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# A hypergeometric transformation associated with the Appell function $F_4$

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Recently in these proceedings (see (5), p. 175) by applying Goldstein's form of Parseval's relation ((4), p. 105) to the operational pairs ((3), p. 284)

$$\left.\begin{aligned}\phi(p) &= 2pI_\nu(2bp^{\frac{1}{2}})\,K_\nu(2cp^{\frac{1}{2}})\\ &\doteqdot \frac{1}{t}\exp\left\{-\frac{b^2+c^2}{t}\right\}I_\nu\left(\frac{2bc}{t}\right) = h(t),\end{aligned}\right\}$$

where $R[(b\pm c)^2] > 0$ and ((3), p. 293)

$$\left.\begin{aligned}\psi(p) &= \frac{p^{\lambda+1}\,\Gamma(\frac{1}{2}-\lambda+\mu)}{a^{\frac{1}{2}}\,\Gamma(1+2\mu)}\,e^{a/2p}M_{\lambda,\mu}\left(\frac{a}{p}\right)\\ &\doteqdot t^{-\lambda-\frac{1}{2}}I_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}}) = g(t),\end{aligned}\right\}$$

where $R(\frac{1}{2}-\lambda+\mu) > 0$, it was proved that if

$$I = \int_0^\infty t^{-\lambda-1}\exp\left[(\tfrac{1}{2}a-b^2-c^2)\,t\right]M_{\lambda,\mu}(at)\,I_\nu(2bct)\,dt,$$

then

$$I = \frac{a^{\mu+\frac{1}{2}}b^\nu\,\Gamma(\frac{1}{2}-\lambda+\mu+\nu)}{c^{1-2\lambda+2\mu+\nu}\Gamma(\nu+1)}\,F_4\left[\tfrac{1}{2}-\lambda+\mu,\tfrac{1}{2}-\lambda+\mu+\nu;\,2\mu+1,\nu+1;\,\frac{a}{c^2},\frac{b^2}{c^2}\right], \tag{1}$$

provided $R(\frac{1}{2}-\lambda+\mu+\nu) > 0$ and $R[(b\pm c)^2] > R(a), 0$.

In terms of confluent hypergeometric functions we observe that

$$I = \frac{a^{\mu+\frac{1}{2}}(bc)^\nu}{\Gamma(\nu+1)}\int_0^\infty t^{\mu+\nu-\lambda-\frac{1}{2}}\exp\left[-(b+c)^2t\right]\times {}_1F_1[\tfrac{1}{2}-\lambda+\mu;\,2\mu+1;\,at]\,{}_1F_1[\nu+\tfrac{1}{2};\,2\nu+1;\,4bct]\,dt,$$

and on evaluating the integral on the right by means of Dr L. J. Slater's formula ((6), p. 54; see also p. 1 for the various notations adopted)

$$\int_0^\infty e^{-st}t^{d-1}\,{}_AF_B[(a);\,(b);\,kt]\,{}_{A'}F_{B'}[(a');\,(b');\,k't]\,dt = \frac{\Gamma(d)}{s^d}\sum_{m=0}^\infty\frac{((a)_m)\,(d)_m}{m!\,((b)_m)}\left(\frac{k}{s}\right)^m{}_{A'+1}F_{B'}\left[(a'),d+m;\,(b');\,\frac{k'}{s}\right], \tag{2}$$

$$R(d),\quad R(s) > 0,$$

with $A = A' = B = B' = 1$, we have

$$I = \frac{a^{\mu+\frac{1}{2}}(bc)^\nu\,\Gamma(\frac{1}{2}-\lambda+\mu+\nu)}{(b+c)^{1-2\lambda+2\mu+2\nu}\,\Gamma(\nu+1)}\times F_2\left[\tfrac{1}{2}-\lambda+\mu+\nu,\tfrac{1}{2}-\lambda+\mu,\nu+\tfrac{1}{2};\,2\mu+1,2\nu+1;\,\frac{a}{(b+c)^2},\frac{4bc}{(b+c)^2}\right], \tag{3}$$

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where $F_2$ is Appell's double hypergeometric function of the second type defined by means of (1) (p. 15)

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

provided, for convergence, $|x|+|y|<1$.

Equivalence of (1) and (3) gives us an interesting quadratic transformation of the $F_4$-function, viz.

$$F_4(a,b;c,a-b+1;x,y^2)$$
$$=(1+y)^{-2a}F_2\left[a,b,a-b+\tfrac{1}{2};c,2a-2b+1;\frac{x}{(1+y)^2},\frac{4y}{(1+y)^2}\right], \tag{4}$$

which holds inside the common domain of convergence of the series involved.

This formula is essentially a generalization of the well-known Goursat's quadratic transformation

$${}_2F_1(a,b;a-b+1;z^2)=(1+z)^{-2a}\,{}_2F_1\left[a,a-b+\tfrac{1}{2};2a-2b+1;\frac{4z}{(1+z)^2}\right], \tag{5}$$

valid in a neighbourhood of $z=0$. Erdélyi ((2), p. 382) obtained (4) by transforming the Pochhammer's double-loop contour integral associated with $F_2$, while it was proved decades earlier in (1) (p. 27) by series manipulation.

In view of (4), Dr Slater's generalized formula (2) furnishes a direct method to evaluate (1) without recourse to operational calculus. On the other hand, if (1) is given, (2) yields the quadratic transformation itself.

Finally, we record, without details of proof, as a necessary consequence of (5) (p. 175 (3)) the hypergeometric transformation

$$F_4(\lambda-\mu,\lambda+\mu;\nu,\nu;\tfrac{1}{2}x^2,\tfrac{1}{2}y^2)$$
$$=\sum_{\mu,-\mu}\frac{\Gamma(-2\mu)\,\Gamma(\nu)}{\Gamma(\lambda-\mu)\,\Gamma(\nu-\lambda-\mu)}[\tfrac{1}{2}(1-x+y)(1+x-y)]^{-\lambda-\mu}$$
$$\times F_2\left[\lambda+\mu,\nu-\lambda+\mu,\nu-\tfrac{1}{2};2\mu+1,2\nu-1;\frac{2}{(1-x+y)(1+x-y)},\frac{4xy}{(1-x+y)(1+x-y)}\right], \tag{6}$$

which does not appear to have been noticed earlier.

It is not difficult to give appropriate extensions of (4) and (6) to hold for Lauricella's multiple hypergeometric functions (see (1), p. 114), nor does it seem worth while to mention the various steps through which our transformation (6) is reduced to the elegant formula

$${}_2F_1(\lambda-\mu,\lambda+\mu;\nu;z)=\sum_{\mu,-\mu}\frac{\Gamma(-2\mu)\,\Gamma(\nu)}{\Gamma(\lambda-\mu)\,\Gamma(\nu-\lambda-\mu)}[\tfrac{1}{2}-z]^{-\lambda-\mu}$$
$$\times\,{}_2F_1\left[\lambda+\mu,\nu-\lambda+\mu;2\mu+1;\frac{2}{1-2z}\right],$$

valid when both $|z|<1$ and $|\tfrac{1}{2}-z|>1$.

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