Bull. Aust. Math. Soc. **114** (2026), 349–355
doi:10.1017/S0004972725100841

LINEAR INDEPENDENCE OF QUANTUM TRANSLATES

MANSI MISHRA

(Received 17 November 2025; accepted 1 December 2025; first published online 26 January 2026)

Abstract

If A is in the p-Schatten class on $\mathbb{R}^n$, $1 \le p \le 4n/(2n-1)$, then the quantum translates of A are linearly independent. Moreover, there exists a nonzero operator in the p-Schatten class on $\mathbb{R}^n$, $p > 4n/(2n-1)$, whose quantum translates are linearly dependent.

2020 *Mathematics subject classification*: primary 43A80; secondary 22D10, 22E30, 43A05, 53D55.

Keywords and phrases: Schatten class, quantum translates, linearly independent.

1. Introduction

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. If $x \in \mathbb{R}^n$, the translate of f by x is the function $T_x f : \mathbb{R}^n \to \mathbb{C}$ given by

$$T_x f(y) = f(x+y), \quad y \in \mathbb{R}^n.$$

In [1], Edgar and Rosenblatt considered the set of translates of a nonzero function in $L^p(\mathbb{R}^n)$ by distinct elements of $\mathbb{R}^n$ and gave conditions under which this set is linearly independent. They proved that any nonzero function in $L^p(\mathbb{R})$, $1 \le p < \infty$, has linearly independent translates. They also proved that if $n \ge 2$ and $1 \le p < 2n/(n-1)$, then a nonzero function in $L^p(\mathbb{R}^n)$ has linearly independent translates (see [1, Corollary 2.7]). Moreover, they showed that there exists a nonzero function in $L^p(\mathbb{R}^n)$, $n \ge 2$, that has linearly dependent translates when $p > 2n/(n-1)$ (see [1, Example 2.4]). Furthermore, in [9], Rosenblatt proved that a nonzero function in $L^p(\mathbb{R}^n)$, $p = 2n/(n-1)$, has linearly independent translates.

In this paper, we will prove noncommutative analogues of these results. Recall that the reduced Heisenberg group G is the set of triples

$$\{(x, y, z) \mid x, y \in \mathbb{R}^n, z \in \mathbb{C}, |z| = 1\}$$

with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', zz' e^{\pi i(x \cdot y' - y \cdot x')}).$$

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

According to the Stone–von Neumann theorem [2, Theorem 1.50], there is a unique irreducible unitary representation ρ of G such that

$$\rho(0,0,z) = zI.$$

The standard realisation of this representation is on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ by the action

$$(\rho(x,y,z)\varphi)(t) = ze^{\pi i(x\cdot y+2y\cdot t)}\varphi(t+x).$$

Let $S^p(\mathcal{H})$ denote the p-Schatten class of $\mathcal{H}$. There is an action of $\mathbb{R}^{2n}$ on $S^p(\mathcal{H})$ called quantum translation which is defined by

$$(x_1,y_1)\cdot X = \rho(x_1,y_1,1)X\rho(x_1,y_1,1)^{-1}.$$

In [11, 12], the quantum translation of an operator was defined, and it was shown that if $X \in S^2(\mathcal{H})$ is nonzero, then the quantum translates of X are linearly independent. The motivation for quantum translation may be found in [11].

The following theorem proved in Section 3 is the main result of this paper.

THEOREM 1.1. *Let $A \in S^p(\mathcal{H})$, $1 \le p \le 4n/(2n-1)$, be a nonzero operator and let $(x_1,y_1),\dots,(x_k,y_k)$ be distinct elements of $\mathbb{R}^{2n}$. Then, $\{(x_1,y_1)\cdot A,\dots,(x_k,y_k)\cdot A\}$ is a linearly independent set.*

In Section 4, we give an explicit example of a nonzero operator in $S^p(\mathcal{H})$, $p > 4n/(2n-1)$, whose quantum translates are linearly dependent.

2. Weyl transform

Recall that if $X \in S^1(\mathcal{H})$, the *Fourier–Wigner transform* of X is the function on $\mathbb{R}^{2n}$ defined by

$$\alpha(X)(x,y) = \operatorname{tr}(\rho(-x,-y,1)X)$$

and if $f \in L^1(\mathbb{R}^{2n})$, the *Weyl transform* of f is the operator $W(f) \in \mathcal{B}(\mathcal{H})$ defined by

$$W(f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,y)\rho(x,y,1)\,dx\,dy,$$

and that these are mutual inverses (see [2]).

We now discuss the relation between the quantum translates and the Weyl transform (see [11] for a detailed discussion). Recall that the Weyl correspondence of a function f on $\mathbb{R}^{2n}$ is defined by

$$\operatorname{Weyl}(f) = W(\mathcal{F}^{-1}(f)),$$

that is, the Weyl transform of the inverse Fourier transform of f. It can be shown that

$$\operatorname{Weyl}(T_{(x,y)}f) = (x,y)\cdot\operatorname{Weyl}(f), \quad (x,y) \in \mathbb{R}^{2n}.$$

The definition of the Weyl transform has been extended to tempered distributions (see, for example, [4, 5]) and we briefly recall this now. Let $\mathcal{S}(\mathbb{R}^{2n})$ denote the Schwartz

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

class functions on $\mathbb{R}^{2n}$ and let $\mathcal{S}'(\mathbb{R}^{2n})$ denote the space of tempered distributions on $\mathcal{S}(\mathbb{R}^{2n})$. If $k \in \mathcal{S}(\mathbb{R}^{2n})$, the integral operator $X \in \mathcal{B}(\mathcal{H})$ defined by

$$(X\varphi)(x) = \int_{\mathbb{R}^n} k(x,y)\varphi(y)\,dy$$

is called a Schwartz operator on $\mathcal{H}$. The class of Schwartz operators on $\mathcal{H}$ is denoted by $\mathcal{S}(\mathcal{H})$ and is a Fréchet space with respect to an appropriate sequence of seminorms. Let $\mathcal{S}'(\mathcal{H})$ denote the topological dual of $\mathcal{S}(\mathcal{H})$. It is well known that the Weyl transform is a topological isomorphism from $\mathcal{S}(\mathbb{R}^{2n})$ to $\mathcal{S}(\mathcal{H})$. If $T \in \mathcal{S}'(\mathbb{R}^{2n})$, we define its Weyl transform $W(T) \in \mathcal{S}'(\mathcal{H})$ by

$$W(T)(Y) = T(\alpha(Y)^{\check{}}), \quad Y \in \mathcal{S}(\mathcal{H}),$$

where $\alpha(Y)^{\check{}}(x,y) = \alpha(Y)(-x,-y)$. The distributional Weyl transform is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^{2n})$ to $\mathcal{S}'(\mathcal{H})$, and its inverse is the distributional Fourier–Wigner transform, which is defined by

$$\alpha(X)(\varphi) = X(W(\varphi^{\check{}})), \quad X \in \mathcal{S}'(\mathcal{H}), \varphi \in \mathcal{S}(\mathbb{R}^{2n}).$$

The following result was proved in [7].

THEOREM 2.1. *Let T be a compactly supported distribution on $\mathbb{R}^{2n}$. Let $\widehat{T}$ denote the Fourier transform of T. For $1 \le p \le \infty$, $W(T) \in S^p(\mathcal{H})$ if and only if $\widehat{T} \in L^p(\mathbb{R}^{2n})$. Moreover, if K is a compact set in $\mathbb{R}^{2n}$, then there exists a constant C_K such that*

$$C_K^{-1}\|\widehat{T}\|_p \le \|W(T)_{S^p}\| \le C_K\|\widehat{T}\|_p,$$

whenever $\operatorname{supp}(T) \subseteq K$. *Furthermore, $W(T)$ is compact if and only if $\widehat{T} \in C_0(\mathbb{R}^{2n})$.*

It is well known that if μ is a smooth measure on a hypersurface in $\mathbb{R}^n$, $n \ge 2$, whose Gaussian curvature is nonzero everywhere, then

$$|\widehat{\mu}(\xi)| \le A|\xi|^{(1-n)/2},$$

where A is a constant independent of ξ, and hence, $\widehat{\mu} \in L^p(\mathbb{R}^n)$ if $p > 2n/(n-1)$ (see, for example, [10, page 348, Theorem 1] and [3, Theorem 7.7.14]). The following is an immediate consequence of Theorem 2.1.

THEOREM 2.2. *Suppose S is a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, whose Gaussian curvature is nonzero everywhere. Let μ be a smooth measure on S. Then, $W(\mu)$ is a compact operator. Moreover, $W(\mu) \in S^p(\mathcal{H})$ if and only if $p > 4n/(2n-1)$.*

3. Proof of the main result

We prove Theorem 1.1 in this section.

Let $A \in S^p(\mathcal{H})$, $1 \le p \le \infty$, and let $T_A \in \mathcal{S}'(\mathcal{H})$ be the map

$$T_A(X) = \operatorname{tr}(XA).$$

The Fourier–Wigner transform $\alpha(A)$ is defined in a distributional sense, that is, $\alpha(A) = \alpha(T_A)$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

Let $e : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ be the map

$$e((x', y'), (x, y)) = e^{2\pi i(x'\cdot y - y'\cdot x)}.$$

Let $(x_1, y_1), \ldots, (x_k, y_k)$ be distinct elements of $\mathbb{R}^{2n}$ and let $c_1, \ldots, c_k$ be nonzero scalars. Define the difference operator D on $S^p(\mathcal{H})$, $1 \le p \le \infty$, by

$$DA = c_1(x_1, y_1) \cdot A + \cdots + c_k(x_k, y_k) \cdot A, \quad A \in S^p(\mathcal{H}).$$

Let $\psi \in \mathcal{S}(\mathbb{R}^{2n})$. Then,

$$\begin{aligned}
\alpha((x_1, y_1) \cdot A)(\psi) &= T_{(x_1,y_1)\cdot A}(W(\psi^\sim)) \\
&= \operatorname{tr}(W(\psi^\sim)(x_1, y_1) \cdot A) \\
&= \operatorname{tr}(W(\psi^\sim)\rho(x_1, y_1, 1)A\rho(x_1, y_1, 1)^{-1}) \\
&= \operatorname{tr}\Big(\iint \psi(-x, -y)\rho(x, y, 1)\rho(x_1, y_1, 1)A\rho(x_1, y_1, 1)^{-1}\, dx\, dy\Big) \\
&= \operatorname{tr}\Big(\iint e((x, y), (x_1, y_1))\psi(-x, -y)\rho(x_1, y_1, 1)\rho(x, y, 1)A\rho(x_1, y_1, 1)^{-1}\, dx\, dy\Big) \\
&= \operatorname{tr}\Big(\iint e((x, y), (x_1, y_1))\psi(-x, -y)\rho(x, y, 1)A\, dx\, dy\Big) \\
&= \operatorname{tr}(W(e((x_1, y_1), \cdot)\psi)^\sim A) \\
&= \alpha(T_A)(e((x_1, y_1), \cdot)\psi) \\
&= e((x_1, y_1), \cdot)\alpha(A)(\psi).
\end{aligned}$$

Therefore,

$$\alpha(DA) = (c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))\alpha(A). \tag{3.1}$$

The trigonometric polynomial

$$c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot)$$

is the characteristic trigonometric polynomial of the difference equation $DA = 0$.

We will first give the proof of Theorem 1.1 for the case $p \in [1, 2]$.

LEMMA 3.1. *Let $A \in S^p(\mathcal{H})$, $1 \le p \le 2$, be a nonzero operator and let $(x_1, y_1), \ldots, (x_k, y_k)$ be distinct elements of $\mathbb{R}^{2n}$. Then, $\{(x_1, y_1) \cdot A, \ldots, (x_k, y_k) \cdot A\}$ is a linearly independent set.*

PROOF. Since $S^p(\mathcal{H}) \subseteq S^2(\mathcal{H})$ for $1 \le p \le 2$, it suffices to prove the result for $p = 2$.

Let $c_1, \ldots, c_k \in \mathbb{C}$ be such that

$$DA = c_1(x_1, y_1) \cdot A + \cdots + c_k(x_k, y_k) \cdot A = 0.$$

By the Plancherel theorem for W, we may write $A = W(f)$ for some nonzero $f \in L^2(\mathbb{R}^{2n})$. By (3.1),

$$DA = W((c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))f).$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

Therefore, $(c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))f = 0$. Since f is a nonzero function in $L^2(\mathbb{R}^{2n})$, there exists $\varepsilon > 0$ such that $|f| > \varepsilon$ on a set of positive measure. Therefore, the analytic function

$$c_1 e((x_1, y_1), (x, y)) + \cdots + c_k e((x_k, y_k), (x, y)) = 0$$

on a set of positive measure and thus it is identically zero. Since

$$\{e((x_1, y_1), \cdot), \ldots, e((x_k, y_k), \cdot)\}$$

is linearly independent, it follows that $c_1 = \cdots = c_k = 0$. □

Recall that the *symplectic Fourier transform* of a function $f \in L^1(\mathbb{R}^{2n})$ is the function on $\mathbb{R}^{2n}$ defined by

$$\check{f}(\xi, \eta) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y) e^{2\pi i(\xi \cdot y - \eta \cdot x)}\, dx\, dy.$$

More generally (see [5]), if T is a tempered distribution on $\mathbb{R}^{2n}$, the symplectic Fourier transform of T is given by

$$\check{T}(\varphi) = T((\check{\varphi})^{\check{}}), \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n}).$$

Let $g(x, y) = e^{-\pi(|x|^2 + |y|^2)/2}$, $(x, y) \in \mathbb{R}^{2n}$. For $X \in \mathcal{S}'(\mathcal{H})$, let $\beta(X) = g\alpha(X)$ and $\check{\beta}(X)$ denote the symplectic Fourier transform of $\beta(X)$. It was shown in [7] that $\check{\beta}$ is a bounded linear map from $S^p(\mathcal{H})$ to $L^p(\mathbb{R}^{2n})$, $1 \le p \le \infty$. We will use this fact in the proof of the main result.

Rosenblatt proved that the linear translates of a nonzero function in $L^p(\mathbb{R}^n)$, $p = 2n/(n-1)$, are linearly independent by using the following two results (see [9]).

LEMMA 3.2. *If $f \in L^p(\mathbb{R}^n)$, where $2 < p \le 2n/(n-1)$ and $n > 1$, then*

$$\int_{\{R<|x|<2R\}} |f(x)|^2\, dx = o(R) \quad \text{as } R \to \infty.$$

THEOREM 3.3. *Let u be a tempered distribution supported on a real analytic set K of codimension $k > 0$ and assume that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that for every analytic manifold M contained in A and every $x_0 \in M$, there is an open cone $\Lambda \subset \mathbb{R}^n$ that contains some normal of M at x_0 and such that on $\Lambda_R = \{x \in \Lambda \mid R < |x| < 2R\}$,*

$$\lim_{R \to \infty} R^{-k} \int_{\Lambda_R} |\hat{u}(x)|^2\, dx = 0.$$

Then, $u = 0$.

We will now use these results to prove that quantum translates of a nonzero operator in $S^p(\mathcal{H})$, where $2 < p \le 4n/(2n-1)$, are linearly independent.

Let $A \in S^p(\mathcal{H})$, $2 < p \le 4n/(2n-1)$, be a nonzero operator such that $DA = 0$. It follows that

$$0 = \alpha(DA) = (c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot))\alpha(A). \tag{3.2}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

Let Z_k denote the zero set of the characteristic trigonometric polynomial

$$c_1 e((x_1, y_1), \cdot) + \cdots + c_k e((x_k, y_k), \cdot).$$

Case 1: Z_k is of positive measure. Since the analytic function $c_1 e((x_1, y_1), (x, y)) + \cdots + c_k e((x_k, y_k), (x, y))$ vanishes on a set of positive measure, it follows that it is identically zero. Since the set

$$\{e((x_1, y_1), \cdot), \ldots, e((x_k, y_k), \cdot)\}$$

is linearly independent, it follows that $c_1 = \cdots = c_k = 0$. Therefore, the quantum translates of A are linearly independent.

Case 2: Z_k is of measure zero. Then, Z_k is an analytic submanifold of codimension one (see, for example, [13]). By (3.2), $\alpha(A)$ is supported in Z_k and hence, $\beta(A)$ is supported in Z_k. Take $u = \beta(A)$ in Theorem 3.3. Since $A \in S^p(\mathcal{H})$, it follows that $\check{u} = \check{\beta}(A) \in L^p(\mathbb{R}^{2n})$. Therefore, $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^{2n})$. By Lemma 3.2, u satisfies the growth condition required in Theorem 3.3 for every cone Λ. Therefore, $u = 0$ and so $\alpha(A) = 0$. It follows that $A = 0$.

4. Example

We will now find an explicit example of a nonzero operator $A \in S^p(\mathcal{H})$, $p > 4n/(2n-1)$, whose quantum translates are linearly dependent. The argument used here is similar to that given by Edgar and Rosenblatt in [1, Example 2.4]. The same example was given in [6] to prove that there exists a nonzero operator in $S^p(\mathcal{H})$, $p > n \geq 6$, whose quantum translates are linearly dependent, and in [8] to prove that there exists a nonzero compact operator on $\mathcal{H}$ whose quantum translates are linearly dependent.

Consider the difference equation given by

$$2(2n-1)A = \sum_{j=1}^{n} [(e_j, \mathbf{0}) \cdot A + (-e_j, \mathbf{0}) \cdot A + (\mathbf{0}, e_j) \cdot A + (\mathbf{0}, -e_j) \cdot A], \tag{4.1}$$

where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Then, the characteristic trigonometric polynomial

$$p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 2(2n-1) - 2 \sum_{j=1}^{n} (\cos(2\pi x_j) + \cos(2\pi y_j))$$

has a zero set which is a disjoint union of compact $(2n-1)$-dimensional surfaces of positive Gaussian curvature. Let S_{2n} be the connected component of the zero set containing the points with all coordinates zero except for one which is $\pm 1/4$. Let σ be the measure on S_{2n} induced by the Lebesgue measure on $\mathbb{R}^{2n}$. Then, by Theorem 2.2, $W(\sigma) \in S^p(\mathcal{H})$ for $p > 4n/(2n-1)$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841

It follows from (3.1) that $A = W(\sigma)$ is a nonzero solution of the difference equation (4.1). Hence, for $p > 4n/(2n-1)$, there exists an operator in $S^p(\mathcal{H})$ whose quantum translates are linearly dependent.

References

[1] G. A. Edgar and J. M. Rosenblatt, 'Difference equations over locally compact abelian groups', *Trans. Amer. Math. Soc.* **253** (1979), 273–289.

[2] G. B. Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies, 122 (Princeton University Press, Princeton, NJ, 1989).

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*, 2nd edn, Grundlehren der mathematischen Wissenschaften, 256 (Springer-Verlag, Berlin, 1990).

[4] M. Keyl, J. Kiukas and R. F. Werner, 'Schwartz operators', *Rev. Math. Phys.* **28**(3) (2016), Article no. 1630001, 60 pages.

[5] J. M. Maillard, 'On the twisted convolution product and the Weyl transformation of tempered distributions', *J. Geom. Phys.* **3**(2) (1986), 230–261.

[6] M. Mishra and M. K. Vemuri, 'The Weyl transform of a measure', *Proc. Indian Acad. Sci. Math. Sci.* **133**(2) (2023), Article no. 29, 11 pages.

[7] M. Mishra and M. K. Vemuri, 'The Weyl transform of a compactly supported distribution', Preprint, 2024, arXiv:2409.16835.

[8] M. Mishra and M. K. Vemuri, 'The Weyl transform of a smooth measure on a real-analytic submanifold', *Bull. Aust. Math. Soc.* **112**(1) (2025), 184–193.

[9] J. Rosenblatt, 'Linear independence of translations', *J. Aust. Math. Soc. Ser. A* **59**(1) (1995), 131–133.

[10] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Mathematical Series, 43 (Princeton University Press, Princeton, NJ, 1993); with the assistance of T. S. Murphy.

[11] S. Thangavelu, 'On Paley–Wiener theorems for the Heisenberg group', *J. Funct. Anal.* **115**(1) (1993), 24–44.

[12] R. Werner, 'Quantum harmonic analysis on phase space', *J. Math. Phys.* **25**(5) (1984), 1404–1411.

[13] H. Whitney, *Complex Analytic Varieties* (Addison-Wesley, Reading, MA, 1972).

MANSI MISHRA, IISER Bhopal, Bhopal, India
e-mail: mansimishra.rs.mat19@itbhu.ac.in

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:07:37, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972725100841