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Hölder shadowing on finite intervals

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Abstract. For any $\theta, \omega > 1/2$, we prove that, if any d-pseudotrajectory of length $\sim 1/d^{\omega}$ of a diffeomorphism $f \in C^2$ can be d^{θ}-shadowed by an exact trajectory, then f is structurally stable. Previously it was conjectured [S. M. Hammel *et al*. Do numerical orbits of chaotic dynamical processes represent true orbits. *J. Complexity* **3** (1987), 136–145; S. M. Hammel *et al*. Numerical orbits of chaotic processes represent true orbits. *Bull. Amer. Math. Soc.* **19** (1988), 465–469] that for $\theta = \omega = 1/2$, this property holds for a wide class of non-uniformly hyperbolic diffeomorphisms. In the proof, we introduce the notion of a sublinear growth property for inhomogeneous linear equations and prove that it implies exponential dichotomy.

1. *Introduction and main results*

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well-developed part of the global theory of dynamical systems (see the monographs [**17**, **21**] and [**25**] for overviews of modern results). The shadowing problem is related to the following question: Under which conditions, for any pseudotrajectory of f, does there exist a close trajectory?

The study of this problem was originated by Anosov [**1**] and Bowen [**3**]. This theory is closely related to the classical theory of structural stability. It is well known that a diffeomorphism has a shadowing property in a neighborhood of a hyperbolic set [**1**, **3**] and a structurally stable diffeomorphism has a shadowing property on the whole manifold [**15**, **27**, **28**].

Let M be a smooth compact manifold of class C^{∞}, without boundary, with a Riemannian metric dist. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$. For an interval $I = (a, b)$, where $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{+\infty\}$ and $d > 0$, a sequence of points $\{y_k\}_{k \in I}$ is called a *d-pseudotrajectory* if the following inequalities hold:

$$\mathrm{dist}(y_{k+1}, f(y_k)) < d, \quad k \in \mathbb{Z}, \quad k, k+1 \in I.$$

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Definition 1.1. We say that f has the *standard shadowing property* (StSh) if for any $\varepsilon > 0$ there exists $d > 0$ such that, for any d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$, there exists a trajectory $\{x_k\}_{k\in\mathbb{Z}}$, such that

$$\operatorname{dist}(x_k, y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1}$$

In this case, we say that pseudotrajectory $\{y_k\}$ is ε-*shadowed* by $\{x_k\}$.

Currently, pseudotrajectories of finite length are not much investigated. This problem is strongly related to the dependence between ε and d in Definition 1.1. In the present paper, we study shadowing properties on finite intervals with polynomial dependences of ε and d. We give an upper bound for the lengths of shadowable pseudotrajectories for non-hyperbolic systems.

The most well-known definition relating ε and d is the following.

Definition 1.2. We say that f has the *Lipschitz shadowing property* (LipSh) if there exist constants $d_0, L > 0$ such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$, there exists a trajectory $\{x_k\}_{k\in\mathbb{Z}}$ such that inequalities (1) hold with $\varepsilon = Ld$.

Analyzing the proofs of the first shadowing results by Anosov [**1**] and Bowen [**3**], it is easy to see that in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz (and the same holds in the case of a structurally stable diffeomorphism [**21**]). Recently [**24**] the following was proved.

THEOREM 1.1. *A diffeomorphism $f \in C^1$ has a Lipschitz shadowing property if and only if it is structurally stable.*

Let us also mention the works [**16**, **23**], where a similar result was proved for the case of periodic and variational shadowing properties.

At the same time, it is easy to give an example of a diffeomorphism that is not structurally stable but has a standard shadowing property (see [**23**], for instance).

Let us introduce the following definition, which is the central notion under investigation in the present article.

Definition 1.3. We say that f has the *finite Hölder shadowing property* with exponents $\theta \in (0, 1)$, $\omega \geq 0$ (FinHolSh(θ, ω)) if there exist constants $d_0, L, C > 0$ such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in[0,Cd^{-\omega}]}$, there exists a trajectory $\{x_k\}_{k\in[0,Cd^{-\omega}]}$, such that

$$\operatorname{dist}(x_k, y_k) < Ld^{\theta}, \quad k \in [0, Cd^{-\omega}].$$

The main result of the present paper is the following.

THEOREM 1.2. *If a diffeomorphism $f \in C^2$ satisfies* FinHolSh(θ, ω) *with*

$$\theta > 1/2, \quad \theta + \omega > 1, \tag{2}$$

then f is structurally stable.

Note that previously Hammel, Yorke and Grebogi, based on results of numerical experiments, conjectured the following [**9**, **10**].

Conjecture 1.1. A typical dissipative map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies FinHolSh$(1/2, 1/2)$.

This conjecture suggests to us that Theorem 1.2 cannot be improved.

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Theorem 1.2 has an interesting consequence, even for the case of infinite pseudotrajectories.

Definition 1.4. We say that f has the *Hölder shadowing property* with exponent $\theta \in (0, 1)$ (HolSh(θ)) if there exist constants $d_0, L > 0$ such that for any $d < d_0$ and d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$, there exists a trajectory $\{x_k\}_{k\in\mathbb{Z}}$ such that inequalities (1) hold with $\varepsilon = Ld^\theta$.

It is easy to see that for any $\theta \in (0, 1)$ and $\omega \geq 0$, the following inclusions hold:

$$\mathrm{SS} = \mathrm{LipSh} \subset \mathrm{HolSh}(\theta) = \mathrm{FinHolSh}(\theta, +\infty) \subset \mathrm{FinHolSh}(\theta, \omega),$$

where SS denotes the set of structurally stable diffeomorphisms and LipSh, HolSh, FinHolSh denote sets of diffeomorphisms satisfying the corresponding shadowing properties.

The following theorem is a straightforward consequence of Theorem 1.2.

THEOREM 1.3. *If a diffeomorphism $f \in C^2$ satisfies* HolSh(θ) *with $\theta > 1/2$, then f is structurally stable.*

Note that this theorem generalizes Theorem 1.1. Let us also mention a related work [**12**], where some consequences of Hölder shadowing for one-dimensional maps were proved.

It is worth mentioning a relation between Theorem 1.3 and a question suggested by Katok.

Question 1.1. Is every diffeomorphism that is Hölder conjugate to an Anosov diffeomorphism itself Anosov?

Recently, it was shown that, in general, the answer to Question 1.1 is negative [**8**]. However, the following positive result was proved in [**7**, **8**].

THEOREM 1.4. *A C^2-diffeomorphism that is conjugate to an Anosov diffeomorphism via Hölder conjugacy h is Anosov itself, provided that the product of Hölder exponents for h and h^{-1} is greater than* $1/2$.

It is easy to show that diffeomorphisms that are Hölder conjugate to a structurally stable one satisfy the Hölder shadowing property. As a consequence of Theorem 1.3, we prove that a C^2-diffeomorphism that is conjugate to a structurally stable diffeomorphism via Hölder conjugacy h is structurally stable itself, provided that the product of Hölder exponents for h and h^{-1} is greater than $1/2$, which generalizes Theorem 1.4.

To illustrate that Theorems 1.2 and 1.3 are almost sharp, we give the following example.

Example 1.1. There exists a non-structurally stable C^∞-diffeomorphism $f : S^1 \to S^1$ satisfying HolSh($1/3$) and FinHolSh($1/2$, $1/2$).

It is easy to see that the identity map satisfies FinHolSh(θ, ω) provided that $\theta + \omega \leq 1$.

The paper is organized as follows.

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In §2, we introduce the main technical tools of the proof, i.e. the notion of the slow growth property of an inhomogeneous linear equation, and recall the notion of exponential dichotomy. We state Lemma 2.1 and Theorem 2.3, which are main steps of the proof, and show that Theorem 1.2 is their consequence.

In §§3 and 4, we give the proofs of Lemma 2.1 and Theorem 2.3.

In §5, we describe Example 1.1 and prove its properties.

2. *Slow growth property and exponential dichotomy*

Consider Euclidean spaces $E_{n\in\mathbb{Z}}$ of dimension m and a sequence $\mathcal{A} = \{A_{n\in\mathbb{Z}} : E_n \to E_{n+1}\}$ of linear isomorphisms, satisfying, for some $R > 0$, the following inequalities:

$$\|A_n\|, \|A_n^{-1}\| < R, \quad n \in \mathbb{Z}. \tag{3}$$

Definition 2.1. We say that a sequence $\mathcal{A}$ has a *slow growth* property with exponent $\gamma > 0$ ($\mathcal{A} \in \mathrm{SG}(\gamma)$) if there exists a constant $L > 0$ such that, for any $i \in \mathbb{Z}$, $N > 0$ and a sequence $\{w_k \in E_k\}_{k\in[i+1,i+N]}$, $|w_k| \le 1$, there exists a sequence $\{v_k \in E_k\}_{k\in[i,i+N]}$ satisfying

$$v_{k+1} = A_k v_k + w_{k+1}, \quad k \in [i, i+N-1], \tag{4}$$

$$|v_k| \le LN^{\gamma}, \quad k \in [i, i+N]. \tag{5}$$

If $\mathcal{A} \in \mathrm{SG}(\gamma)$ with $\gamma \in [0, 1)$, we say that it has a *sublinear growth* property. If $\mathcal{A} \in \mathrm{SG}(0)$, we say that it has a *bounded solution* property.

We have not found analogs of the notion of slow growth property in the literature. However, the notion of bounded solution property has been widely investigated, for example see [**2**, **4**, **6**, **13**, **18**–**20**, **26**].

We prove the following relations between shadowing and sublinear growth properties.

LEMMA 2.1. *If f satisfies the assumptions of Theorem 1.2, then there exists $\gamma \in (0, 1)$ such that, for any trajectory $\{p_k\}_{k\in\mathbb{Z}}$, the sequence $\{A_k = Df(p_k)\}$ satisfies* $\mathrm{SG}(\gamma)$.

To characterize sequences satisfying the sublinear growth property, we need the notion of exponential dichotomy (see [**6**], and for some generalizations see [**2**]).

Definition 2.2. We say that a sequence $\mathcal{A}$ has *exponential dichotomy* on $\mathbb{Z}^+$ if there exist numbers $C > 0$, $\lambda \in (0, 1)$ and a decomposition $E_k = E_k^{s,+} \oplus E_k^{u,+}$, $k \ge 0$, such that

$$E_{k+1}^{\sigma,+} = A_k E_k^{\sigma,+}, \quad k \ge 0, \sigma \in \{s, u\},$$

$$|A_{k+l-1} \cdot \cdots \cdot A_k v_k^s| \le C\lambda^l |v_k^s|, \quad k \ge 0, l > 0, v_k^s \in E_k^{s,+}, \tag{6}$$

$$|A_{k+l-1} \cdots A_k v_k^u| \ge \frac{1}{C}\lambda^{-l} |v_k^u|, \quad k \ge 0, l > 0, v_k^u \in E_k^{u,+}. \tag{7}$$

Similarly, we say that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$ if there exist numbers $C > 0$, $\lambda \in (0, 1)$ and a decomposition $E_k = E_k^{s,-} \oplus E_k^{u,-}$, $k \le 0$, such that

$$E_{k+1}^{\sigma,-} = A_k E_k^{\sigma,-}, \quad k < 0, \sigma \in \{s, u\},$$

$$|A_{k+l-1} \cdot \cdots \cdot A_k v_k^s| \le C\lambda^l |v_k^s|, \quad l > 0, l + k < 0, v_k^s \in E_k^{s,-},$$

$$|A_{k+l-1} \cdot \cdots \cdot A_k v_k^u| \ge \frac{1}{C}\lambda^{-l} |v_k^u|, \quad l > 0, l + k < 0, v_k^u \in E_k^{u,-}.$$

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Denote by $P_k^{s,+}$ the projection with the range $E_k^{s,+}$ and kernel $E_k^{u,+}$. Similarly we define $P_k^{u,+}$, $P_k^{s,-}$, $P_k^{u,-}$.

Remark 2.1. It is easy to show that there exists $H > 0$ such that (see, for instance, [**11**, Lemma 3.1], [**30**, Remark 2.3]),

$$|P_k^{\sigma,a} v_k| \le H|v_k|, \quad v_k \in E_k, \sigma \in \{s, u\}, a \in \{+, -\}, k \in \mathbb{Z}^a.$$

Remark 2.2. In Definition 2.2, we do not require the uniqueness of $E_k^{s,+}$, $E_k^{s,-}$, $E_k^{u,+}$, $E_k^{u,-}$. At the same time, if $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$, then $E_k^{s,+}$ is uniquely defined, and if $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$, then $E_k^{u,-}$ is uniquely defined [**20**, Proposition 2.3].

Recently, the following were shown [**30**, Theorem 1, 2].

THEOREM 2.1. *A sequence $\mathcal{A}$ has a bounded solution property if and only if the following two conditions hold.*
(ED) *$\mathcal{A}$ has exponential dichotomy both on $\mathbb{Z}^+$ and $\mathbb{Z}^-$.*
(TC) *The corresponding spaces $E_0^{s,+}$, $E_0^{u,-}$ satisfy the following transversality condition:*

$$E_0^{s,+} + E_0^{u,-} = E_0.$$

THEOREM 2.2. *The following statements are equivalent.*
(i) *$\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$ ($\mathbb{Z}^-$).*
(ii) *There exists $L > 0$ such that, for any sequence $\{w_k \in E_k\}$, $k \ge 0$ ($k \le 0$), satisfying $|w_k| \le 1$, there exists a sequence $\{v_k \in E_k\}_{k\in\mathbb{Z}}$, such that $|v_k| \le L$ and*

$$v_{k+1} = A_k v_k + w_{k+1}, \tag{8}$$

for $k \ge 0$ ($k \le 0$).

Remark 2.3. Such results were also considered in [**5**, **11**, **20**, **29**]; however, we were not able to find in the earlier literature statements that imply Theorems 2.1 and 2.2. Similar results not for sequences of isomorphisms but for inhomogeneous linear systems of differential equations were obtained in [**6**, **13**, **17**, **26**]. The relation between discrete and continuous settings is discussed in [**22**].

In this paper, we prove the following theorem, which is interesting in itself, without relation to shadowing property.

THEOREM 2.3. *If a sequence $\mathcal{A}$ has a sublinear growth property, then it satisfies properties* (ED) *and* (TC).

As a consequence of this theorem, we conclude that the sublinear growth property and bounded solution property are in fact equivalent.

Remark 2.4. Note that sequences $\mathcal{A} \in \mathrm{SG}(1)$ do not necessarily satisfy condition (ED). A trivial example in arbitrary an dimension is $\mathcal{A} = \{A_k = \mathrm{Id}\}$.

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Let us now give the proof of Theorem 1.2.

For a point $p \in M$, we define the following two subspaces of T_pM:

$$B^+(p) = \{v \in T_pM \,:\, |Df^k(p)v| \to 0,\ k \to +\infty\},$$

$$B^-(p) = \{v \in T_pM \,:\, |Df^k(p)v| \to 0,\ k \to -\infty\}.$$

THEOREM 2.4. (Mañé [**14**]) *Diffeomorphism f is structurally stable if and only if*

$$B^+(p) + B^-(p) = T_pM, \quad p \in M.$$

Theorem 1.2 is a consequence of Lemma 2.1 and Theorems 2.3 and 2.4.

3. *Proof of Lemma 2.1*

Let exp be the standard exponential mapping on the tangent bundle of M, and let $\exp_x : T_xM \to M$ be the corresponding exponential mapping at a point x. Denote by $B(r, x)$ the ball in M of radius r centered at a point x, and by $B_T(r, x)$ the ball in T_xM of radius r centered at the origin.

There exists $\varepsilon > 0$ such that, for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(\varepsilon, x)$ onto its image, and $\exp_x^{-1}$ is a diffeomorphism of $B(\varepsilon, x)$ onto its image. In addition, we may assume that ε has the following property.

If $v, w \in B_T(\varepsilon, x)$, then

$$\frac{\operatorname{dist}(\exp_x(v), \exp_x(w))}{|v - w|} \le 2, \tag{9}$$

and if $y, z \in B(\varepsilon, x)$, then

$$\frac{|\exp_x^{-1}(y) - \exp_x^{-1}(z)|}{\operatorname{dist}(y, z)} \le 2. \tag{10}$$

Let $L, C, d_0 > 0$ and $\theta \in (1/2, 1)$, $\omega > 0$ be the constants from the definition of FinHolSh. Define $\alpha = \theta - 1/2$. Inequalities (2) imply that

$$\alpha \in (0, 1/2), \quad 1/2 - \alpha < \omega. \tag{11}$$

Since M is compact, and $f \in C^2$, there exists $S > 0$ such that

$$\operatorname{dist}(f(\exp_x(v)), \exp_{f(x)}(Df(x)v)) \le S|v|^2, \quad x \in M,\ v \in T_xM,\ |v| < \varepsilon \tag{12}$$

(we additionally decrease ε, if necessary).

Fix $i \in \mathbb{Z}$ and $N > 0$. For an arbitrary sequence $\{w_k \in T_{p_k}M\}_{k\in[i+1,i+N+1]}$ with $|w_k| \le 1$, consider the following equation:

$$v_{k+1} = A_k v_k + w_{k+1}, \quad k \in [i, i + N]. \tag{13}$$

For any sequence $\{v_k \in T_{p_k}M\}_{k\in[i,i+N+1]}$, define $\|\{v_k\}\| = \max_{k\in[i,i+N+1]} |v_k|$. For any sequence $\{w_k \in T_{p_k}M\}_{k\in[i+1,i+N+1]}$, consider the set

$$E(i, N, \{w_k\}) = \{\{v_k\}_{k\in[i,i+N+1]} \text{ satisfies (13)}\}.$$

Define

$$F(i, N, \{w_k\}) = \min_{\{v_k\}\in E(i,N,\{w_k\})} \|\{v_k\}\|. \tag{14}$$

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Since $\|\cdot\| \geq 0$ is a continuous function on the linear space of sequences $\{v_k\}$, and the set $E(i, N, \{w_k\})$ is closed, it follows that the value $F(i, N, \{w_k\})$ is well defined. Note that a sequence $\{v_k\} \in E(i, N, \{w_k\})$ is determined by the value v_i. Consider the sequence $\{v_k\}$, corresponding to $v_i = 0$. It is easy to see that $|v_{i+k}| \leq 1 + R + R^2 + \cdots + R^k$ for $k \in [0, N+1]$, where $R = \max_{x \in M} \|Df(x)\|$. Hence, $F(i, N, \{w_k\}) \leq 1 + R + R^2 + \cdots + R^{2N}$, for any $\{|w_k| \leq 1\}$. It is easy to see that $F(i, N, \{w_k\})$ is continuous with respect to $\{w_k\}$, and hence

$$Q = Q(i, N) = \max_{\{w_k\},\, |w_k| \leq 1} F(i, N, \{w_k\}) \tag{15}$$

is well defined.

Let us choose sequences $\{w_k\}$ and $\{v_k\} \in F(i, N, \{w_k\})$, such that

$$Q(i, N) = F(i, N, \{w_k\}), \quad F(i, N, \{w_k\}) = \|\{v_k\}\|.$$

The definition of Q and linearity of equation (13) imply the following two properties.

(Q1) For any sequence $\{w'_k\}_{k \in [i+1, i+N+1]}$, there exists a sequence $\{v'_k\}_{k \in [i, i+N+1]}$ satisfying

$$v'_{k+1} = A_k v'_k + w'_{k+1}, \quad \|\{v'_k\}\| \leq Q(i, N)\|\{w'_k\}\|.$$

(Q2) For any sequence $\{v_k\}_{k \in [i, i+N+1]}$, satisfying (13), the following inequality holds:

$$\|\{v_k\}\| \geq Q(i, N).$$

Relations (11) imply that there exists $\beta > 0$, such that the following conditions holds:

$$0 < (2+\beta)(1/2 - \alpha) < 1, \quad (2+\beta)\omega > 1. \tag{16}$$

Define

$$\gamma = \frac{1}{(2+\beta)\omega} \in (0, 1), \quad \gamma' = 1 - (2+\beta)(1/2 - \alpha) > 0,$$

$$d = \frac{\varepsilon}{Q^{2+\beta}}. \tag{17}$$

Let us prove that there exist $L' > 0$ independent of i and N, such that

$$Q(i, N) \leq L' N^{\gamma}. \tag{18}$$

Below, we consider two cases.

Case 1. $C((S+2)d)^{-\omega} < N$. Then $Q < (\varepsilon^{\omega}(S+2)^{\omega}/C)^{\gamma} N^{\gamma}$ and inequality (18) is proved.

Case 2. $C((S+2)d)^{-\omega} \geq N$. Below, we prove an even stronger statement: there exists $L' > 0$ (independent of i and N), such that

$$Q(i, N) \leq L'. \tag{19}$$

Considering the trajectory $\{p'_k = f^{-i}(p_k)\}$, we can assume, without loss of generality, that $i = 0$.

Consider the sequence

$$y_k = \exp_{p_k}(d v_k), \quad k \in [0, N].$$

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Let us show that $\{y_k\}$ is an $(S+2)d$-pseudotrajectory. For $k \in [0, N]$, equations (9), (12) and inequalities $|dv_k| < \varepsilon$, $(dQ)^2 < d$ imply the following:

$$\begin{aligned}\operatorname{dist}(f(y_k), y_{k+1}) &= \operatorname{dist}(f(\exp_{p_k}(dv_k)), \exp_{p_{k+1}}(d(A_k v_k + w_{k+1}))\\ &\le \operatorname{dist}(f(\exp_{p_k}(dv_k)), \exp_{p_{k+1}}(dA_k v_k))\\ &\quad + \operatorname{dist}(\exp_{p_{k+1}}(dA_k v_k), \exp_{p_{k+1}}(d(A_k v_k + w_k)))\\ &\le S|dv_k|^2 + 2d \le (S+2)d. \end{aligned} \tag{20}$$

We may assume that

$$Q > ((S+2)\varepsilon/d_0)^{1/(2+\beta)}. \tag{21}$$

Indeed, the right-hand side of (21) does not depend on N, and if Q is smaller than the right-hand side of (21), then we have already proved (18). In the text below, we make similar remarks several times, to ensure that Q is large enough.

Inequality (21) implies that $(S+2)d < d_0$. Since $f \in \operatorname{FinHolSh}(1/2+\alpha, \omega)$, and the assumption of case 2 holds, it follows that the pseudotrajectory $\{y_k\}_{k\in[0,N]}$ can be $L((S+2)d)^{1/2+\alpha}$-shadowed by a trajectory $\{x_k\}_{k\in[0,N]}$.

By reasoning similar to (21), we may assume that $L((S+2)d)^{1/2+\alpha} < \varepsilon/2$. Inequalities (9) and (21) imply that for $k \in [0, N]$, the following inequalities hold:

$$\operatorname{dist}(p_k, x_k) \le \operatorname{dist}(p_k, y_k) + \operatorname{dist}(y_k, x_k) \le 2d|v_k| + L((S+2)d)^{1/2+\alpha} < \varepsilon.$$

Hence, $c_k = \exp_{p_k}^{-1}(x_k)$ is well defined.

Define $L_1 = L(S+2)^{1/2+\alpha}$. Since $\operatorname{dist}(y_k, x_k) < L_1 d^{1/2+\alpha}$, inequalities (10) imply that

$$|dv_k - c_k| < 2L_1 d^{1/2+\alpha}. \tag{22}$$

Hence,

$$|c_k| < Qd + 2L_1 d^{1/2+\alpha}. \tag{23}$$

By reasoning similar to (21), we can assume that $|c_k| < \varepsilon$.

Since $f(x_k) = x_{k+1}$, inequalities (10) and (12) imply that for $k \in [0, N]$, the following relations hold:

$$\begin{aligned}|c_{k+1} - A_k c_k| &< 2\operatorname{dist}(\exp_{p_{k+1}}(c_{k+1}), \exp_{p_{k+1}}(A_k c_k))\\ &= 2\operatorname{dist}(f(\exp_{p_k}(c_k)), \exp_{p_{k+1}}(A_k c_k)) \le 2S|c_k|^2. \end{aligned} \tag{24}$$

Inequalities (16), (17), and (23) imply that $|c_k| < L_2 Qd$ for some $L_2 > 0$, independent of N.

Let $t_{k+1} = c_{k+1} - A_k c_k$. By inequality (24), it follows that

$$|t_k| \le 2S|c_k|^2 \le L_3(Qd)^2,$$

for some $L_3 > 0$, independent of N. Property (Q1) implies that there exists a sequence $\{\tilde{c}_k \in T_{p_k}M\}$ satisfying

$$\tilde{c}_{k+1} - A_k \tilde{c}_k = t_{k+1}, \quad |\tilde{c}_k| \le QL_3(Qd)^2, \quad k \in [0, N].$$

Consider the sequence $r_k = c_k - \tilde{c}_k$. Obviously, it satisfies the following conditions:

$$r_{k+1} = A_k r_k, \quad |r_k - c_k| \le QL_3(Qd)^2, \quad k \in [0, N]. \tag{25}$$

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Consider the sequence $e_k = (1/d)(dv_k - r_k)$. Equations (22) and (25) imply that

$$e_{k+1} = A_k e_k + w_k, \quad k \in [0, N] \tag{26}$$

and

$$|e_k| = \left| \frac{1}{d}((dv_k - c_k) - (r_k - c_k)) \right| \le L_1 d^{-1/2+\alpha} + L_3 Q^3 d, \quad k \in [0, N].$$

Property (Q2) implies that

$$L_1 d^{-1/2+\alpha} + L_3 Q^3 d \ge Q.$$

By (17), the last inequality is equivalent to

$$L_4 Q^{-(2+\beta)(-1/2+\alpha)} + L_5 Q^{1-\beta} \ge Q,$$

where L_4, $L_5 > 0$ do not depend on N. This inequality and (16) imply that

$$L_4 Q^{1-\gamma'} + L_5 Q^{1-\beta} \ge Q.$$

Hence,

$$L_4 Q^{1-\gamma'} \ge Q/2 \quad \text{or} \quad L_5 Q^{1-\beta} \ge Q/2$$

and

$$Q \le \max((2L_4)^{1/\gamma'}, (2L_5)^{1/\beta}).$$

We have proved that there exists $L' > 0$ such that (19) holds. This completes the proof of Case 2 and Lemma 2.1.

4. *Proof of Theorem 2.3*

Let us first prove the following.

LEMMA 4.1. *If a sequence $\mathcal{A}$ satisfies the slow growth property and* (ED) *then it satisfies* (TC).

Proof. Let L, $\gamma > 0$ be the constants from the definition of slow growth property and let $C > 0$, $\lambda \in (0, 1)$ be the constants from the definition of exponential dichotomy on $\mathbb{Z}^{\pm}$. Let H be the constant from Remark 2.1 for exponential dichotomies on $\mathbb{Z}^{\pm}$. Assume that $E_0^{s,+} + E_0^{u,-} \ne E_0$. Let us choose a vector $\eta \in E_0 \setminus (E_0^{s,+} + E_0^{u,-})$ satisfying $|\eta| = 1$. Define $a = \operatorname{dist}(\eta, E_0^{s,+} + E_0^{u,-})$. Consider the sequence $\{w_k \in E_k\}_{k\in\mathbb{Z}}$, defined by the formula

$$w_k = \begin{cases} 0, & k \ne 0, \\ \eta, & k = 0. \end{cases}$$

Take $N > 0$ and an arbitrary solution $\{v_k\}_{k\in[-N,N]}$ of

$$v_{k+1} = A_k v_k + w_k, \quad k \in [-N, N-1]. \tag{27}$$

Define $v_k^{s,+} = P_k^{s,+} v_k$, $v_k^{u,+} = P_k^{u,+} v_k$, for $k \ge 0$. Since $w_k = 0$ for $k > 0$, we conclude that

$$|v_N^{u,+}| \ge \frac{1}{C} \lambda^{-(N-1)} |v_0^{u,+}|$$

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and hence

$$|v_N| \geq \frac{1}{H}\frac{1}{C}\lambda^{-(N-1)}|v_0^{u,+}|. \tag{28}$$

Similarly, we define $v_k^{s,-} = P_k^{s,-}v_k$, $v_k^{u,-} = P_k^{u,-}v_k$, for $k \leq 0$, and conclude that

$$|v_{-N}| \geq \frac{1}{H}\frac{1}{C}\lambda^{-(N-1)}|v_{-1}^{s,-}|. \tag{29}$$

Equality (27) implies that

$$v_0 = A_{-1}v_{-1} + \eta$$

and hence

$$\max(\operatorname{dist}(v_0, E_0^{s,+} + E_0^{u,-}), \operatorname{dist}(A_{-1}v_{-1}, E_0^{s,+} + E_0^{u,-})) \geq a/2.$$

From this inequality, it is easy to conclude that

$$v_0^{u,+} \geq \frac{1}{H}\frac{a}{2} \quad \text{or} \quad v_{-1}^{s,-} \geq \frac{1}{R}\frac{1}{H}\frac{a}{2}. \tag{30}$$

Inequalities (28)–(30) imply that

$$\max(|v_N|, |v_{-N}|) \geq \frac{1}{H}\frac{1}{C}\lambda^{-(N-1)}\frac{1}{R}\frac{1}{H}\frac{a}{2}.$$

Note that for large enough N, the right-hand side of this inequality is greater than $L(2N + 1)^{\gamma}$, which contradicts the sublinear growth property. □

Now let us pass to the proof of Theorem 2.3. We prove this statement by induction over m (dimension of the Euclidean spaces). First, we prove the following.

LEMMA 4.2. *Theorem 2.3 holds for* $m = 1$.

Proof. Choose a vector $e_0 \in E_0$, $|e_0| = 1$, and consider the sequence $\{e_k \in E_k\}_{k\in\mathbb{Z}}$, defined by the relations

$$e_{k+1} = \frac{A_k e_k}{|A_k e_k|}, \quad e_{-k-1} = \frac{A_{-k-1}^{-1}e_{-k}}{|A_{-k-1}^{-1}e_{-k}|}, \quad k \geq 0. \tag{31}$$

Let $\lambda_k = |A_k e_k|$. Inequalities (3) imply that

$$\lambda_k \in (1/R, R), \quad k \in \mathbb{Z}. \tag{32}$$

Define

$$\Pi(k, l) = \lambda_k \cdot \cdots \cdot \lambda_{k+l-1}, \quad k \in \mathbb{Z}, l \geq 1. \tag{33}$$

Let us prove the following lemma, which is the heart of the proof of Theorem 2.3.

LEMMA 4.3. *If* $m = 1$ *and* $\mathcal{A}$ *satisfies the sublinear growth property, then there exists* $N > 0$ *such that, for any* $i \in \mathbb{Z}$,

$$\Pi(i, N) > 2 \quad \text{or} \quad \Pi(i + N, N) < 1/2.$$

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Proof. The proof follows the ideas of [**16**].

Let us fix $i \in \mathbb{Z}$, $N > 0$, and consider the sequence

$$w_k = -e_k, \quad k \in [i+1, i+2N+1].$$

By the sublinear growth property, there exists a sequence $\{v_k\}_{k\in[i,i+2N+1]}$ satisfying

$$v_{k+1} = A_k v_k + w_k, \quad |v_k| \le L(2N+1)^\gamma, \quad k \in [i, i+2N].$$

Let $v_k = a_k e_k$, where $a_k \in \mathbb{R}$; then

$$a_{k+1} = \lambda_k a_k - 1, \quad |a_k| \le L(2N+1)^\gamma, \quad k \in [i, i+2N]. \tag{34}$$

These relations easily imply the following.

PROPOSITION 4.1. *If $a_k \le 0$ for some $k \in [i, i+2N-1]$, then $a_{k+1} < 0$.*

Below, we prove the following: there exists a large $N > 0$ (depending only on R, L, and γ) such that:

Case 1. If $a_{i+N-1} \ge 0$ then $\Pi(i, N) > 2$.

Case 2. If $a_{i+N-1} < 0$ then $\Pi(i+N, N) < 1/2$.

We give the proof of case 1 in detail; the second case is similar. Proposition 4.1 implies that $a_i, \dots, a_{i+N-2} > 0$, $a_{i+N-1} \ge 0$. Relation (34) implies that

$$\lambda_k = \frac{a_{k+1}+1}{a_k}, \quad k \in [i, i+N-1].$$

The following relations hold:

$$\begin{aligned}\Pi(i, N) &= \frac{a_{i+1}+1}{a_i}\frac{a_{i+2}+1}{a_{i+1}}\cdots\frac{a_{i+N-1}+1}{a_{i+N-2}} \\ &= \frac{1}{a_i}\frac{a_{i+1}+1}{a_{i+1}}\frac{a_{i+2}+1}{a_{i+2}}\cdots\frac{a_{i+N-2}+1}{a_{i+N-2}}(a_{i+N-1}+1) \\ &= \frac{a_{i+N-1}+1}{a_i}\prod_{k=i+1}^{i+N-2}\frac{a_k+1}{a_k} \ge \frac{1}{L(2N+1)^\gamma}\left(1+\frac{1}{L(2N+1)^\gamma}\right)^{N-2}.\end{aligned}$$

Denote the latter expression by $G_\gamma(N)$. The inclusion $\gamma \in (0, 1)$ implies that

$$\lim_{N\to+\infty} G_\gamma(N) = +\infty \tag{35}$$

and, for large enough N, the inequality $G_\gamma(N) > 2$ holds, which completes the proof of case 1.

Remark 4.1. In relation (35), we essentially use that $\gamma \in (0, 1)$; for $\gamma \ge 1$, it does not hold. □

LEMMA 4.4. *Let N be the number from Lemma 4.3.*

(i) *If $\Pi(i, N) > 2$, then $\Pi(i-N, N) > 2$.*

(ii) *If $\Pi(i, N) < 1/2$, then $\Pi(i+N, N) < 1/2$.*

Proof. We prove statement (i); the second one is similar. Lemma 4.3 implies that either $\Pi(i-N, N) > 2$ or $\Pi(i, N) < 1/2$. By the assumptions of Lemma 4.4, the second case is not possible, and hence $\Pi(i-N, N) > 2$. □

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Now let us complete the proof of Lemma 4.2. It is easy to conclude from Lemmas 4.3 and 4.4 that one of the following cases holds.

Case 1. For all $i \in \mathbb{Z}$, the inequality $\Pi(i, N) > 2$ holds. Then

$$\Pi(i, l) \geq R^{N-1}(2^{1/N})^l, \quad i \in \mathbb{Z}, l > 0$$

and hence $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^{\pm}$, with the splitting

$$E_k^{s,\pm} = \{0\}, \quad E_k^{u,\pm} = \langle e_k \rangle, \quad k \in \mathbb{Z}.$$

Case 2. For all $i \in \mathbb{Z}$, the inequality $\Pi(i, N) < 1/2$ holds. Similarly to the previous case, $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^{\pm}$, with the splitting

$$E_k^{s,\pm} = \langle e_k \rangle, \quad E_k^{u,\pm} = \{0\}.$$

Case 3. There exist $i_1, i_2 \in \mathbb{Z}$ such that

$$\Pi(i_1, N) > 2, \quad \Pi(i_2, N) < 1/2.$$

Similarly to Case 1, the inequality

$$\Pi(k, l) \geq R^{N-1}(2^{1/N})^l, \quad k + l < i_1, l > 0$$

holds and hence

$$\Pi(k, l) \geq R^{|i_1|+N-1}(2^{1/N})^l, \quad k + l < 0, l > 0.$$

The last inequality implies that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$, with the splitting

$$E_k^{s,-} = \{0\}, \quad E_k^{u,-} = \langle e_k \rangle, \quad k \leq 0.$$

Similarly, $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$, with the splitting

$$E_k^{s,+} = \langle e_k \rangle, \quad E_k^{u,+} = \{0\}, \quad k \geq 0.$$

In all of these cases, Lemma 4.2 is proved. □

Now let us continue the proof of Theorem 2.3. Assume that Theorem 2.3 is proved for $\dim E_k \leq m$. Below, we prove it for $\dim E_k = m + 1$.

Let us choose a unit vector $e_0 \in E_0$ and consider the vectors $\{e_k\}_{k\in\mathbb{Z}}$ defined by relations (31). Define $\lambda_k = |A_k e_k|$. Similarly to Lemma 4.2, inclusions (32) hold. For $k \in \mathbb{Z}$, let S_k be the orthogonal complement of e_k in E_k, and let Q_k be the orthogonal projection onto S_k. Note that $\dim S_k = m$. Consider the linear operators $B_k : S_k \to S_{k+1}$, $D_k : S_k \to \langle e_{k+1} \rangle$, defined by the following:

$$B_k = Q_{k+1}A_k, \quad D_k = (\mathrm{Id} - Q_{k+1})A_k, \quad k \in \mathbb{Z}.$$

Note that $B_k^{-1} = Q_{k-1}A_k^{-1}$ and

$$\|B_k\|, \|B_k^{-1}\|, \|D_k\| < R. \tag{36}$$

For any vector $b \in E_k$, define $b^{\perp} = P_k b, \quad b^1 = b - b^{\perp}$. We also write $b = (b^{\perp}, b^1)$. In such notation, equations (4) are equivalent to

$$v_{k+1}^{\perp} = B_k v_k^{\perp} + w_{k+1}^{\perp}, \tag{37}$$

$$v_{k+1}^{1} = \lambda_k v_k^{1} + D_k v_k^{\perp} + w_{k+1}^{1}. \tag{38}$$

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Let us prove that the sequence $\{B_k\}$ satisfies property SG(γ). Indeed, fix $i \in \mathbb{Z}$, $N > 0$ and consider an arbitrary sequence $\{w_k^\perp \in S_k\}_{k\in[i+1,i+N+1]}$ with $|w_k^\perp| \le 1$. Consider the sequence $\{w_k \in E_k\}_{k\in[i+1,i+N+1]}$, defined by $w_k = w_k^\perp$. By the sublinear growth property, there exists a sequence $\{v_k \in E_k\}_{k\in[i,i+N+1]}$ satisfying equations (4) and (5), and hence (37). Recalling that $|v_k^\perp| \le |v_k|$, we conclude that the sequence $\{B_k\}$ satisfies the sublinear growth property, and hence, by the induction assumption, if satisfies conditions (ED) and (TC) from Theorem 2.1.

Below, we prove that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$. Let $\{B_i\}$ satisfy exponential dichotomy on $\mathbb{Z}^+$ with constants $C > 0$, $\lambda \in (0, 1)$ and splitting $S_k = S_k^{s,+} \oplus S_k^{u,+}$. Let H_1 be the constant from Remark 2.1 for this splitting.

First, we prove that there exists a large $N > 0$ such that, for any $i \ge 2N$, the following inequality holds:

$$\Pi(i, N) > 2 \quad \text{or} \quad \Pi(i - N, N) < 1/2, \tag{39}$$

where $\Pi(k, l)$ is defined by (33).

Let us choose $N > 0$, satisfying

$$C\lambda^N H_1 L(4N)^\gamma < 1/(4R), \tag{40}$$

and consider some $i \ge 2N$. Define a sequence $\{w_k = -e_k\}_{k\in[i-2N,i+2N]}$. By the slow growth property, there exists a sequence $\{v_k = (v_k^\perp, v_k^1)\}_{k\in[i-2N,i+2N+1]}$ satisfying the following, for $k \in [i - 2N, i + 2N]$:

$$v_{k+1}^\perp = B_k v_k^\perp, \tag{41}$$

$$v_{k+1}^1 = \lambda_k v_k^1 + D_k v_k^\perp - 1, \tag{42}$$

$$|v_k| < L(4N)^\gamma. \tag{43}$$

Define $v_k^\perp = v_k^{\perp,s} + v_k^{\perp,u}$, where $v_k^{\perp,s} \in S_k^{s,+}$, $v_k^{\perp,u} \in S_k^{u,+}$. Applying relations (41) and (43), and Remark 2.1, we conclude that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < H_1 L(4N)^\gamma, \quad k \in [i - 2N, i + 2N].$$

Exponential dichotomy of $\{B_i\}$ implies that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < C\lambda^N H_1 L(4N)^\gamma, \quad k \in [i - N, i + N].$$

By inequality (40), we conclude that

$$|v_k^{\perp,s}|, |v_k^{\perp,u}| < 1/(4R), \quad k \in [i - N, i + N]$$

and hence

$$|v_k^\perp| < 1/(2R), \quad k \in [i - N, i + N]. \tag{44}$$

Define $b_k = D_k v_k^\perp - 1$. Inequalities (36) and (44) imply that

$$b_k \in (-3/2, -1/2), \quad k \in [i - N, i + N].$$

Using these inclusions, relations (42) and (43), and arguing similarly to Lemma 4.3 (increasing N if necessary), we conclude relation (39).

Arguing similarly to the proof of Lemma 4.2, we conclude that the linear operators generated by λ_i have exponential dichotomy on $\mathbb{Z}^+$.

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Let us show that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$. Consider an arbitrary sequence

$$\{w_k = (w_k^\perp, w_k^1) \in E_k\}_{k\geq 0}, \quad |w_k| \leq 1.$$

Since $\{B_k\}$ has exponential dichotomy on $\mathbb{Z}^+$, by Theorem 2.2, there exists a sequence $\{v_k^\perp \in S_k\}_{k\geq 0}$, satisfying (37) and $|v_k| \leq L_1$, where $L_1 > 0$ does not depend on $\{w_k\}$. Inequality (36) implies that

$$|D_k v_k^\perp + w_{k+1}^1| \leq L_1 R + 1, \quad k \geq 0.$$

Since linear operators generated by λ_k have exponential dichotomy on $\mathbb{Z}^+$, by Theorem 2.2, there exists $\{v_k^1 \in \mathbb{R}\}$ such that, for $k \geq 0$ equalities (38) hold, and $|v_k^1| \leq L_2(L_1 R + 1)$, where L_2 does not depend on $\{w_k\}$.

Hence, for $k \geq 0$, the sequence $v_k = (v_k^\perp, v_k^1)$ satisfies (8) and

$$|v_k| \leq |v_k^\perp| + |v_k^1| \leq L_2(L_1 R + 1) + L_1.$$

Theorem 2.2 implies that $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^+$.

Similarly, $\mathcal{A}$ has exponential dichotomy on $\mathbb{Z}^-$, and hence satisfies property (ED). By Lemma 4.1, the sequence $\mathcal{A}$ also satisfies property (TC). This completes the induction step and the proof of Theorem 2.3.

5. *Example 1.1*

Consider a diffeomorphism $f : S^1 \to S^1$ constructed as follows.

(i) The non-wandering set of f consists of two fixed points $s, u \in S^1$.

(ii) In some neighborhood U_s of s, there exists a coordinate system such that $f|_{U_s}(x) = x/2$.

(iii) In some neighborhood U_u of u, there exists a coordinate system such that $f|_{U_u}(x) = x + x^3$.

(iv) In $S^1 \setminus (U_s \cup U_u)$, the map is chosen to be C^∞ and to satisfy the following condition: there exists $N > 2$ such that

$$f^N(S^1 \setminus U_u) \subset U_s, \quad f^{-N}(S^1 \setminus U_s) \subset U_u, \quad f^2(U_u) \cap U_s = \emptyset.$$

THEOREM 5.1. *If $f : S^1 \to S^1$ satisfies the above properties (i)–(iv), then $f \in \mathrm{HolSh}(1/3)$ and $f \in \mathrm{FinHolSh}(1/2, 1/2)$.*

Proof. First, let us prove a technical statement.

LEMMA 5.1. *Define $g(x) = x + x^3$. If $|x - y| \geq \varepsilon$, then*

$$|g(x) - g(y)| \geq \varepsilon + \varepsilon^3/4.$$

Proof. Using inequality $x^2 + xy + y^2 > (x - y)^2/4$, we deduce that

$$\begin{aligned} |g(x) - g(y)| &= |x + x^3 - y - y^3| = |(x - y)(1 + x^2 + xy + y^2)| \\ &\geq |(x - y)||1 + (x - y)^2/4| \geq \varepsilon(1 + \varepsilon^2/4). \end{aligned}$$

□

We divide the proof of Theorem 5.1 into several propositions.

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PROPOSITION 5.1. *Conditions (ii) and (iii) imply that there exists $d_1 > 0$ such that*

$$B(d_1, f(U_s)) \subset U_s, \quad B(d_1, f^{-1}(U_u)) \subset U_u, \quad B(d_1, f(S^1 \setminus U_u)) \subset S^1 \setminus U_u. \quad (45)$$

Since $f|_{U_s}$ is hyperbolically contracting, there exist $L > 0$ and $d_2 \in (0, d_1)$ such that, for any d-pseudotrajectory $\{y_k\}$ with $d < d_2$ and $y_0 \in S^1 \setminus U_u$, the following conditions hold.

(i) $\{y_k\}_{k \geq 0} \subset S^1 \setminus U_u$.

(ii) $\operatorname{dist}(f^k(x_0), y_k) < Ld$, *for* $x_0 \in B(d, y_0)$, $k \geq 0$.

(iii) *If* $\{y_k\}_{k \in \mathbb{Z}} \subset S^1 \setminus U_u$, *then* $\{y_k\}_{k \in \mathbb{Z}}$ *can be* Ld*-shadowed by a trajectory.*

PROPOSITION 5.2. *For any d-pseudotrajectory $\{y_k\}_{k \leq 0}$ with $d < d_1$ and $y_0 \in U_u$, the following inequality holds:*

$$\operatorname{dist}(y_k, f^k(y_0)) < 2d^{1/3}, \quad k \leq 0. \quad (46)$$

Proof. Proposition 5.1 implies that $y_k \in U_u$ for $k < 0$. Assume (46) does not hold. Let

$$l = \max\{k \leq 0 : \operatorname{dist}(y_k, f^k(y_0)) \geq 2d^{1/3}\}.$$

Note that $l < 0$. Lemma 5.1 implies that

$$\operatorname{dist}(f(y_l), f^{l+1}(y_0)) > 2d^{1/3} + 2d.$$

Hence, $\operatorname{dist}(y_{l+1}, f^{l+1}(y_0)) > 2d^{1/3}$, which contradicts the choice of l. □

PROPOSITION 5.3. *If $\{y_k\}_{k \in \mathbb{Z}} \subset U_u$ is a d-pseudotrajectory with $d < d_1$, then*

$$\operatorname{dist}(y_k, u) < 2d^{1/3}, \quad k \in \mathbb{Z}. \quad (47)$$

Proof. Let us identify y_k with its coordinate in the system introduced in (iii) above, and consider $Y = \sup_{k \in \mathbb{Z}} |y_k|$. Assume that $Y > 2d^{1/3}$; then there exists $k \in \mathbb{Z}$ such that

$$|y_k| > \max(2d^{1/3}, Y - d/2).$$

Without loss of generality, we may assume that $y_k > 0$. Since $y_k \in U_u$, the following holds:

$$f(y_k) - y_k = y_k^3 > 2d.$$

Hence, $y_{k+1} - y_k > (f(y_k) - y_k) - d > d$ and $y_{k+1} > Y + d/2$, which contradicts the choice of Y. Inequalities (47) are proved. □

PROPOSITION 5.4. *For any d-pseudotrajectory $\{y_k\}_{k \in [0,n]}$ with $d < d_1$ and $y_n \in U_u$, the following inequality holds:*

$$\operatorname{dist}(y_{n-k}, f^{-k}(y_n)) \leq dk, \quad k \in [0, n]. \quad (48)$$

Proof. Proposition 5.1 implies that $y_k \in U_u$ for $k \in [0, n]$. Assume that equation (48) does not hold. Define

$$l = \min\{k \in [0, n] : \operatorname{dist}(y_{n-k}, f^{-k}(y_n)) > dk\}.$$

Note that $l > 0$. Lemma 5.1 implies that

$$\operatorname{dist}(f(y_{n-l}), f^{-l+1}(y_n)) > ld$$

and hence

$$\operatorname{dist}(y_{n-l+1}, f^{-l+1}(y_n)) > (l-1)d,$$

which contradicts the choice of l. □

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Now we are ready to complete the proof of Theorem 5.1.

First, let us prove that $f \in \mathrm{HolSh}(1/3)$. Consider an arbitrary d-pseudotrajectory $\{y_k\}_{k\in\mathbb{Z}}$, with $d < d_2$. Let us prove that it can be $Ld^{1/3}$-shadowed by a trajectory.

If $\{y_k\} \subset U_u$, then, by Proposition 5.3, it can be $2d^{1/3}$-shadowed by $\{x_k = u\}$.

If $\{y_k\} \subset S^1 \setminus U_u$, then, by Proposition 5.1, it can be Ld-shadowed.

In the other cases, there exists l such that $y_l \in U_u$ and $y_{l+1} \notin U_u$. By Proposition 5.2,

$$\mathrm{dist}(y_k,\, f^{k-l}(y_l)) < 2d^{1/3}, \quad k \le l.$$

By Proposition 5.1,

$$\mathrm{dist}(y_k,\, f^{k-l}(y_l)) < Ld, \quad k \ge l+1.$$

Hence, $\{y_k\}$ is $Ld^{1/3}$-shadowed by the trajectory $\{x_k = f^{k-l}(y_l)\}$.

Now let us prove that $f \in \mathrm{FinHolSh}(1/2,\, 1/2)$. Consider an arbitrary d-pseudotrajectory $\{y_k\}_{k\in[0,1/d^{1/2}]}$, with $d < d_2$. Let us prove that it can be $Ld^{1/2}$-shadowed by a trajectory.

If $\{y_k\} \subset U_u$, then, by Proposition 5.4, it can be $d^{1/2}$-shadowed by $\{x_k = f^{k-n}(y_n)\}$.

If $\{y_k\} \subset S^1 \setminus U_u$, then, by Proposition 5.1, it can be Ld-shadowed.

In the other cases, there exists l such that $y_l \in U_u$ and $y_{l+1} \notin U_u$. From Proposition 5.4, it is easy to conclude that

$$\mathrm{dist}(y_k,\, f^{k-l}(y_l)) < d^{1/2}, \quad k \le l.$$

Proposition 5.1 implies that

$$\mathrm{dist}(y_k,\, f^{k-l}(y_l)) < Ld, \quad k \ge l+1.$$

Hence, $\{y_k\}$ is $Ld^{1/2}$-shadowed by the trajectory $\{x_k = f^{k-l}(y_l)\}$. □

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