mathbf{z}, \tag{6.1.7}$$

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and the set of all $\mathbf{x}^*$ with this property is called the *subgradient*, which will be denoted by $\nabla F(\mathbf{x})$. Of course, when F is not subdifferentiable at $\mathbf{x}$, this set is empty. Thus, the subgradient of F at $\mathbf{x}$, if it exists, consists of the slopes of all affine linear functions that are not larger than F and agree with F at $\mathbf{x}$. As such, this is a global condition, as the inequality is required for all $\mathbf{z} \in \mathbb{R}^N$, but one can also conceive a local version. For our purposes, however, the global condition will suffice. In this setting, the eigenvalue equation (6.1.3) will become

$$0 \in \nabla F(\mathbf{x}_0) - \lambda \nabla G(\mathbf{x}_0). \tag{6.1.8}$$

Later, we shall need the following easy result whose proof we omit.

Lemma 6.1.1. *For a piecewise linear function $F : \mathbb{R}^N \to \mathbb{R}$ with finite pieces, if F is linear on a convex subset Ω, then $\nabla F(\mathbf{x}) \subset \nabla F(\mathbf{y})$ for any relative interior point $\mathbf{x}$ of Ω and any relative boundary point $\mathbf{y}$ of $\overline{\Omega}$.*

6.2 Definitions and Basic Properties of Lovász Extensions

In this section, we present the various definitions and important properties of Lovász extensions. We shall start by looking at the original Lovász extension. For simplicity, we shall work throughout this chapter with a finite and nonempty set $V = \{1, \dots, N\}$ and its power set $\mathcal{P}(V)$. We identify every $A \in \mathcal{P}(V)$ with its indicator vector $\mathbf{1}_A \in \{0,1\}^N \subset \mathbb{R}^N$. These indicator vectors can be identified with the vertices of the unit cube $[0,1]^N$ in $\mathbb{R}^N$. A function $f : \mathcal{P}(V) \to \mathbb{R}$ with $f(\emptyset) = 0$ then can be considered as a function on those indicator vectors, that is, on the vertices of the unit cube, and we can then extend its domain to all of $\mathbb{R}^N$. Of course, some prescription is necessary for having an extension that, on one hand, preserves qualitative properties of f and, on the other hand, possesses a simple and convenient mathematical structure. In particular, we want to transfer optimization problems from the discrete set $\mathcal{P}(V)$ to $\mathbb{R}^N$ with its Euclidean structure where we can apply the tools of calculus.

This is achieved by the *Lovász extension*, which extends the domain of f to the whole Euclidean space[1] $\mathbb{R}^V$ in a particular manner.

We shall assume

$$f(\emptyset) = 0, \quad g(\emptyset) = 0 \tag{6.2.1}$$

[1] Some other versions in the literature only extend the domain to the cube $[0,1]^V$ or the nonnegative orthant $\mathbb{R}^V_{\geq 0}$. In fact, many works on Boolean lattices identify $\mathcal{P}(V)$ with the discrete cube $\{0,1\}^N$.

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for all functions f, g occurring in the sequel.

Definition 6.2.1. For $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a bijection such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(N)}$ and $\sigma(0) = 0$, where $x_0 := 0$. The *Lovász extension* of f is defined by

$$f^L(\mathbf{x}) = \sum_{i=0}^{N-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V^{\sigma(i)}(\mathbf{x})), \tag{6.2.2}$$

where $V^0(\mathbf{x}) = V$ and $V^{\sigma(i)}(\mathbf{x}) := \{j \in V : x_j > x_{\sigma(i)}\}$, for $i = 1, \ldots, N-1$.

There are several equivalent expressions. We can write (6.2.2) in an integral form as

$$f^L(\mathbf{x}) = \int_{\min_{1 \leq i \leq N} x_i}^{\max_{1 \leq i \leq N} x_i} f(V^t(\mathbf{x}))dt + f(V) \min_{1 \leq i \leq N} x_i \tag{6.2.3}$$

$$= \int_{-\infty}^{0} (f(V^t(\mathbf{x})) - f(V))dt + \int_0^{+\infty} f(V^t(\mathbf{x}))dt, \tag{6.2.4}$$

where $V^t(\mathbf{x}) = \{i \in V : x_i > t\}$. If we apply the Möbius transformation, this becomes

$$f^L(\mathbf{x}) = \sum_{A \subset V} \sum_{B \subset A} (-1)^{\#A - \#B} f(B) \bigwedge_{i \in A} x_i, \tag{6.2.5}$$

where $\bigwedge_{i \in A} x_i$ is the minimum over $\{x_i : i \in A\}$.

Note that by these expressions, $f^L(\mathbf{1}_A) = f(A)$ for any $A \subset V$. This is why we call f^L an "extension", and moreover, f^L is actually a piecewise linear extension of f.

We now discuss some simple **examples**. We take $N = 2$ and, as indicated above, we identify $\emptyset$ with $(0,0)$, $\{1\}$ with $(1,0)$, $\{2\}$ with $(0,1)$, and $\{1,2\}$ with $(1,1)$ in $\mathbb{R}^2$.

1. We consider the function defined by $f_1(0,0) = 0$, $f_1(1,0) = 0$, $f_1(0,1) = 1$, and $f_1(1,1) = 0$. For $\mathbf{x} = (x_1, x_2)$, we have

$$f_1^L(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 \geq x_2, \\ x_2 - x_1 & \text{if } x_2 \geq x_1. \end{cases}$$

   We observe that f_1^L is convex.
2. Let $f_2(0,0) = 0$, $f_2(1,0) = 0$, $f_2(0,1) = 0$, and $f_2(1,1) = 1$. For $\mathbf{x} = (x_1, x_2)$, we have

$$f_2^L(\mathbf{x}) = \begin{cases} x_2 & \text{if } x_1 \geq x_2, \\ x_1 & \text{if } x_2 \geq x_1. \end{cases}$$

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We observe that f_2^L is not convex, even though the function f_2 also possesses some convex extension. The reason is, as we shall see in a moment, that f_2, in contrast to f_1, is not submodular.

Definition 6.2.2. A function $f : \mathcal{P}(V) \to \mathbb{R}$ is *modular* if

$$f(A) + f(B) = f(A \cup B) + f(A \cap B),$$

and it is *submodular* if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B),$$

for all $A, B \in \mathcal{P}(V)$.

Some further **examples** that are important for us:

3. Suppose that f is additive in the sense that $f(i_1, \ldots, i_N) = \sum_j f(0, \ldots, i_j, 0, \ldots, 0)$, where $i_j \in \{0, 1\}$ (and, as always, $f(0, \ldots, 0) = 0$), and we have identified $\mathcal{P}(V)$ with $\{0, 1\}^V$. Then,

$$f^L(\mathbf{x}) = \sum_j f(0, \ldots, i_j, 0, \ldots, 0) x_j.$$

To prove this, by symmetry it suffices to consider the case

$$0 < x_1 < x_2 < \ldots < x_N.$$

Then, the contribution of x_j comes from

$$(x_{j+1} - x_j) f(0, \ldots, 0, 1_{j+1}, \ldots 1_N) + (x_j - x_{j-1}) f(0, \ldots, 0, 1_j, \ldots 1_N),$$

where 1_k indicates the entry 1 in position k. By additivity, the contribution of x_j therefore is $f(0, \ldots, 0, 1_j, 0, \ldots, 0) x_j$. In particular, when f is defined on $\mathcal{P}(V)$ for the vertex set V of a graph, and $f(A) = \operatorname{vol} A$, then

$$f^L(\mathbf{x}) = \sum_i \deg i \; x_i, \quad \text{for } f(A) = \operatorname{vol} A. \tag{6.2.6}$$

4. Another example that is relevant in our context is $f(A) = |E(A, V \setminus A)|$, the number of edges between a subset A of the vertex set V of a graph and its complement $V \setminus A$. We note that this f is symmetric in the sense that $f(A) = f(V \setminus A)$. In particular, $f(V) = f(\emptyset) = 0$. When $x_i < x_j$, then when $i \sim j$, a contribution from that edge appears in $V^t(\mathbf{x})$ when $t > x_i$ and disappears when $t > x_j$. Therefore, from (6.2.4), we obtain

$$f^L(\mathbf{x}) = \sum_{i \sim j} |x_j - x_i|, \quad \text{for } f(A) = |E(A, V \setminus A)|. \tag{6.2.7}$$

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5. Given a constant function $f \equiv c$, we use (6.2.3) to derive

$$f^L(\mathbf{x}) = c \min_{1\le i\le N} x_i + \int_{\min_{1\le i\le N} x_i}^{\max_{1\le i\le N} x_i} c\, dt = c \max_{1\le i\le N} x_i.$$

6. As a reformulation of (6.2.2), we have

$$\begin{aligned} f^L(\mathbf{x}) = {} & (x_{\overline{\sigma}(1)} - x_{\overline{\sigma}(2)}) f(V^{(2)}(\mathbf{x})) + \cdots \\ & + (x_{\overline{\sigma}(N-1)} - x_{\overline{\sigma}(N)})\, f(V^{(N)}(\mathbf{x})) + x_{\overline{\sigma}(N)} f(V), \end{aligned}$$

where $\overline{\sigma}\colon V \to V$ is a permutation such that $x_{\overline{\sigma}(1)} \ge \ldots \ge x_{\overline{\sigma}(N)}$, and

$$V^{(k)}(\mathbf{x}) \coloneqq \left\{ j \in V\colon x_j > x_{\overline{\sigma}(k)} \right\},$$

for $k = 2, \ldots, N$. Now, given $k \in \{1, \ldots, N\}$, let $f\colon \mathcal{P}(V) \to \mathbb{R}$ be defined as

$$f(A) := \begin{cases} 1, & \text{if } |A| \ge k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by letting $x_{\overline{\sigma}(k)}$ indicate the k-th largest component of $\mathbf{x}$, we have

$$\begin{aligned} f^L(\mathbf{x}) &= (x_{\overline{\sigma}(k)} - x_{\overline{\sigma}(k+1)}) f(V^{(k+1)}(\mathbf{x})) + \cdots \\ &\quad + (x_{\overline{\sigma}(N-1)} - x_{\overline{\sigma}(N)}) f(V^{(N)}(\mathbf{x})) + x_{\overline{\sigma}(N)} \\ &= (x_{\overline{\sigma}(k)} - x_{\overline{\sigma}(k+1)}) + \cdots + (x_{\overline{\sigma}(N-1)} - x_{\overline{\sigma}(N)}) + x_{\overline{\sigma}(N)} \\ &= x_{\overline{\sigma}(k)}. \end{aligned}$$

7. Finally, we leave it for the reader to check (or to consult [50, 132]) that

$$f^L(\mathbf{x}) = \min_{t\in\mathbb{R}} \|\mathbf{x} - t\mathbf{1}\|_{L^1}, \quad \text{for } f(A) = \min(\operatorname{vol} A, \operatorname{vol}(V \setminus A)). \quad (6.2.8)$$

We now list some fundamental properties of the Lovász extension [189].

Proposition 6.2.1. *(a) f^L is positively one-homogeneous, PL (piecewise linear), and Lipschitz continuous.*

(b) For all $t \in \mathbb{R}$ and for all $\mathbf{x} \in \mathbb{R}^N$,

$$f^L(\mathbf{x} + t\mathbf{1}_V) = f^L(\mathbf{x}) + t f(V).$$

Thus, f^L is linear along the direction $\mathbf{1}_V$.

(c) f^L is convex if and only if $f\colon \mathcal{P}(V) \to \mathbb{R}$ is submodular.

(d) A continuous function $F\colon \mathbb{R}^N \to \mathbb{R}$ is the Lovász extension of some $f\colon \mathcal{P}(V) \to \mathbb{R}$ if and only if $F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$ whenever $\mathbf{x}$ and $\mathbf{y}$ are comonotonic, meaning that

$$(x_i - x_j)(y_i - y_j) \ge 0,$$

for all $i, j \in V$.

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(e) *A one-homogeneous function* $F: \mathbb{R}^N \to \mathbb{R}$ *is the Lovász extension of some submodular function if and only if*

$$F(\mathbf{x} + t\mathbf{1}_V) = F(\mathbf{x}) + tF(\mathbf{1}_V),$$

for all $t \in \mathbb{R}$ *and for all* $\mathbf{x} \in \mathbb{R}^N$, *and*

$$F(\mathbf{x}) + F(\mathbf{y}) \geq F(\mathbf{x} \vee \mathbf{y}) + F(\mathbf{x} \wedge \mathbf{y}),$$

where the i*th components of* $\mathbf{x} \vee \mathbf{y}$ *and* $\mathbf{x} \wedge \mathbf{y}$ *are* $(\mathbf{x} \vee \mathbf{y})_i = \max\{x_i, y_i\}$ *and* $(\mathbf{x} \wedge \mathbf{y})_i = \min\{x_i, y_i\}$, *respectively*.

A useful variant of the Lovász extension is the *disjoint-pair Lovász extension*, which was introduced by Qi [231] and has been systematically investigated by Fujishige [101] and Murota [221] in the context of discrete convex analysis (or submodular analysis). Specifically, with $\mathcal{P}(V \times V)$ being the power set of $V \times V$, for a function $f: \mathcal{P}(V \times V) \to \mathbb{R}$, its disjoint-pair Lovász extension is defined by the integral formulation

$$f^L(\mathbf{x}) = \int_0^{\|\mathbf{x}\|_\infty} f(V_+^t(\mathbf{x}), V_-^t(\mathbf{x}))dt, \tag{6.2.9}$$

where $V_\pm^t(\mathbf{x}) = \{i \in V: \pm x_i > t\}$, for all $t \geq 0$. If we naturally assume

$$f(\emptyset, \emptyset) = 0, \tag{6.2.10}$$

we can write (6.2.9) as

$$f^L(\mathbf{x}) = \int_0^{\infty} f(V_+^t(\mathbf{x}), V_-^t(\mathbf{x}))dt. \tag{6.2.11}$$

Equivalently,

$$f^L(\mathbf{x}) = \sum_{i=0}^{N-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V_{\sigma(i)}^+(\mathbf{x}), V_{\sigma(i)}^-(\mathbf{x})), \tag{6.2.12}$$

where $\sigma: V \cup \{0\} \to V \cup \{0\}$ is a bijection such that $|x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \ldots \leq |x_{\sigma(N)}|$ and $\sigma(0) = 0$, where $x_0 := 0$, and

$$V_{\sigma(i)}^\pm(\mathbf{x}) := \{j \in V: \pm x_j > |x_{\sigma(i)}|\}, \quad i = 0, 1, \ldots, N-1.$$

Note that to define f^L, we only use the information on disjoint pairs $(A, B) \subset V \times V$, not a general subset $S \subset V \times V$. Thus, we shall focus on

$$\mathcal{P}_2(V) := \{(A, B): A, B \subset V, A \cap B = \emptyset\},$$

a subfamily of $\mathcal{P}(V \times V)$. We regard $\mathcal{P}_2(V)$ as $\{-1, 0, 1\}^N$ by identifying the disjoint pair (A, B) with the ternary (indicator) vector $\mathbf{1}_A - \mathbf{1}_B$.

Analogously to (d) in Proposition 6.2.1, we have

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Proposition 6.2.2. *A continuous function F is the disjoint-pair Lovász extension of some function $f\colon \mathcal{P}_2(V) \to \mathbb{R}$, if and only if $F(\mathbf{x}) + F(\mathbf{y}) = F(\mathbf{x}+\mathbf{y})$ whenever $\mathbf{x}$ and $\mathbf{y}$ are absolutely comonotonic, that is, $x_i y_i \geq 0$ for all i, and*

$$(|x_i| - |x_j|)(|y_i| - |y_j|) \geq 0$$

for all i, j.

Proof From (6.2.12), we have that F is the disjoint-pair Lovász extension of some function $f\colon \mathcal{P}_2(V) \to \mathbb{R}$ if and only if

$$\lambda F(\mathbf{x}) + (1-\lambda)F(\mathbf{y}) = F(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}),$$

for all absolutely comonotonic vectors $\mathbf{x}$ and $\mathbf{y}$ and for all $\lambda \in [0,1]$. Therefore, we only need to prove sufficiency.

Given $\mathbf{x} \in \mathbb{R}^V$, since $s\mathbf{x}$ and $t\mathbf{x}$ are absolutely comonotonic for $s, t \geq 0$, we have that

$$F(s\mathbf{x}) + F(t\mathbf{x}) = F((s+t)\mathbf{x}),$$

which yields a Cauchy equation on the half-line. Thus, the continuity assumption implies the linearity of F on the ray $\mathbb{R}^+\mathbf{x}$, which implies the property $F(t\mathbf{x}) = tF(\mathbf{x})$, for all $t \geq 0$, and hence

$$\lambda F(\mathbf{x}) + (1-\lambda)F(\mathbf{y}) = F(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}),$$

for any absolutely comonotonic vectors $\mathbf{x}$ and $\mathbf{y}$, and for all $\lambda \in [0,1]$. □

6.3 Eigenvalue Problem for a Pair of Lovász Extensions

As explained in Section 6.1, we now reformulate our eigenvalue problem.

Definition 6.3.1. $\lambda \in \mathbb{R}$ is called an *eigenvalue*, and $\mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ a corresponding *eigenvector*, of the function pair (f^L, g^L), if

$$\mathbf{0} \in \nabla f^L(\mathbf{x}) - \lambda \nabla g^L(\mathbf{x}).$$

We also call $(\lambda, \mathbf{x})$ an *eigenpair*.

Proposition 6.3.1. *Given $f, g\colon \mathcal{P}(V) \to \mathbb{R}$, for any eigenvalue λ of (f^L, g^L) there exists $A \in \mathcal{P}(V)\setminus\{\emptyset\}$ such that $\lambda = f(A)/g(A)$ and $\mathbf{1}_A$ is a corresponding eigenvector. In addition, given any two distinct real numbers a and b, every eigenvalue has an eigenvector in $\{a,b\}^N$. Moreover, we have that:*

- *If $g(V) \neq 0$, then $\lambda = f(V)/g(V)$ is the only possible eigenvalue of (f^L, g^L);*

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- *If $g(V) = 0$ and (f^L, g^L) has at least one eigenvalue, then $f(V) = 0$, and, in this case, (f^L, g^L) may have many distinct eigenvalues.*

Proof Let $(\lambda, \mathbf{x})$ be an eigenpair of (f^L, g^L). Since f^L is one-homogeneous and linear along the direction $\mathbf{1}$, without loss of generality we can assume that $\mathbf{x}$ lies in the relative interior of the simplex $\triangle$ with vertices $\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_k}$, where $A_1, \ldots, A_k$ are upper-level sets of $\mathbf{x}$. By applying Lemma 6.1.1 to the piecewise linear functions f^L and g^L, we infer that

$$\mathbf{0} \in \nabla f^L(\mathbf{x}) - \lambda \nabla g^L(\mathbf{x}) \subset \nabla f^L(\mathbf{1}_{A_i}) - \lambda \nabla g^L(\mathbf{1}_{A_i}),$$

for any $i = 1, \ldots, k$, meaning that each $\mathbf{1}_{A_i}$ is an eigenvector of (f^L, g^L).

By definition of Lovász extension, we can also check that, for all $\mathbf{v} \in \nabla f^L(\mathbf{x})$,

$$\sum_{i \in V} v_i = f(V).$$

Therefore, for all $\mathbf{x}$,

$$\nabla f^L(\mathbf{x}) \cdot \mathbf{1} = f(V).$$

The same identity holds for g, that is, $\nabla g^L(\mathbf{x}) \cdot \mathbf{1} = g(V)$. This implies that, for any eigenpair $(\lambda, \mathbf{x})$,

$$f(V) - \lambda g(V) = (\nabla f^L(\mathbf{x}) - \lambda \nabla g^L(\mathbf{x})) \cdot \mathbf{1} = 0.$$

Therefore $\lambda = f(V)/g(V)$ if $g(V) \neq 0$, and $f(V) = 0$ if $g(V) = 0$.

For the case when $f(V) = g(V) = 0$, we may consider the following example. Let $V = \{1, 2\}$ and consider two functions $f, g \colon \mathcal{P}(V) \to \mathbb{R}$ defined by $f(\{1\}) = g(\{2\}) = 2$, $f(\{2\}) = g(\{1\}) = 1$, and $f(\{1,2\}) = f(\emptyset) = g(\{1,2\}) = g(\emptyset) = 0$. Then, one can check that

$$f^L(\mathbf{x}) = \begin{cases} x_2 - x_1, & \text{if } x_1 \leq x_2, \\ 2(x_1 - x_2), & \text{if } x_1 \geq x_2, \end{cases} \text{ and } g^L(\mathbf{x}) = \begin{cases} 2(x_2 - x_1), & \text{if } x_1 \leq x_2, \\ x_1 - x_2, & \text{if } x_1 \geq x_2. \end{cases}$$

Moreover, it can be verified that $(2, \mathbf{1}_{\{1\}} - \mathbf{1}_{\{2\}})$ and $(1/2, \mathbf{1}_{\{2\}} - \mathbf{1}_{\{1\}})$ are two eigenpairs of (f^L, g^L). Hence, in this example, the function pair (f^L, g^L) has at least two distinct eigenvalues: $1/2$ and 2. □

The case where $g(V) = 0$ will be the interesting one for us. Recall the quantity

$$\eta(S) = \frac{|E(S, V \backslash S)|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \backslash S)\}} = \frac{f(S)}{g(S)}$$

from (4.3.12), for a given $S \subset V$. In this case, the pair (f^L, g^L) will have several distinct eigenvalues. Since the quantity $\eta(S)$ considers edges between pairs

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of sets, namely S and $V\backslash S$, the eigenvectors of (f^L, g^L) directly correspond to subsets of V, and therefore, the first nontrivial eigenvector of (f^L, g^L) relates to the subset S realizing the minimum of $\eta(\cdot)$. In particular, we need to work with disjoint pair extensions, and this is addressed in Theorem 6.3.1.

Lemma 6.3.1. *Let C and Ω be two convex cones in $\mathbb{R}^N$ that satisfy*

$$\Omega \cap ((-C) \cup C) = \{\mathbf{0}\}.$$

Let also

$$C^* \coloneqq \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, \mathbf{y}) \leq 0 \, \forall \mathbf{y} \in C\}$$

be the polar cone of C. Then, $\Omega^* \cap C^* \neq \{\mathbf{0}\}$ and $\Omega^* \cap (-C^*) \neq \{\mathbf{0}\}$.

Proof We have that $\Omega^* \cap C^* = (\Omega \cup C)^* \supset (\Omega + C)^*$, where $\Omega + C$ is the Minkowski summation of C and Ω. If $\Omega + C = \mathbb{R}^N$, then for any $-\mathbf{c} \in (-C)\backslash\{\mathbf{0}\}$, there exist $\mathbf{a} \in \Omega \setminus \{\mathbf{0}\}$ and $\mathbf{c}' \in C \setminus \{\mathbf{0}\}$ such that $\mathbf{a} + \mathbf{c}' = -\mathbf{c}$. This implies that $\mathbf{a} = -\mathbf{c}' - \mathbf{c} \in -C \setminus \{\mathbf{0}\}$, which contradicts the condition that $\Omega \cap (-C) = \{\mathbf{0}\}$. Therefore, the convex cone $\Omega + C$ is not the whole space $\mathbb{R}^N$, which implies that $(\Omega + C)^* \neq \{\mathbf{0}\}$. Hence, $\Omega^* \cap C^* \neq \{\mathbf{0}\}$, and similarly, $\Omega^* \cap (-C^*) \neq \{\mathbf{0}\}$. □

Theorem 6.3.1. *Given $f, g : \mathcal{P}_2(V) \to \mathbb{R}$, every eigenvalue of (f^L, g^L) has an eigenvector of the form $\mathbf{1}_A - \mathbf{1}_B$. Moreover,*

1. *If, for any $(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}$ we have that*

   $$2f(A,B) = f(A, V\backslash A) + f(V\backslash B, B) \text{ and } 2g(A,B) = g(A, V\backslash A) + g(V\backslash B, B),$$

   then every eigenvalue of (f^L, g^L) has an eigenvector of the form $\mathbf{1}_A - \mathbf{1}_{V\backslash A}$.
2. *If g is a constant function, then for any $A \subset V$, $\mathbf{1}_A - \mathbf{1}_{V\backslash A}$ is an eigenvector.*

Proof The general result on the disjoint-pair Lovász extension is similar to Proposition 6.3.1, and thus, we omit the proof. We prove the other statements in a sequence of steps.

1. We first prove that, in this case, for any $A_1 \subset \ldots \subset A_k$ with $(A_1, A_k) \neq (\emptyset, V)$, f^L is linear on $\operatorname{conv}\{\mathbf{1}_{A_i} - \mathbf{1}_{V\backslash A_i} : i = 1, \ldots, k\}$.
   Proposition 6.2.2 implies that f^L is linear on $\operatorname{conv}\{\mathbf{1}_{A_i} - \mathbf{1}_{B_i} : i = 1, \ldots, k\}$ whenever $A_1 \subset \ldots \subset A_k$, $B_1 \subset \ldots \subset B_k$, and $A_k \cap B_k = \emptyset$. Thus, for any $\sigma, \tau : \{1, \ldots, k\} \to \{1, \ldots, k\}$ with

   $$\sigma(1) \leq \ldots \leq \sigma(k) \leq \tau(k) \leq \ldots \leq \tau(1),$$

   f^L is linear on $\operatorname{conv}\{\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\backslash A_{\tau(i)}} : i = 1, \ldots, k\}$. Observe that

   $$\operatorname{conv}\{\mathbf{1}_{A_i} - \mathbf{1}_{V\backslash A_i} : i = 1, \ldots, k\}$$

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$$= \bigcup_{\sigma,\tau} \operatorname{conv}\{\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}} : i = 1,\dots,k\}$$
$$= \bigcup_{l=1}^{k} \bigcup_{\sigma(1)=1,\sigma(k)=\tau(k)=l,\tau(1)=k} \operatorname{conv}\{\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}} : i = 1,\dots,k\},$$

where in the second line, we can further assume that each

$$\operatorname{conv}\{\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}} : i = 1,\dots,k\}$$

is a $(k-1)$-dimensional simplex, and there are exactly $\sum_{l=1}^{k}\binom{k-1}{l-1} = 2^{k-1}$ simplexes of dimension $(k-1)$. Since, furthermore,

$$\begin{aligned} 2f^L(\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}}) &= 2f(A_{\sigma(i)}, V\setminus A_{\tau(i)}) \\ &= f(A_{\sigma(i)}, V\setminus A_{\sigma(i)}) + f(A_{\tau(i)}, V\setminus A_{\tau(i)}) \\ &= f^L\left(\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\sigma(i)}}\right) + f^L\left(\mathbf{1}_{A_{\tau(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}}\right) \end{aligned}$$

and

$$\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\sigma(i)}} + \mathbf{1}_{A_{\tau(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}} = 2(\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}}),$$

f^L must be linear on the segment

$$\operatorname{conv}\{\mathbf{1}_{A_{\sigma(i)}} - \mathbf{1}_{V\setminus A_{\sigma(i)}}, \mathbf{1}_{A_{\tau(i)}} - \mathbf{1}_{V\setminus A_{\tau(i)}}\}.$$

Therefore, one can check that f^L is linear on the simplex

$$\operatorname{conv}\{\mathbf{1}_{A_i} - \mathbf{1}_{V\setminus A_i} : i = 1,\dots,k\}.$$

Now, given an eigenvalue λ, let $\mathbf{1}_A - \mathbf{1}_B$ be a corresponding eigenvector, for some $(A,B) \in \mathcal{P}_2(V)\setminus\{(\emptyset,\emptyset)\}$. By the above argument, it can be verified that both $\mathbf{1}_A - \mathbf{1}_{V\setminus A}$ and $\mathbf{1}_{V\setminus B} - \mathbf{1}_B$ are eigenvectors for λ.

2. We shall prove that, for any $f\colon \mathcal{P}_2(V) \to [0,+\infty)$ and for any $A \subset V$, the vector $\mathbf{1}_A - \mathbf{1}_{V\setminus A}$ is an eigenvector of $(f^L, \|\cdot\|_\infty)$. In particular, by definition of the eigenvalue problem on $(f^L, \|\cdot\|_\infty)$, we only need to prove that

$$\nabla f^L(\mathbf{x}) \cap \left((-\mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}) \cup \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}\right) \neq \emptyset$$

for any $\mathbf{x} \in \{-1,1\}^N$, where

$$\mathbb{R}^N_{\operatorname{sign}(\mathbf{x})} := \{\mathbf{y} \in \mathbb{R}^N : y_i x_i \geq 0\ \forall i\} = \operatorname{cone}(\nabla\|\mathbf{x}\|_\infty).$$

We first assume that f is positive-definite, that is, $f(A,B) > 0$ whenever $(A,B) \neq (\emptyset,\emptyset)$, and we shall apply Lemma 6.3.1 to this case.

Assume by contradiction that, for some $\mathbf{x} \in \{-1,1\}^N$,

$$\nabla f^L(\mathbf{x}) \cap ((-\mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}) \cup \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}) = \emptyset.$$

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Then,

$$\operatorname{cone}(\nabla f^L(\mathbf{x})) \cap ((-\mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}) \cup \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}) = \{\mathbf{0}\}$$

and, by Lemma 6.3.1,

$$\operatorname{cone}^*(\nabla f^L(\mathbf{x})) \cap \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})} = \operatorname{cone}^*(\nabla f^L(\mathbf{x})) \cap (-\mathbb{R}^N_{\operatorname{sign}(\mathbf{x})})^* \neq \{\mathbf{0}\}.$$

Now, fix a set $S \subset \mathbb{R}^N$. The *normal cone* of S at $\mathbf{x}$ is

$$\mathsf{N}_{\mathbf{x}}(S) \coloneqq \{\mathbf{y} \in \mathbb{R}^N : (\mathbf{y}, \mathbf{x}' - \mathbf{x}) \leq 0, \forall \mathbf{x}' \in S\}.$$

The *tangent cone* $\mathsf{T}_{\mathbf{x}}(S)$ of S at $\mathbf{x}$ is the closure of the set

$$\{\mathbf{z} \in \mathbb{R}^N : \mathbf{x} + \varepsilon \mathbf{z} \in S, \text{ for some } \varepsilon > 0\}.$$

Since f is positive-definite, if f is further assumed to be submodular, from the fact that f^L is convex and positive-definite, we have

$$\operatorname{cone}(\nabla f^L(\mathbf{x})) = \mathsf{N}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq f^L(\mathbf{x})\}).$$

It follows that

$$\operatorname{cone}^*(\nabla f^L(\mathbf{x})) \subset \mathsf{T}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq f^L(\mathbf{x})\}),$$

implying that

$$\mathsf{T}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq f^L(\mathbf{x})\}) \cap \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})} \neq \{\mathbf{0}\}.$$

The above relation still holds if f is just positive-definite.

Now, since $\mathbf{x} \in \{-1, 1\}^N$, we can write $\mathbf{x} = \mathbf{1}_{A_N} - \mathbf{1}_{B_N}$, with $A_N \sqcup B_N = V$. Also, for any given permutation $\sigma \colon \{1, \ldots, N\} \to \{1, \ldots, N\} = V = A_N \sqcup B_N$, we define $A_i \coloneqq A_N \setminus \{\sigma(i+1), \ldots, \sigma(N)\}$ and $B_i \coloneqq B_N \setminus \{\sigma(i+1), \cdots, \sigma(N)\}$, for $i = 1, \ldots, N-1$.

Now, for $i = 1, \ldots, N$, let

$$\mathbf{x}^i \coloneqq \frac{\mathbf{1}_{A_i} - \mathbf{1}_{B_i}}{f(A_i, B_i)}.$$

Since $f(A_i, B_i) > 0$, we have $f^L(\mathbf{x}^i) = 1$. Therefore,

$$\mathsf{T}_{\mathbf{x}^N}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq 1\}) = \mathsf{T}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq f^L(\mathbf{x})\}).$$

Hence, without loss of generality, we may assume that $\mathbf{x} = \mathbf{x}^N$.

Now, the definition of f^L implies that

$$\operatorname{conv}(\mathbf{0}, \mathbf{x}^1, \ldots, \mathbf{x}^N) \subset \{\mathbf{y} \colon f^L(\mathbf{y}) \leq 1\}.$$

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We write $\triangle_\sigma := \operatorname{conv}(\mathbf{0}, \mathbf{x}^1, \ldots, \mathbf{x}^N)$, since the construction of $\mathbf{x}^1, \ldots, \mathbf{x}^N$ depends on the permutation σ. For any

$$\mathbf{y} := \sum_{i=1}^{N} t_i \mathbf{x}^i \in \operatorname{conv}(\mathbf{0}, \mathbf{x}^1, \ldots, \mathbf{x}^N) \setminus \{\mathbf{x}\},$$

we have that

$$(\mathbf{y} - \mathbf{x})_{\sigma(N)} x_{\sigma(N)} = -(1 - t_N) x_{\sigma(N)}^2 < 0,$$

and thus, $\mathbf{y} - \mathbf{x} \notin \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})}$. Hence,

$$\mathsf{T}_{\mathbf{x}}(\triangle_\sigma) \cap \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})} = \{\mathbf{0}\}.$$

Since

$$\mathsf{T}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq 1\}) = \bigcup_{\sigma} \mathsf{T}_{\mathbf{x}}(\triangle_\sigma),$$

it follows that

$$\mathsf{T}_{\mathbf{x}}(\{\mathbf{y} \colon f^L(\mathbf{y}) \leq 1\}) \cap \mathbb{R}^N_{\operatorname{sign}(\mathbf{x})} = \{\mathbf{0}\}.$$

This is a contradiction. Therefore, we have proved that, for any positive-definite function f on $\mathcal{P}_2(V)$, every vector $\mathbf{x} \in \{-1, 1\}^N$ is an eigenvector of the pair $(f^L, \|\cdot\|_\infty)$, that is, there exists $\mathbf{u} \in \nabla \|\mathbf{x}\|_\infty$ and $\lambda \in \mathbb{R}$ such that $\lambda \mathbf{u} \in \nabla f^L(\mathbf{x})$. Applying Euler's identity on one-homogeneous convex functions, we have that $\langle \mathbf{u}, \mathbf{x} \rangle = \|\mathbf{x}\|_\infty$ and $\langle \lambda \mathbf{u}, \mathbf{x} \rangle = f^L(\mathbf{x})$, which implies that

$$\lambda = \frac{f^L(\mathbf{x})}{\|\mathbf{x}\|_\infty} > 0.$$

We now consider the general case where $f \geq 0$. Fix a sequence $\{f_n\}_{n \geq 1}$ of positive-definite functions on $\mathcal{P}_2(V)$ such that $f_n \to f$, as n tends to $+\infty$. It can then be verified that, for any $\mathbf{v}_n \in \nabla f_n^L(\mathbf{x})$, all limit points of $\{\mathbf{v}_n\}_{n \geq 1}$ belong to $\nabla f^L(\mathbf{x})$. We know that there exist $\mathbf{u}_n \in \nabla \|\mathbf{x}\|_\infty$ and

$$\lambda_n = \frac{f_n^L(\mathbf{x})}{\|\mathbf{x}\|_\infty} > 0$$

such that $\lambda_n \mathbf{u}_n \in \nabla f_n^L(\mathbf{x})$. This implies that, for any limit point $\mathbf{u}$ of $\{\mathbf{u}_n\}_{n \geq 1}$, $\mathbf{u} \in \nabla \|\mathbf{x}\|_\infty$ and $\lambda \mathbf{u} \in \nabla f^L(\mathbf{x})$, where $\lambda := \lim_{n \to +\infty} \lambda_n$. Therefore, $(\lambda, \mathbf{x})$ is an eigenpair of $(f^L, \|\cdot\|_\infty)$.

□

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We now introduce the (f,g)–*Cheeger constant* corresponding to a given pair (f,g) of functions $f\colon \mathcal{P}(V)\to\mathbb{R}$ and $g\colon \mathcal{P}(V)\to\mathbb{R}_+$ as

$$\mathrm{Ch}(f,g) \coloneqq \min_{A\in\mathcal{P}(V)\setminus\{\emptyset,V\}} \frac{f(A)}{\min\{g(A),g(V\setminus A)\}}. \tag{6.3.1}$$

Theorem 6.3.2. *If $f\colon \mathcal{P}(V)\to\mathbb{R}$ is symmetric, that is, $f(A)=f(V\setminus A)$ for all A, while $g\colon \mathcal{P}(V)\to\mathbb{R}_+$ is submodular and nondecreasing, then*

$$\mathrm{Ch}(f,g) = \min_{(A,B)\in\mathcal{P}_2(V)\setminus\{(\emptyset,\emptyset)\}} \max\left\{\frac{f(A)}{g(A)},\frac{f(B)}{g(B)}\right\}. \tag{6.3.2}$$

Proof From the symmetry of f, it follows that

$$\frac{f(A)}{\min\{g(A),g(V\setminus A)\}} = \max\left\{\frac{f(A)}{g(A)},\frac{f(V\setminus A)}{g(V\setminus A)}\right\},$$

and this yields

$$\mathrm{Ch}(f,g) \geq \min_{(A,B)\in\mathcal{P}_2(V)\setminus\{(\emptyset,\emptyset)\}} \max\left\{\frac{f(A)}{g(A)},\frac{f(B)}{g(B)}\right\}.$$

To see that (6.3.2) is equivalent to (6.3.1), we need to show that, for any $A,B\neq\emptyset$ with $A\cap B=\emptyset$,

$$\max\left\{\frac{f(A)}{g(A)},\frac{f(B)}{g(B)}\right\} \geq \min\left\{\frac{f(A)}{\min\{g(A),g(V\setminus A)\}},\frac{f(B)}{\min\{g(B),g(V\setminus B)\}}\right\}.$$

To see this, assume the contrary, by contradiction. Then, $g(A)>g(V\setminus A)$ and $g(B)>g(V\setminus B)$, implying that $g(A)+g(B)>g(V\setminus A)+g(V\setminus B)$. Since $A\subset V\setminus B$ and g is nondecreasing, this implies that $g(A)\leq g(V\setminus B)$. Similarly, $g(B)\leq g(V\setminus A)$, which leads to a contradiction. This proves the claim. □

For a simple graph G, by letting $f(A)=|\partial A|$ and $g(A)=\mathrm{vol}\,(A)$, the Cheeger constant $h(G)$ equals $\mathrm{Ch}(f,g)$.

Theorem 6.3.3. *Let $f_s,g_s\colon \mathcal{P}_2(V)\to\mathbb{R}$ be defined by $f_s(A,B)=f(A)+f(B)$ and $g_s(A,B)=g(A)+g(B)$. Then,*

$$\mathrm{Ch}(f,g)=\lambda_2(f_s^L,g_s^L),$$

that is, the (f,g)–Cheeger constant equals the second eigenvalue of (f_s^L,g_s^L).

Proof Let $g_m(A)\coloneqq\min\{g(A),g(V\setminus A)\}$. Since g is nondecreasing, $g(V^t(\mathbf{x}))$ must be nonincreasing with respect to $t\in\mathbb{R}$, that is, $g(V^{t_1}(\mathbf{x}))\geq g(V^{t_2}(\mathbf{x}))$ if $t_1\leq t_2$. Hence, the function $t\mapsto g(V^t(\mathbf{x}))-g(V\setminus V^t(\mathbf{x}))$ is nonincreasing and left-continuous on $t\in\mathbb{R}$. Since

$$\lim_{t\to-\infty}\left(g(V^t(\mathbf{x}))-g(V\setminus V^t(\mathbf{x}))\right)=g(V)-g(\emptyset)\geq 0$$

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and

$$\lim_{t\to+\infty}(g(V^t(\mathbf{x}))-g(V\setminus V^t(\mathbf{x})))=g(\emptyset)-g(V)\le 0,$$

there exists $t_0\in\mathbb{R}$ such that

$$g(V^t(\mathbf{x}))\ge g(V\setminus V^t(\mathbf{x})),\quad \forall t<t_0,$$

and

$$g(V^t(\mathbf{x}))\le g(V^{t_0}(\mathbf{x}))\le g(V\setminus V^t(\mathbf{x})),\quad \forall t\ge t_0.$$

Therefore, the Lovász extension g_m^L of g_m satisfies

$$\begin{aligned}
g_m^L(\mathbf{x}) &= \int_{\min\mathbf{x}}^{\max\mathbf{x}} g_m(V^t(\mathbf{x}))dt+\min\mathbf{x}\, g_m(V)\\
&= \int_{\min\mathbf{x}}^{t_0} g(V\setminus V^t(\mathbf{x}))dt+\int_{t_0}^{\max\mathbf{x}} g(V^t(\mathbf{x}))dt \quad (\text{as } g(\emptyset)=0)\\
&= \int_{\min(\mathbf{x}-t_0\mathbf{1})}^{0} g(V\setminus V^t(\mathbf{x}-t_0\mathbf{1}))dt+\int_0^{\max(\mathbf{x}-t_0\mathbf{1})} g(V^t(\mathbf{x}-t_0\mathbf{1}))dt\\
&= \int_{-\|\mathbf{x}-t_0\mathbf{1}\|_\infty}^{0} g(V\setminus V^t(\mathbf{x}-t_0\mathbf{1}))dt+\int_0^{\|\mathbf{x}-t_0\mathbf{1}\|_\infty} g(V^t(\mathbf{x}-t_0\mathbf{1}))dt\\
&= \int_0^{\|\mathbf{x}-t_0\mathbf{1}\|_\infty} g(V^t(\mathbf{x}-t_0\mathbf{1}))+g(V\setminus V^{-t}(\mathbf{x}-t_0\mathbf{1}))dt\\
&= g_s^L(\mathbf{x}-t_0\mathbf{1})\\
&= \min_{t\in\mathbb{R}} g_s^L(\mathbf{x}-t\mathbf{1}).
\end{aligned}$$

Since f is symmetric, this implies that $f_s^L(\mathbf{x})=f^L(\mathbf{x})=f_m^L(\mathbf{x})$.

Moreover, since g is positive, submodular, and nondecreasing, it follows that g_s is bisubmodular. Therefore, $g_s(A,B)$ is submodular in A for any fixed B, and it is submodular in B for any fixed A. Thus, as the Lovász extension of a submodular function, g_s^L is a convex function. Therefore, we have

$$\min_{\mathbf{x}\perp\mathbf{1}}\frac{f_s^L(\mathbf{x})}{\min_{t\in\mathbb{R}} g_s^L(\mathbf{x}-t\mathbf{1})}=\min_{\mathbf{x}\in\mathbb{R}_+^N:\ \min x=0}\frac{f_m^L(\mathbf{x})}{g_m^L(\mathbf{x})}=\min_{A\neq\emptyset,V}\frac{f_m(A)}{g_m(A)}=\mathrm{Ch}(f,g),$$

which is the second eigenvalue of the pair (f_s^L,g_s^L). □

If we remove the symmetry assumption of f in Theorem 6.3.2, we can then express the (f,g)–Cheeger constant as

$$\mathrm{Ch}(f,g)=\min_{A\in\mathcal{P}(V)\setminus\{\emptyset,V\}}\frac{\min\{f(A),f(V\setminus A)\}}{\min\{g(A),g(V\setminus A)\}}.$$

From here on, we assume for simplicity that $g\colon\mathcal{P}(V)\to\mathbb{R}_+$ is modular.

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Theorem 6.3.4. *Let $g^L(|\cdot|)$ be the function defined by $\mathbf{x} \mapsto g^L(|\mathbf{x}|)$, for all $\mathbf{x} \in \mathbb{R}^N$, where $|\mathbf{x}| := (|x_1|, \ldots, |x_N|)$. If $f(V) = 0$, then*

$$\mathrm{Ch}(f,g) = \lambda_2(f^L, g^L(|\cdot|)),$$

that is, the (f,g)–Cheeger constant equals the second eigenvalue of $(f^L, g^L(|\cdot|))$.

Proof Since $f \geq 0$, f^L is non-negative on $\mathbb{R}^N$. By Proposition 6.3.1 and Theorem 6.3.1, we may take an eigenvector of the form $\mathbf{1}_A - \mathbf{1}_B$ corresponding to the second eigenvalue. Let now $\hat{f}(A,B) := f(A \cup B)$ and

$$\hat{g}(A,B) := \min\{g(A \cup B), g(V \setminus (A \cup B))\}.$$

Then, $\hat{f}^L(\mathbf{x}) = f^L(|\mathbf{x}|)$ and $\hat{g}^L(\mathbf{x}) = \min_{t \in \mathbb{R}} g^L(||\mathbf{x}| - t|)$, where $||\mathbf{x}| - t| := (||x_1| - t|, \ldots, ||x_N| - t|)$ for $\mathbf{x} := (x_1, \ldots, x_N)$. Therefore,

$$\begin{aligned}
\mathrm{Ch}(f,g) &= \min_{A \in \mathcal{P}(V) \setminus \{\emptyset, V\}} \frac{f(A)}{\min\{g(A), g(V \setminus A)\}} \\
&= \min_{(A,B) \in \mathcal{P}_2(V), A \cup B \neq \emptyset, V} \frac{\hat{f}(A,B)}{\hat{g}(A,B)} \\
&= \inf_{\mathbf{x} \in \mathbb{R}^N :\ |\mathbf{x}| \text{ nonconstant}} \frac{\hat{f}^L(\mathbf{x})}{\hat{g}^L(\mathbf{x})} \\
&= \inf_{\mathbf{x} \in \mathbb{R}^N :\ |\mathbf{x}| \text{ nonconstant}} \frac{f^L(|\mathbf{x}|)}{\min_{t \in \mathbb{R}} g^L(||\mathbf{x}| - t|)} \\
&= \inf_{\mathbf{x} \in \mathbb{R}^N_+ :\ \mathbf{x} \text{ nonconstant}} \frac{f^L(\mathbf{x})}{\min_{t \in \mathbb{R}} g^L(|\mathbf{x} - t|)} \\
&= \inf_{\mathbf{x} \in \mathbb{R}^N :\ \mathbf{x} \text{ nonconstant}} \frac{f^L(\mathbf{x})}{\min_{t \in \mathbb{R}} g^L(|\mathbf{x} - t|)}
\end{aligned}$$

is the second eigenvalue of $(f^L, g^L(|\cdot|))$, where we use the fact that f^L is translation invariant along $\mathbf{1}$. □

Remark 6.3.1. In Section 6.2, we have seen that, for a simple graph G, by letting $f(A) = |\partial A|$ and $g(A) = \mathrm{vol}\,(A)$, we have

$$f^L(\mathbf{x}) = f_s^L(\mathbf{x}) = \sum_{ij \in E} |x_i - x_j|,$$

$$g^L(\mathbf{x}) = \sum_{i \in V} \deg i \; x_i$$

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and

$$g_s^L(\mathbf{x}) = g^L(|\mathbf{x}|) = \sum_{i\in V} \deg i \; |x_i|.$$

Both Theorems 6.3.3 and 6.3.4 say, in this case, that $h(G)$ equals the second eigenvalue of the function pair $(\sum_{ij\in E} |x_i - x_j|, \sum_{i\in V} \deg i \; |x_i|)$. As we shall see in Section 7.2, this is the second eigenvalue of the graph 1-Laplacian.

As a slight generalization of an oriented hypergraph, a *chemical hypergraph* further allows for *catalysts*, that is, vertices that are both inputs and outputs of a hyperedge. We shall now work on a chemical hypergraph satisfying $e_{in} \neq \emptyset \neq e_{out}$ and $\#(e_{in} \cup e_{out}) \geq 2$ for all $e \in E$, where e_{in} denotes the set of all the inputs of e, and e_{out} is the set of the outputs of e.

We introduce the *Cheeger constant* on a chemical hypergraph as

$$h \coloneqq \min_{A\in\mathcal{P}(V)\setminus\{\emptyset,V\}} \frac{\#(\partial A)}{\min\{\mu(A), \mu(V\setminus A)\}}, \tag{6.3.3}$$

where $\mu(A) \coloneqq \sum_{i\in A} \mu(i)$ with $\mu(i) > 0$, and the boundary set is

$$\partial A \coloneqq \{e \in E \colon e_{in} \cap A \neq \emptyset \neq e_{out} \setminus A \text{ or } e_{out} \subset A \subset V \setminus e_{in}\}. \tag{6.3.4}$$

We note that the above boundary set is not necessarily symmetric, that is, $\partial A \neq \partial(V \setminus A)$ in general.

Let $G = (V, E)$ be a simple graph, and let $\mathcal{N}(i)$ denote the set of neighbors of i.

- If $e_{in}^i = \mathcal{N}(i)$, $e_{out}^i = \{i\}$ and $\mu(A) = \#A$, then the boundary set (6.3.4) has the same cardinality as the *external vertex boundary*

$$\partial_{\text{ext}} A \coloneqq \{j \in V \setminus A \mid \{j, i\} \in E \text{ for some } i \in A\}$$

  from Section 4.3. Moreover, the corresponding Cheeger constant (6.3.3) equals

$$\inf_{\langle \mathbf{x}, \mathbf{1}\rangle = 0, \mathbf{x}\neq \mathbf{0}} \frac{\sum_{i\in V} \left(\max_{j\in\{i\}\cup\mathcal{N}(i)} x_j - x_i \right)}{\min_{t\in\mathbb{R}} \|\mathbf{x} - t\mathbf{1}\|_1}.$$

- If $e_{in}^i = \{i\}$ and $e_{out}^i = \mathcal{N}(i)$, then the boundary set (6.3.4) has the same cardinality as the *internal vertex-boundary*

$$\partial_{\text{int}} A \coloneqq \{i \in A \mid \{i, j\} \in E \text{ for some } j \in V \setminus A\}.$$

- If $e_{in}^i = e_{out}^i = \mathcal{N}(i)$ for all $i \in V$, then the boundary set (6.3.4) has the same cardinality as the *vertex-boundary*

$$\partial_{\text{ver}} A \coloneqq \partial_{\text{ext}} A \cup \partial_{\text{int}} A = V(E(A, V \setminus A)) = V(\partial A).$$

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6.4 Lovász Extension and Cheeger Inequalities

In Theorem 6.3.4, we have seen that the (f,g)–Cheeger constant

$$\mathrm{Ch}(f,g) = \min_{A\in\mathcal{P}(V)\setminus\{\emptyset,V\}} \frac{\min\{f(A), f(V\setminus A)\}}{\min\{g(A), g(V\setminus A)\}}$$

equals the second eigenvalue $\lambda_2(f^L, g^L(|\cdot|))$ of a pair of Lovász extensions. We now want to consider pth powers and derive inequalities between the Cheeger constant and the second eigenvalue. For $p=1$, these will reduce to the equalities derived above. For $p>1$, we need to utilize Hölder-type inequalities.

The cases of interest for us are covered by functions f and g of the form

$$f(A) \coloneqq \sum_{e\in E} w_e f_e(A) \qquad \text{and} \qquad g(A) \coloneqq \sum_{e\in E} g_e(A),$$

where $\{w_e\}_{e\in E}$ is a family of positive real numbers, while

$$\{f_e : \mathcal{P}(V)\to\mathbb{R}_{\geq 0}\}_{e\in E} \quad \text{and} \quad \{g_e : \mathcal{P}(V)\to\mathbb{R}_{\geq 0}\}_{e\in E}$$

are two families of functions satisfying

$$f_e(\emptyset) = f_e(V) = 0 = g_e(\emptyset) \quad \text{and} \quad g_e(A) = \sum_{i\in A} g_e(\{i\}), \quad \forall A\subset V.$$

We also let

$$F_p(\mathbf{x}) \coloneqq \sum_{e\in E} (f_e^L(\mathbf{x}))^p \quad \text{and} \quad G_p(\mathbf{x}) \coloneqq \sum_{e\in E} g_e^L(|\mathbf{x}|^p),$$

where $|\mathbf{x}|^p \coloneqq (|x_1|^p,\dots,|x_N|^p)$. By Theorem 6.3.1, the second eigenvalue of the function pair (F_p,G_p) is

$$\lambda_2(F_p,G_p) = \inf_{\mathbf{x}\ \text{nonconstant}} \frac{\sum_{e\in E}(f_e^L(\mathbf{x}))^p}{\min_{c\in\mathbb{R}} \sum_{e\in E} g_e^L(|\mathbf{x}-c\mathbf{1}|^p)}. \tag{6.4.1}$$

If we assume that $g(\{i\}) \coloneqq \sum_{e\in E} g_e(\{i\}) > 0$ for any $i\in V$, the following Cheeger inequalities hold.

Theorem 6.4.1. *In the above setting, we have*

$$\left(\frac{2}{c}\right)^{p-1} \frac{h^p}{p^p} \leq \lambda \leq 2^{p-1} C\cdot h, \tag{6.4.2}$$

where $h \coloneqq \mathrm{Ch}(f,g)$, $\lambda \coloneqq \lambda_2(F_p,G_p)$, $C \coloneqq \max_{e,A} f_e(A)^{p-1}$,

$$c \coloneqq \max_{i\in V} \frac{\sum_{e\in E_i} w_e}{g(i)}, \quad \textit{and} \quad E_i \coloneqq \{e\in E : \exists S\subset V : f_e(S\setminus\{i\}) \neq f_e(S)\}.$$

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Proof Let $A \in \mathcal{P}(V) \setminus \{\emptyset, V\}$ be such that

$$h = \frac{f(A)}{\min\{g(A), g(V \setminus A)\}}.$$

By taking the nonconstant vector $\mathbf{x} = \mathbf{1}_A$ in (6.4.1), we get

$$\lambda \leq \frac{\sum_{e \in E} w_e (f_e^L(\mathbf{1}_A))^p}{\min_{c \in \mathbb{R}} \sum_{e \in E} g_e^L(|\mathbf{1}_A - c\mathbf{1}|^p)}.$$

Since g_e is modular and $g = \sum_{e \in E} g_e$, we have

$$\sum_{e \in E} g_e^L(|\mathbf{1}_A - c\mathbf{1}|^p) = \sum_{i \in V} g(i)|(\mathbf{1}_A)_i - c|^p,$$

and

$$\min_{c \in \mathbb{R}} \sum_{i \in V} g(i)|(\mathbf{1}_A)_i - c|^p = \frac{g(A)g(V \setminus A)}{(g(A)^{\frac{1}{p-1}} + g(V \setminus A)^{\frac{1}{p-1}})^{p-1}}.$$

Therefore,

$$\begin{aligned}
\lambda &\leq \frac{\sum_{e \in E} w_e (f_e^L(\mathbf{1}_A))^p}{g(A)g(V \setminus A)/(g(A)^{\frac{1}{p-1}} + g(V \setminus A)^{\frac{1}{p-1}})^{p-1}} \\
&= \sum_{e \in E} w_e (f_e(A))^p \left(\sqrt[p-1]{\frac{1}{g(A)}} + \sqrt[p-1]{\frac{1}{g(V \setminus A)}} \right)^{p-1} \\
&\leq \max_{e,A} f_e(A)^{p-1} \sum_{e \in E} w_e f_e(A) 2^{p-1} \frac{1}{\min\{g(A), g(V \setminus A)\}} \\
&= \frac{2^{p-1} C f(A)}{\min\{g(A), g(V \setminus A)\}} \\
&= 2^{p-1} C \cdot h.
\end{aligned}$$

We shall now prove the lower bound for λ in (6.4.2). For simplicity, we identify a vector $\mathbf{x} \in \mathbb{R}^V$ with the function $\mathbf{x} \colon V \to \mathbb{R}$. Given $j \in V$, let $\deg(j) := \sum_{e \in E_j} w_e$. Then, for all j, we have that $\deg(j) \leq c g(j)$.

Also, we rewrite the *Lovász extension* (6.2.2) as

$$f^L(\mathbf{x}) = \sum_{i=0}^{N-1} (x_{(i+1)} - x_{(i)}) f(V_i(\mathbf{x})), \tag{6.4.3}$$

where $x_{(0)} := 0$ and $V_0(\mathbf{x}) := V$, and equivalently, as

$$f^L(\mathbf{x}) = \sum_{i=0}^{k-1} (x_{[i+1]} - x_{[i]}) f(V_{[i]}(\mathbf{x})),$$

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where

$$k \coloneqq 1 + \sum_{i=0}^{N-1} \operatorname{sign}(|f(V_i(\mathbf{x})) - f(V_{i+1}(\mathbf{x}))|),$$

while $\{[1], \ldots, [k]\} \subset V$ satisfy:

- $x_{[0]} = 0$,
- $x_{[1]} < \ldots < x_{[k]}$,
- $V_{[0]}(\mathbf{x}) \coloneqq V$,
- $V_{[i]}(\mathbf{x}) \coloneqq \{j \in V : x_j > x_{[i]}\}$, for $i \geq 1$,
- $f(V_{[i]}(\mathbf{x})) \neq f(V_{[i+1]}(\mathbf{x}))$, for all $i = 0, \ldots, k-1$.

We call the set $\{[1], \ldots, [k]\}$ a *simple index set* for f at $\mathbf{x}$. For a fixed vector $\mathbf{x}$, we also let $\operatorname{si}(f) \coloneqq \{i : f(V_{[i]}(\mathbf{x})) \neq 0\}$. One can then check that

$$f^L(\mathbf{x}) = \sum_{i \in \operatorname{si}(f)} (x_{[i+1]} - x_{[i]}) f(V_{[i]}(\mathbf{x})). \tag{6.4.4}$$

Given $p \geq 1$ and $\mathbf{x} \in \mathbb{R}^N_{\geq 0}$, by (6.4.4) and since f_e is non-negative, we have

$$\begin{aligned}
&\sum_{e \in E} w_e f_e^L(\mathbf{x}^p) && (6.4.5)\\
&= \sum_{e \in E} w_e \sum_{i \in \operatorname{si}(f_e)} (x^p_{[i+1]_e} - x^p_{[i]_e}) f_e(V_{[i]_e}(\mathbf{x}^p))\\
&\leq \sum_{e \in E} w_e \sum_{i \in \operatorname{si}(f_e)} p(x_{[i+1]_e} - x_{[i]_e}) \left(\frac{x^p_{[i+1]_e} + x^p_{[i]_e}}{2}\right)^{\frac{1}{p'}} f_e(V_{[i]_e}(\mathbf{x})) && (6.4.6)\\
&\leq \frac{p}{2^{\frac{1}{p'}}} \left(\sum_{e \in E} w_e \sum_{i \in \operatorname{si}(f_e)} f_e(V_{[i]_e}(\mathbf{x}))^p (x_{[i+1]_e} - x_{[i]_e})^p\right)^{\frac{1}{p}}\\
&\quad \left(\sum_{e \in E} w_e \sum_{i \in \operatorname{si}(f_e)} (x^p_{[i+1]_e} + x^p_{[i]_e})\right)^{\frac{1}{p'}} && (6.4.7)\\
&\leq \frac{p}{2^{\frac{1}{p'}}} \left(\sum_{e \in E} w_e \left(\sum_{i \in \operatorname{si}(f_e)} f_e(V_{[i]_e}(\mathbf{x}))(x_{[i+1]_e} - x_{[i]_e})\right)^p\right)^{\frac{1}{p}} \left(\sum_{j \in V} \widetilde{\deg}(j) x_j^p\right)^{\frac{1}{p'}}\\
&\leq \left(\frac{c}{2}\right)^{\frac{1}{p'}} p \left(\sum_{e \in E} w_e (f_e^L(\mathbf{x}))^p\right)^{\frac{1}{p}} \left(\sum_{j \in V} g(j) x_j^p\right)^{\frac{1}{p'}}, && (6.4.8)
\end{aligned}$$

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where p' is the Hölder conjugate of p, that is, $\frac{1}{p}+\frac{1}{p'}=1$; $\{[1]_e,\dots,[i]_e,\dots\}$ is a simple index set for f_e at $\mathbf{x}$;

$$\widetilde{\deg}(j):=\sum_{e\in\widetilde{E}_j} w_e, \text{ and } \widetilde{E}_j:=\{e\in E\colon j=[i]_e \text{ or } j=[i+1]_e \text{ for some } i\in \mathrm{si}(f_e)\}.$$

The first above inequality (6.4.6) follows from the inequality in Lemma 3 in [5]. The second inequality (6.4.7) uses Hölder's inequality, and the last inequality (6.4.8) follows from the fact that

$$\sum_{j\in V\colon x_j\geq t}\widetilde{\deg}(j)\leq \sum_{j\in V\colon x_j\geq t}\deg(j)\leq c\sum_{j\in V\colon x_j\geq t} g(j),\quad \text{for any } t\in\mathbb{R}. \tag{6.4.9}$$

In fact, for any $e\in\widetilde{E}_j$, we may assume that $j=[i]_e$, which implies that

$$f_e(V_{[i-1]_e}(\mathbf{x}))\neq f_e(V_{[i]_e}(\mathbf{x})).$$

Clearly, if $j'\neq j$ and $x_{[i]_e}\leq x_{j'}<x_{[i+1]_e}$, then $e\notin\widetilde{E}_{j'}$, since $j'\neq[i']_e$ for any i'. That is, e is counted exactly once in $\widetilde{E}_j$, for each j such that $x_j\in[x_{[i]_e},x_{[i+1]_e})$. We now want to show that there exists j' such that

$$e\in E_{j'}\quad\text{and}\quad x_{[i]_e}\leq x_{j'}<x_{[i+1]_e}.$$

To see this, suppose the contrary. Then, for each j' such that $x_{[i]_e}\leq x_{j'}<x_{[i+1]_e}$, we have that $e\notin E_{j'}$. Consider

$$\{j_1,\dots,j_l\}=\{j\colon x_{[i]_e}\leq x_j<x_{[i+1]_e}\}=V_{[i-1]_e}\setminus V_{[i]_e}.$$

Since $e\notin E_{j_1}\cup\ldots\cup E_{j_l}$, it follows that

$$\begin{aligned} f(V_{[i-1]_e}(\mathbf{x})) &= f(V_{[i-1]_e}(\mathbf{x})\setminus\{j_1\})=\ldots=f(V_{[i-1]_e}(\mathbf{x})\setminus\{j_1,\dots,j_l\})\\ &= f(V_{[i]_e}(\mathbf{x})), \end{aligned}$$

which is a contradiction. As a consequence, e is counted at least once in E_j, for all j such that $x_j\in[x_{[i]_e},x_{[i+1]_e})$. This proves the inequality in (6.4.9). Therefore,

$$\sum_{j\in V}\widetilde{\deg}(j)x_j^p=\int_0^\infty\sum_{j\in V\colon x_j^p\geq t}\widetilde{\deg}(j)dt\leq c\int_0^\infty\sum_{j\in V\colon x_j^p\geq t} g(j)dt=c\sum_{j\in V} g(j)x_j^p,$$

which implies (6.4.8).

We infer that, for $\mathbf{x}\in\mathbb{R}^N_{\geq 0}\setminus\{\mathbf{0}\}$,

$$\frac{\sum_{e\in E} w_e f_e^L(\mathbf{x}^p)}{\|\mathbf{x}\|_{p,g}^p}:=\frac{\sum_{e\in E} w_e f_e^L(\mathbf{x}^p)}{\sum_{j\in V} g(j)x_j^p}\leq p\left(\frac{c}{2}\right)^{\frac{1}{p'}}\left(\frac{\sum_{e\in E} w_e(f_e^L(\mathbf{x}))^p}{\sum_{j\in V} g(j)x_j^p}\right)^{\frac{1}{p}}.$$

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Similarly, for $\mathbf{x} \in \mathbb{R}^N_{\leq 0} \setminus \{\mathbf{0}\}$, if we let $|\mathbf{x}| = (|x_1|, \ldots, |x_N|)$ and we take $\tilde{f}_e(S) = f_e(V \setminus S)$, for all $S \subset V$, we have that

$$\begin{aligned}\frac{\sum_{e\in E} w_e \tilde{f}_e^L(|\mathbf{x}|^p)}{\|\mathbf{x}\|_{p,g}^p} &= \frac{\sum_{e\in E} w_e f_e^L(-|\mathbf{x}|^p)}{\|-|\mathbf{x}|\|_{p,g}^p} \\ &= \frac{\sum_{e\in E} w_e f_e^L(-|\mathbf{x}|^p)}{\sum_{j\in V} g(j)|x_j|^p} \\ &\leq p\left(\frac{c}{2}\right)^{\frac{1}{p'}} \left(\frac{\sum_{e\in E} w_e (f_e^L(\mathbf{x}))^p}{\sum_{j\in V} g(j)|x_j|^p}\right)^{\frac{1}{p}}.\end{aligned}$$

Now, for $\mathbf{x} \in \mathbb{R}^N \setminus (\mathbb{R}^N_{\geq 0} \cup \mathbb{R}^N_{\leq 0})$, write $\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_-$, with $(x_+)_i = \max\{x_i, 0\}$ and $(x_-)_i = \min\{x_i, 0\}$ for all $i \in V$. We derive that

$$\begin{aligned}\frac{\sum_{e\in E} w_e (f_e^L(\mathbf{x}))^p}{\sum_{j\in V} g(j)|x_j|^p} &= \frac{\sum_{e\in E} w_e |f_e^L(\mathbf{x}_+) + f_e^L(\mathbf{x}_-)|^p}{\sum_{j\in V} g(j)|x_{+,j}|^p + \sum_{j\in V} g(j)|x_{-,j}|^p} \\ &\geq \min\left\{\frac{\sum_{e\in E} w_e (f_e^L(\mathbf{x}_+))^p}{\|\mathbf{x}_+\|_{p,g}^p}, \frac{\sum_{e\in E} w_e (f_e^L(\mathbf{x}_-))^p}{\|\mathbf{x}_-\|_{p,g}^p}\right\} \\ &\geq \frac{1}{p^p}\left(\frac{2}{c}\right)^{\frac{p}{p'}} \min\left\{\frac{\sum_{e\in E} w_e f_e^L(\mathbf{x}_+^p)}{\|\mathbf{x}_+^p\|_{1,g}}, \frac{\sum_{e\in E} w_e \tilde{f}_e^L(|\mathbf{x}_-|^p)}{\|\mathbf{x}_-^p\|_{1,g}}\right\}^p \qquad (6.4.10) \\ &\geq \frac{(2/c)^{p-1}}{p^p} \min\left\{\frac{f(A_+)}{g(A_+)}, \frac{\tilde{f}(A_-)}{g(A_-)}\right\}^p \\ &\geq \frac{(2/c)^{p-1}}{p^p} \min\left\{\frac{\hat{f}(A_+)}{g(A_+)}, \frac{\hat{f}(A_-)}{g(A_-)}\right\}^p,\end{aligned}$$

for some nonempty subset $A_\pm \subset \operatorname{supp}(\mathbf{x}_\pm)$, where

$$\hat{f}(S) \coloneqq \min\{f(S), f(V \setminus S)\} = \min\{f(S), \tilde{f}(S)\}.$$

For any nonconstant vector $\hat{\mathbf{x}}$, since g is a counting measure function, there exists $c \in \{\hat{x}_1, \ldots, \hat{x}_N\} \subset \mathbb{R}$ such that the vector $\mathbf{x} = \hat{\mathbf{x}} - c\mathbf{1}$ satisfies

$$g(\operatorname{supp}(\mathbf{x}_+)) \leq \frac{1}{2} g(V) \quad \text{and} \quad g(\operatorname{supp}(\mathbf{x}_-)) \leq \frac{1}{2} g(V).$$

Therefore, since $f_e(V) = 0$, we obtain that

$$f_e^L(\mathbf{x}) = f_e^L(\hat{\mathbf{x}}) - c f_e(V) = f_e^L(\hat{\mathbf{x}}).$$

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Therefore,

$$\frac{\sum_{e\in E} w_e(f_e^L(\hat{\mathbf{x}}))^p}{\min\limits_{c\in\mathbb{R}} \sum_{e\in E} g_e^L(|\hat{\mathbf{x}} - c\mathbf{1}|^p)} \geq \frac{\sum_{e\in E} w_e(f_e^L(\mathbf{x}))^p}{\sum_{e\in E} g_e^L(|\mathbf{x}|^p)} \geq \frac{(2/c)^{p-1}}{p^p} \cdot \frac{\hat{f}(A)^p}{g(A)^p},$$

for some nonempty subset $A \subset \operatorname{supp}(\mathbf{x}_+) \subset \operatorname{supp}(\mathbf{x})$ or $A \subset \operatorname{supp}(\mathbf{x}_-) \subset \operatorname{supp}(\mathbf{x})$. As a consequence, $g(A) \leq \frac{1}{2}g(V)$, or equivalently, $g(A) \leq g(V \setminus A)$. One can then check that

$$\begin{aligned}\min_{A\neq\emptyset,V} \frac{\hat{f}(A)}{\min\{g(A), g(V\setminus A)\}} &= \min_{A\neq\emptyset,V} \frac{\min\{f(A), f(V\setminus A)\}}{\min\{g(A), g(V\setminus A)\}} \\ &= \min_{A\neq\emptyset,V} \frac{f(A)}{\min\{g(A), g(V\setminus A)\}} \\ &= h.\end{aligned}$$

Therefore,

$$\lambda \geq \frac{(2/c)^{p-1}}{p^p} h^p.$$

□

In order to relate Theorem 6.4.1 to our earlier estimates (4.3.15) and (4.3.16), we need particular choices of the functions f_e and g_e. Given a simple graph $G = (V, E)$, consider the functions $f_e, g_e : \mathcal{P}(V) \to \mathbb{R}$ defined by

$$f_e(A) := \begin{cases} 1, & \text{if } e \text{ has one end point in } A \text{ and the other in } V \setminus A, \\ 0, & \text{otherwise,} \end{cases} \tag{6.4.11}$$

and $g_e(A) := \#(e \cap A)$. By the original Lovász extension in Section 6.2, we have $f_e^L(\mathbf{x}) = |x_i - x_j|$ and $g_e^L(\mathbf{x}) = x_i + x_j$, where $\{i, j\} = e \in E$. Therefore,

$$\sum_{e\in E} (f_e^L(\mathbf{x}))^p = \sum_{\{i,j\}\in E} |x_i - x_j|^p$$

and

$$\sum_{e\in E} g_e^L(|\mathbf{x}|^p) = \sum_{\{i,j\}\in E} (|x_i|^p + |x_j|^p) = \sum_{i\in V} \deg(i)|x_i|^p.$$

Hence, in this case, the eigenvalue problem of the function pair (F_p, G_p) coincides with the eigenvalue problem of the normalized p-Laplacian on a graph (see Section 2.6.2 and Chapter 7).

Now, as in Section 6.3, we consider a chemical hypergraph satisfying $e_{in} \neq \emptyset \neq e_{out}$ and $\#(e_{in} \cup e_{out}) \geq 2$, for all $e \in E$. Let $f_e : \mathcal{P}(V) \to \mathbb{R}$ be defined by

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$$f_e(A) \coloneqq \begin{cases} 1, & \text{if } e_{in} \cap A \neq \emptyset \neq e_{out} \setminus A \text{ or } e_{out} \subset A \subset V \setminus e_{in}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.12}$$

Then, the Lovász extension of f_e is

$$f_e^L(\mathbf{x}) = \left| \max_{i \in e_{in}} x_i - \min_{j \in e_{out}} x_j \right|.$$

Moreover,

$$\lambda_2(\Delta_p) \coloneqq \inf_{\mathbf{x} \text{ non-constant}} \frac{\sum_{e \in E} \left| \max_{i \in e_{in}} x_i - \min_{j \in e_{out}} x_j \right|^p}{\min_{t \in \mathbb{R}} \sum_{i \in V} \deg(i) |x_i - t|^p}$$

is the second eigenvalue of the pair

$$\left(\sum_{e \in E} \left| \max_{i \in e_{in}} x_i - \min_{j \in e_{out}} x_j \right|^p, \sum_{i \in V} \deg(i) |x_i|^p \right).$$

Let now $\mu(i) \coloneqq \deg(i)$ in (6.3.3), for all $i \in V$, and let h denote the Cheeger constant in (6.3.3). Then, we have

Theorem 6.4.2 (Cheeger inequality for chemical hypergraphs)**.**

$$2^{p-1} \frac{h^p}{p^p} \leq \lambda_2(\Delta_p) \leq 2^{p-1} h. \tag{6.4.13}$$

In particular, for $p = 1$, we infer that $h = \lambda_2(\Delta_1)$, and this generalizes the identity relating the second eigenvalue of the graph 1-Laplacian to the Cheeger constant on a graph. If we consider, for example, the complete graph of order 4, then the Cheeger constant is $h = 2/3$, while the second eigenvalue of the p-Laplacian is $\lambda_2(\Delta_p) = 2^p/3$ for any $p \in [1,2]$. Clearly,

$$2^{p-1} \frac{(2/3)^p}{p^p} \leq \frac{2^p}{3} = 2^{p-1} \frac{2}{3}.$$

Hence, in this case, the upper bound $2^{p-1}h$ for $\lambda_2(\Delta_p)$ is tight for any $p \in [1,2]$. Also, if we let $p \to 1$, then the lower bound $2^{p-1}h^p/p^p$ for $\lambda_2(\Delta_p)$ tends to $2/3$, which coincides with $\lim_{p \to 1} \lambda_2(\Delta_p) = \lambda_2(\Delta_1)$.

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