Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2022.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

106.A (K. S. Bhanu and M. N. Deshpande)

A set of three positive integers (a, b, c) with $a < b < c$ is a *Diophantine triple* if there are integers x, y, z with $ab + 1 = x^2$, $ac + 1 = y^2$, $bc + 1 = z^2$.

Show that there are an infinite number of Diophantine triples such that x, y, z are in arithmetic progression.

106.B (Michael Fox)

The half-line a has endpoint $A(0, 0, 1)$ and passes through $\alpha(0, 1, 1)$. The half-line b has endpoint $B(0, 0, -1)$ and passes through $\beta(1, 0, -1)$. They are smoothly joined by a curve APB comprising circular arcs AP and BP with equal radii. There is no discontinuity of direction at any point.

Show that there are two such curves, and find the ratio of their lengths, longer to shorter.

106.C (Dorin Marghidanu)

If a, b, c, x, y, z are positive real numbers and $n > 1$, prove that

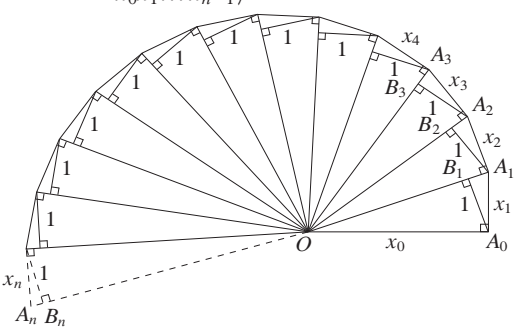
$$(ax + by + cz)^n + (ay + bz + cx)^n + (az + bx + cy)^n \leqslant (a + b + c)^n (x^n + y^n + z^n).$$

106.D (Toyesh Prakash Sharma)

The polygonal spiral $A_0A_1A_2\ldots$ with $OA_0 = x_0 > 1$ and $A_0A_1 = x_1$, $A_1A_2 = x_2$, … is constructed from right-angled triangles of height 1 as shown in the diagram.

(a) Show that $OB_n = x_n\left(x_0x_1\ldots x_{n-1} - \dfrac{1}{x_0x_1\ldots x_{n-1}}\right)$, for $n \geqslant 1$.

(b) Does the spiral go around O an infinite number of times?



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Solutions and comments on **105.E**, **105.F**, **105.G**, **105.H** (July 2021).

105.E (Yasuo Matsuda)

There are 100 items, ranked from 1st place to 100th place.

Choose these items one by one at random and stop once the best 10 have been chosen. Find the expected number of items that need to be chosen in order to achieve this.

Answer: $91\frac{9}{11}$

This popular problem attracted a wide range of solutions with several solvers expressing surprise at how close the answer is to 100.

The problem asks for $E(X)$, where the random variable X takes the value k, $10 \leqslant k \leqslant 100$, if the selection of the best 10 is completed on the kth choice. Most solvers used a probability approach such as the following.

$$\text{Since } \mathrm{P}(X \leqslant k) = \frac{\binom{90}{k-10}}{\binom{100}{k}} = \frac{90!\,k!}{100!\,(k-10)!},$$

$$\mathrm{P}(X=k) = \frac{90!\,k!}{100!\,(k-10)!} - \frac{90!\,(k-1)!}{100!\,(k-11)!} = \frac{90!\,(k-1)!}{100!\,(k-10)!}[k-(k-10)]$$

$$= \frac{90!\,10!}{100!}\frac{(k-1)!}{(k-10)!\,9!} = \frac{\binom{k-1}{9}}{\binom{100}{10}}.$$

(This may also be seen directly: 9 of the best 10 are picked in the first $k-1$ choices, with the 10th picked on the kth choice.)

Then

$$\mathrm{E}(X) = \frac{1}{\binom{100}{10}}\sum_{k=10}^{100} k\binom{k-1}{9} = \frac{10}{\binom{100}{10}}\sum_{k=10}^{100}\binom{k}{10}$$

$$= \frac{10}{\binom{100}{10}}\sum_{k=10}^{100}\binom{k+1}{11} - \binom{k}{11} = \frac{10\binom{101}{11}}{\binom{100}{10}} = \frac{1010}{11} = 91\tfrac{9}{11}.$$

Peter Johnson also calculated $\mathrm{var}(X) = 62\frac{73}{121}$.

M. G. Elliott considered the replacement version of the problem, where the expected value is $100\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10}\right) = 292\frac{113}{126}$.

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Several solvers generalised the above argument to obtain $E_{m,n} = \dfrac{m(n+1)}{m+1}$ as the expected value when choosing the best m from n ranked items.

Stan Dolan and Graham Howlett gave expectation arguments with Stan choosing all n items one by one and using the m 'best' items to partition the $n - m$ 'other' items into $m + 1$ (possibly empty) sets.

By symmetry, the expected size of each of these sets is $\dfrac{n-m}{m+1}$, so the expected number needing to be chosen is $n - \dfrac{n-m}{m+1} = \dfrac{m(n+1)}{m+1}$.

Graham used the recurrence relation

$$E_{m,n} = 1 + \frac{m}{n} E_{m-1,n-1} + \frac{n-m}{n} E_{m,n-1}$$

with boundary conditions $E_{k,k} = k$ and $E_{1,n} = \frac{1}{2}(n+1)$.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, M. Hennings, G. Howlett, P. F. Johnson, J. A. Mundie, S. Riccarelli, J. Siehler, C. Starr and the proposer Yasuo Matsuda.

105.F (Chris Starr)

Consider a right tetrahedron $OABC$ with OA, OB and OC mutually perpendicular with lengths a, b and c respectively.

(a) Let a, b and c be successive terms in the Fibonacci sequence. Prove that the area of triangle ABC is rational.

(b) Let a, b be successive terms in the Fibonacci sequence. Show that there are an infinite number of integer values of c such that the area of triangle ABC is rational.

(a) The area of triangle ABC is given by $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$. (This is known as de Gua's theorem in the literature and may be readily derived by calculating $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.) In a Fibonacci (or generalised Fibonacci) sequence, $c = a + b$ so that

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= \left(a^2 + b^2\right)(a+b)^2 + a^2b^2 \\ &= \left(a^2 + b^2\right)^2 + 2ab\left(a^2 + b^2\right) + a^2b^2 \\ &= \left(a^2 + b^2 + ab\right)^2. \end{aligned}$$

Triangle ABC thus has rational area $\frac{1}{2}\left(a^2 + ab + b^2\right)$. (With $a = F_n$, $b = F_{n+1}$, this can be expressed as $F_{n+1}^2 - \frac{1}{2}(-1)^n$, using standard Fibonacci identities.)

(b) Let $a = F_n$ and $b = F_{n+1}$ be successive Fibonacci numbers. Then we require $z = \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$ to be an integer, so that

$$z^2 - \left(a^2 + b^2\right)c^2 = a^2b^2 \qquad (1)$$

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or, since $F_n^2 + F_{n+1}^2 = F_{2n+1}$,

$$x^2 - F_{2n+1}y^2 = 1 \qquad (2)$$

where $x = \dfrac{z}{ab}$ and $y = \dfrac{c}{ab}$.

Integer solutions of the Pell equation (2) will thus generate integer solutions of (1). And (2) has an infinite number of solutions provided F_{2n+1} is not a square number. But, see [1], the only square Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_{12} = 144$, with $F_1 = 1$ giving the trivial solution $a = F_0 = 0, b = F_1 = 1, z = c \in \mathbb{N}$.

M. G. Elliott and Rob Summerson gave some numerical data for the fundamental solutions of (2).

n	F_{2n+1}	x	y	a	b	c	Area $= \frac{1}{2}z$
1	2	3	2	1	1	2	$\frac{3}{2}$
2	5	9	4	1	2	8	9
3	13	649	180	2	3	1080	1947
4	34	35	6	3	5	90	$\frac{525}{2}$
5	89	500001	53000	5	8	2120000	10000020

Reference

1. J. H. E. Cohn, Square Fibonacci numbers, etc., *Fibonacci Quarterly* **2** (1964) pp. 109-113.

Correct solutions were received from: M. V. Channakeshava, S. Dolan, M. G. Elliott, A. P. Harrison, M. Hennings, G. Howlett (part (a)), J. A. Mundie, Z. Retkes, R. Summerson and the proposer C. Starr.

105.G (Dorin Marghidanu)

If a, b, c, x, y, z are positive real numbers, prove that

$$\frac{H_3(a,b,c)}{A_3(x,y,z)} \leqslant \frac{1}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} + \frac{1}{\frac{y}{a}+\frac{z}{b}+\frac{x}{c}} + \frac{1}{\frac{z}{a}+\frac{x}{b}+\frac{y}{c}} \leqslant \frac{A_3(a,b,c)}{A_3(x,y,z)}$$

where $A_3(m, n, p) = \frac{1}{3}(m + n + p)$, $H_3(m, n, p) = \dfrac{3}{\frac{1}{m}+\frac{1}{n}+\frac{1}{p}}$.

When is there equality?

A wide range of techniques was used by solvers of this enticing inequality, including direct algebraic verification of each side.

For the right-hand inequality, the most common approach was to show first that

$$\frac{1}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} \leqslant \frac{ax+by+cz}{(x+y+z)^2}.$$

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This follows from the Cauchy-Schwarz inequality

$$(x + y + z)^2 \leqslant (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)$$

with equality if, and only if,

$$\left(\sqrt{ax}, \sqrt{by}, \sqrt{cz}\right) = \lambda\left(\sqrt{\frac{x}{a}}, \sqrt{\frac{y}{b}}, \sqrt{\frac{z}{c}}\right) \text{ or } a = b = c.$$

It also follows from the weighted AM-HM inequality

$$\frac{\sum \lambda_i}{\sum \frac{\lambda_i}{\alpha_i}} \leqslant \frac{\sum \lambda_i \alpha_i}{\sum \lambda_i}$$

which gives

$$\frac{x + y + z}{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}} \leqslant \frac{ax + by + cz}{x + y + z},$$

with equality if, and only if, $a = b = c$.

In the same way, we have

$$\frac{1}{\frac{y}{a} + \frac{z}{b} + \frac{x}{c}} \leqslant \frac{ay + bz + cx}{(x + y + z)^2} \text{ and } \frac{1}{\frac{z}{a} + \frac{x}{b} + \frac{y}{c}} \leqslant \frac{az + bx + cy}{(x + y + z)^2}.$$

Adding these three inequalities gives the required result:

$$\frac{1}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} + \frac{1}{\frac{y}{a}+\frac{z}{b}+\frac{x}{c}} + \frac{1}{\frac{z}{a}+\frac{x}{b}+\frac{y}{c}} \leqslant \frac{a+b+c}{x+y+z} = \frac{A_3(a,b,c)}{A_3(x,y,z)}$$

with equality if, and only if, $a = b = c$.

For the left-hand inequality, the AM-HM inequality applied to the middle sum immediately gives

$$\begin{aligned}
&\frac{1}{3}\left(\frac{1}{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}} + \frac{1}{\frac{y}{a} + \frac{z}{b} + \frac{x}{c}} + \frac{1}{\frac{z}{a} + \frac{x}{b} + \frac{y}{c}}\right) \\
&\geqslant \frac{3}{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{y}{a} + \frac{z}{b} + \frac{x}{c} + \frac{z}{a} + \frac{x}{b} + \frac{y}{c}} \\
&= \frac{3}{(x + y + z)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} = \frac{H_3(a, b, c)}{3A_3(x, y, z)}, \text{ as required,}
\end{aligned}$$

but here the case of equality is more subtle.

It requires $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{y}{a} + \frac{z}{b} + \frac{x}{c} = \frac{z}{a} + \frac{x}{b} + \frac{y}{c}$. Subtracting these equations in pairs yields

$$(bc - ab)x + (ac - bc)y + (ab - ac)z = 0$$

$$(bc - ac)x + (ac - ab)y + (ab - bc)z = 0$$

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which solves to give $\lambda x = \lambda y = \lambda z$ where

$$\begin{aligned}\lambda &= (bc - ab)^2 + (ab - ac)(bc - ac)\\ &= a^2b^2 + b^2c^2 + c^2a^2 - a^2bc - ab^2c - abc^2\\ &= \tfrac{1}{2}\left[a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2\right].\end{aligned}$$

We thus have equality when $x = y = z$ or $a = b = c$ (or both), which is less stringent than the equality condition for the right-hand inequality.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, M. Hennings, G. Howlett, H. Ricardo (2 solutions), S. Riccarelli, C. Starr and the proposer Dorin Marghidanu.

105.H (Seán Stewart)

(a) Evaluate $\displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^6}\,dx.$

(b) Evaluate $\displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^5}\,dx.$

Answer: (a) $\dfrac{\pi}{40}$, (b) $\dfrac{1}{3}\ln 2 - \dfrac{1}{12}$.

These two improper integrals proved popular with solvers. Most carefully integrated by parts, but differed as to how they side-stepped any divergent integrals that threatened to appear. The details in the following solution most closely follow those given by Russ Gordon and Graham and Nicholas Jameson.

Let $f(x) = (\sin x - x)^2$. Then $f(x) = O(x^6)$ as $x \to 0$ and $f(x) = O(x^2)$ as $x \to \infty$. So $I_n = \displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^n}dx$ is convergent for $3 < n < 7$, in particular for $n = 4,\ 5,\ 6$.

(a) Integration by parts gives

$$I_6 = 0 + \frac{1}{5}\int_0^\infty \frac{f'(x)}{x^5}\,dx$$

and we can repeat this three more times to obtain

$$I_6 = \frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\int_0^\infty \frac{f^{(4)}(x)}{x^2}dx = \frac{1}{120}\int_0^\infty \frac{f^{(4)}(x)}{x^2}dx$$

But $f^{(4)}(x) = 8\cos x - 8\cos 2x - 2x\sin x$, so

$$I_6 = \frac{1}{15}\int_0^\infty \frac{\cos x - \cos 2x}{x^2}dx - \frac{1}{60}\int_0^\infty \frac{\sin x}{x}dx.$$

We now use the standard integral $\displaystyle\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, see [1, p. 276]. For the first integral, parts gives

$$\int_0^\infty \frac{\cos x - \cos 2x}{x^2}dx = 0 + \int_0^\infty \frac{2\sin 2x - \sin x}{x}dx = (2 - 1)\frac{\pi}{2} = \frac{\pi}{2},$$

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hence $I_6 = \left(\frac{1}{15} - \frac{1}{60}\right)\frac{\pi}{2} = \frac{\pi}{40}$.

It is worth noting that this method also evaluates I_4.

Integrating by parts twice gives

$$I_4 = \int_0^\infty \frac{f(x)}{x^4}dx = \frac{1}{3}\cdot\frac{1}{2}\int_0^\infty \frac{f''(x)}{x^2}dx$$

with $f''(x) = 2\cos 2x - 4\cos x + 2 + 2x\sin x$.

Thus $I_4 = \frac{1}{6}\int_0^\infty \frac{2\cos 2x - 4\cos x + 2}{x^2}dx + \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx$ and, using parts for the first integral, we obtain

$$I_4 = \frac{1}{6}\int_0^\infty \frac{4\sin x - 4\sin 2x}{x}dx + \frac{1}{3}\int_0^\infty \frac{\sin x}{x}dx$$

$$= \left(\frac{2}{3} - \frac{2}{3} + \frac{1}{3}\right)\frac{\pi}{2} = \frac{\pi}{6}.$$

(b) For I_5 we utilise the Frullani integral

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}dx = \ln\left(\frac{b}{a}\right), \qquad \text{see [1, p. 280].}$$

Integrating by parts gives

$$I_5 = \int_0^\infty \frac{f(x)}{x^5}dx = 0 + \frac{1}{4}\int_0^\infty \frac{f'(x)}{x^4}dx = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\int_0^\infty \frac{f^{(3)}(x)}{x^2}dx$$

where $f^{(3)}(x) = 6\sin x - 4\sin 2x + 2x\cos x$

$$= 8\sin x - 4\sin 2x + 2x\cos x - 2\sin x$$

so that

$$I_5 = \frac{1}{6}\int_0^\infty \frac{2\sin x - \sin 2x}{x^2}dx + \frac{1}{12}\int_0^\infty \frac{x\cos x - \sin x}{x^2}dx.$$

For the first integral, parts gives

$$\int_0^\infty \frac{2\sin x - \sin 2x}{x^2}dx = 0 + \int_0^\infty \frac{2\cos x - 2\cos 2x}{x}dx = 2\ln 2$$

and direct evaluation of the second integral gives $\left[\frac{1}{12}\frac{\sin x}{x}\right]_0^\infty = -\frac{1}{12}$.

Thus $I_5 = \frac{1}{3}\ln 2 - \frac{1}{12}$.

Mark Hennings and the proposer, Seán Stewart, used Laplace transform methods to evaluate I_n. We summarise Mark's approach.

$$\mathscr{L}f(u) = \int_0^\infty e^{-xu}(\sin x - x)^2 dx = \int_0^\infty e^{-xu}\left(\frac{1}{2} - \frac{1}{2}\cos 2x - 2x\sin x + x^2\right)dx$$

$$= \frac{1}{2u} - \frac{u}{2(u^2+4)} - \frac{4u}{(u^2+1)^2} + \frac{2}{u^3} = \frac{4(5u^2+2)}{u^3(u^2+1)^2(u^2+4)},$$

using standard Laplace transforms.

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A routine manipulation of double integrals gives

$$I_n = \int_0^\infty \frac{(\sin x - x)^2}{x^n} dx = \frac{1}{\Gamma(n)} \int_0^\infty \left(\int_0^\infty e^{-xu} u^{n-1} du \right) (\sin x - x)^2 \, dx$$
$$= \frac{1}{\Gamma(n)} \int_0^\infty u^{n-1} \mathscr{L} f(u) \, du$$
$$= \frac{4}{\Gamma(n)} \int_0^\infty \frac{(5u^2 + 2) u^{n-4}}{(u^2 + 1)^2 (u^2 + 4)} du.$$

Partial fractions then reduces the evaluation of I_4, I_5, I_6 to the evaluation of standard integrals:

$$I_4 = \frac{2}{3} \int_0^\infty \frac{2}{u^2 + 1} - \frac{2}{u^2 + 4} - \frac{1}{(u^2 + 1)^2} du$$
$$= \frac{2}{3} \left(2 \times \frac{\pi}{2} - 2 \times \frac{\pi}{4} - \frac{\pi}{4} \right) = \frac{\pi}{6},$$
$$I_5 = \frac{1}{30} \int_0^\infty \frac{1}{(u^2 + 1)^2} - \frac{3}{u^2 + 1} + \frac{8}{u^2 + 4} du$$
$$= \frac{1}{30} \left(\frac{\pi}{4} - 3 \times \frac{\pi}{2} + 8 \times \frac{\pi}{4} \right) = \frac{\pi}{40},$$
$$I_6 = \frac{1}{6} \int_0^\infty \frac{2u}{u^2 + 1} - \frac{2u}{u^2 + 4} - \frac{u}{(u^2 + 1)^2} du$$
$$= \frac{1}{6} \left[\ln \left(\frac{u^2 + 1}{u^2 + 4} \right) + \frac{1}{2(u^2 + 1)} \right]_0^\infty = \frac{1}{3} \ln 2 - \frac{1}{12}.$$

Reference

1. G. J. O. Jameson, Sine, cosine and exponential integrals, *Math. Gaz.* **99** (July 2015) pp. 276-289.

Correct solutions were received from: N. Curwen, S. Dolan, R. Gordon, M. Hennings, G. Howlett, G. & N. Jameson, P. Kitchenside, J. A. Mundie, P. Pandey, C. Starr, R. Summerson and the proposer Seán Stewart.

N. J. L.

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Dr. M. N. Deshpande

We recently learned the sad news that Dr. M. N. Deshpande died in November 2021. His academic posts included Professor of Statistics and Director of the Government Institute of Science, Nagpur. He was a prolific author of papers and problems for many journals including *Mathematics in School* and the *Gazette*, where his article on Diophantine triples (with Alan Beardon) in the July 2002 issue and Problem **106.A** above are typical of his work.

His colleague and co-author of **106.A**, Dr. K. S. Bhanu, movingly writes, "In spite of his failing health he kept himself immersed in research, completing a book on Diophantine triples which he was able to see when it was published four days before his death. His passion for problems continued until his very end. We have lost a true academician."

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