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Hénon-like maps with arbitrary stationary combinatorics

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Abstract. We extend the renormalization operator introduced in [A. de Carvalho, M. Martens and M. Lyubich. Renormalization in the Hénon family, I: universality but non-rigidity. *J. Stat. Phys.* **121**(5/6) (2005), 611–669] from period-doubling Hénon-like maps to Hénon-like maps with arbitrary stationary combinatorics. We show that the renormalization picture also holds in this case if the maps are taken to be *strongly dissipative*. We study infinitely renormalizable maps F and show that they have an invariant Cantor set $\mathcal{O}$ on which F acts like a p-adic adding machine for some $p > 1$. We then show, as for the period-doubling case in the work of de Carvalho, Martens and Lyubich [Renormalization in the Hénon family, I: universality but non-rigidity. *J. Stat. Phys.* **121**(5/6) (2005), 611–669], that the sequence of renormalizations has a universal form, but that the invariant Cantor set $\mathcal{O}$ is non-rigid. We also show that $\mathcal{O}$ cannot possess a continuous invariant line field.

1. *Introduction*

1.1. *Hénon renormalization.* In [**3**], de Carvalho *et al* constructed a period-doubling renormalization operator for Hénon-like mappings of the form

$$F(x, y) = (f(x) - \varepsilon(x, y), x). \tag{1.1}$$

Here f is a unimodal map and ε is a real-valued map from the square to the positive real numbers of small size (we shall be more explicit about the maps under consideration in §§2 and 3). They showed that for $|\varepsilon|$ sufficiently small the unimodal renormalization picture carries over to this case. Namely, there exists a unique renormalization fixed point (which actually coincides with the unimodal period-doubling renormalization fixed point) which is hyperbolic with codimension-one stable manifold, consisting of infinitely renormalizable period-doubling maps, and dimension-one local unstable manifold. They later called this regime *strongly dissipative*.

In the same paper they then studied the dynamics of infinitely renormalizable Hénon-like maps F. They showed that such a map has an invariant Cantor set, $\mathcal{O}$, upon which the map acts like an adding machine. This allowed them to define the *average Jacobian*

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given by

$$b = \exp \int_{\mathcal{O}} \log |\mathrm{Jac}_z F| \, d\mu(z) \tag{1.2}$$

where $\mathrm{Jac}_z F = \det \mathrm{D}_z F$ is the Jacobian determinant and μ denotes the unique F-invariant measure on $\mathcal{O}$ induced by the adding machine. This quantity played an important role in their study of the local behaviour of such maps around the Cantor set. They took a distinguished point τ of the Cantor set $\mathcal{O}$ called the *tip*. They examined the dynamics and geometry of the Cantor set asymptotically around τ. Their two main results can then be stated as follows.

THEOREM 1.1. (Universality at the tip) *There exists a universal constant $0 < \rho < 1$ and a universal real analytic real-valued function $a(x)$ such that the following holds: let F be a strongly dissipative, period-doubling, infinitely renormalizable Hénon-like map. Then*

$$\mathcal{R}^n F(x, y) = (f_n(x) - b^{2^n} a(x) y(1 + \mathrm{O}(\rho^n)), x) \tag{1.3}$$

where b denotes the average Jacobian of F and f_n are unimodal maps converging exponentially to the unimodal period-doubling renormalization fixed point.

THEOREM 1.2. (Non-rigidity around the tip) *Let F and $\tilde{F}$ be two strongly dissipative, period-doubling, infinitely renormalizable Hénon-like maps. Let their average Jacobians be b and $\tilde{b}$ and their Cantor sets be $\mathcal{O}$ and $\tilde{\mathcal{O}}$, respectively. Then for any conjugacy $\pi : \mathcal{O} \to \tilde{\mathcal{O}}$ between F and $\tilde{F}$ the Hölder exponent α satisfies*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log b}{\log \tilde{b}}\right). \tag{1.4}$$

In particular, if the average Jacobians b and $\tilde{b}$ differ then there cannot exist a C^1-smooth conjugacy between F and $\tilde{F}$.

For a long time it was assumed that the properties satisfied by the one-dimensional unimodal renormalization theory would also be satisfied by any renormalization theory in any dimension. In particular, the equivalence of the universal (real and complex *a priori* bounds) and rigid (pullback argument) properties in this setting made it natural to think that such a relation would hold more generally. That is, if universality controls the geometry of an attractor and we have a topological conjugacy between two attractors it was expected that such a conjugacy could be promoted to a smooth map, since the geometry of infinitesimally close pairs of orbits cannot differ too much. The above shows that this intuitive reasoning is incorrect.

Let us now outline the structure of the present work. After recalling preliminary results in §2, in §3 we generalize this renormalization operator to other combinatorial types. We show that in this case the renormalization picture also holds if $\bar{\varepsilon}$ is sufficiently small. Namely, for any stationary combinatorics there exists a unique renormalization fixed point, again coinciding with the unimodal renormalization fixed point, which is hyperbolic with codimension-one stable manifold, consisting of infinitely renormalizable maps, and dimension-one local unstable manifold.

We then study the dynamics of infinitely renormalizable maps of stationary combinatorial type and show that such maps have an F-invariant Cantor set $\mathcal{O}$ on which F acts as an adding machine. We note that the strategy to show that the limit set is a

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Cantor set in the period-doubling case does not carry over to maps with general stationary combinatorics. In both cases the construction of the Cantor set is via 'scope maps', defined in §§2 and 3, which we approximate using the so-called 'presentation function' of the renormalization fixed point. In the period-doubling case this is known to be contracting as the renormalization fixed point is convex (see the result of Davie [**2**]) and the unique fixed point lying in the interior of the interval is expanding (see the theorem of Singer [**5**, Ch. 3]). In the case of general combinatorics this is unlikely to be true. The work of Eckmann and Wittwer [**6**] suggests the convexity of fixed points for sufficiently large combinatorial types does not hold. The problem of contraction of branches of the presentation function was also raised in [**8**].

Once this is done we can define the average Jacobian and the tip of an infinitely renormalizable Hénon-like map in a way analogous to the period-doubling case. This then allows us, in §6, to generalize the universality and non-rigidity results stated above to the case of arbitrary combinatorics. We also generalize another result from [**3**], namely that the Cantor set of an infinitely renormalizable Hénon-like map cannot support a continuous invariant line field. Our proof, however, is significantly different. This is because in the period-doubling case they observed a 'flipping' phenomenon where orientations were changed purely because of combinatorics. Their argument clearly breaks down in the more general case where there is no control over such things.

1.2. *Notation and conventions.* Given a function F we denote by $\mathrm{Dom}(F)$ its domain of definition. Typically this will be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. For $i \geq 0$ we denote its ith iterate by $F^{\circ i}$ whenever it is defined. For $S \subset \mathrm{Dom}(F)$ we denote its ith preimage by $F^{\circ -i} : F^{\circ i}(S) \to S \subset \mathbb{R}^n$ whenever this is defined.

We now restrict our attention to the one- and two-dimensional cases, both real and complex. Let $\pi_x, \pi_y : \mathbb{R}^2 \to \mathbb{R}$ denote the projections onto the x- and y- coordinates, respectively. We will identify these with their extensions to $\mathbb{C}^2$. (In fact we identify all real functions with their complex extensions whenever they exist.)

Given $a, b \in \mathbb{R}$, denote the closed interval between them by $[a, b] = [b, a]$. Denote the interval $[0, 1]$ by J. For any interval $T \subset \mathbb{R}$ denote its boundary by ∂T, its left endpoint by $\partial^- T$ and its right endpoint by $\partial^+ T$. Given two intervals $T_0, T_1 \subset J$, denote an affine bijection from T_0 to T_1 by $\iota_{T_0 \to T_1}$. Typically it will be clear from the situation whether we are using the unique orientation-preserving or orientation-reversing bijection.

Let us denote the square $[0, 1] \times [0, 1] = J^2$ by B. We call $S \subset B$ a *rectangle* if it is the Cartesian product of two intervals. Given two rectangles $B_0, B_1 \subset B$, we will denote an affine bijection from B_0 to B_1 preserving horizontal and vertical lines by $I_{B_0 \to B_1}$. Again the orientations of its components will be clear from the situation.

Let $\Omega_x \subseteq \Omega_y \subset \mathbb{C}$ be simply connected domains compactly containing J and let $\Omega = \Omega_x \times \Omega_y$ denote the resulting polydisk containing B.

If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at a point $z \in \mathbb{R}^2$ we will denote the derivative of F at z by $\mathrm{D}_z F$. The *Jacobian* of F at z is given by

$$\mathrm{Jac}_z F = \det \mathrm{D}_z F. \tag{1.5}$$

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Given a bounded region $S \subset \mathbb{R}^2$, we will define the *distortion* of F on S by

$$\mathrm{Dis}(F;\, S) = \sup_{z,\tilde{z}\in S} \log \left| \frac{\mathrm{Jac}_z F}{\mathrm{Jac}_{\tilde{z}} F} \right|. \tag{1.6}$$

2. *Unimodal maps*

Let us now briefly review some parts of one-dimensional unimodal renormalization theory—in particular, the presentation function theory associated with it developed by Rand [**11**], Feigenbaum [**7**], Sullivan [**12**] and Birkhoff *et al* [**1**].

2.1. *The space of unimodal maps.*

Let $\beta > 0$ be a constant, which we will think of as being small. Let $\mathcal{U}_{\Omega_x,\beta}$ denote the space of maps $f \in C^\omega(J,\, J)$ satisfying the following properties:

(i) There is a unique critical point $c_0 = c_0(f)$, which lies in $(0,\, 1-\beta]$.
(ii) f is orientation-preserving to the left of c_0 and orientation-reversing to the right of c_0.
(iii) $f(\partial^+ J) = f(\partial^- J) = 0$ and $c_1 = f(c_0) > c_0$.
(iv) There is a unique fixed point $\alpha = \alpha(f)$ lying in $\mathrm{int}(J)$. Both fixed points are expanding.
(v) f admits a holomorphic extension to the domain Ω_x, upon which it can be factored as $\psi \circ Q$, where $Q : \mathbb{C} \to \mathbb{C}$ is given by $Q(z) = 4z(1-z)$ and $\psi : Q(\Omega_x) \to \mathbb{C}$ is an orientation-preserving univalent mapping which fixes the real axis.

Such maps will be called *unimodal maps*. Given any interval $T \subset \mathbb{R}$, we will say that a map $g : T \to T$ is *unimodal on* T if there exists an affine bijection $h : J \to T$ such that $h^{-1} \circ g \circ h \in \mathcal{U}_{\Omega_x,\beta}$. We will identify all unimodal maps with their holomorphic extensions.

We make following observations: first, the holomorphic extension will be $\mathbb{R}$-symmetric (i.e. $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega_x$); and secondly, there are two fixed points, one with positive multiplier lying in ∂J and the other lying in the interior with negative multiplier.

2.2. *The renormalization operator.*

Definition 2.1. Let $p > 1$ be an integer and let $W = W_p$ denote the set $\{0,\, 1,\, \ldots,\, p-1\}$. A permutation υ of W is said to be *unimodal of length* p if there exists an order-preserving embedding $\mathrm{i} : W \to J$ and a unimodal map $f : J \to J$ such that $f(\mathrm{i}(k-1)) = \mathrm{i}(k \bmod p)$ for all $k \in W$.

Definition 2.2. Let $p > 1$ be an integer. A map $f \in \mathcal{U}_{\Omega_x,\beta}$ has a *renormalization interval of period* p if:

(i) there is a closed subinterval $J^0 \subset J$ such that $f^{\circ p}(J^0) \subset J^0$;
(ii) there exists an affine bijection $h : J \to J^0$ such that

$$\mathcal{R}_{\mathcal{U}} f = h^{-1} \circ f^{\circ p} \circ h : J \to J \tag{2.1}$$

is an element of $\mathcal{U}_{\Omega_x,\beta}$. Note that there are exactly two such affine bijections, but there will only be one such that $\mathcal{R}_{\mathcal{U}} f \in \mathcal{U}_{\Omega_x,\beta}$.

The interval J^0 is called a *renormalization interval of period* p for f. The collection $\{J^w\}_{w\in W}$ is called the *renormalization cycle*.

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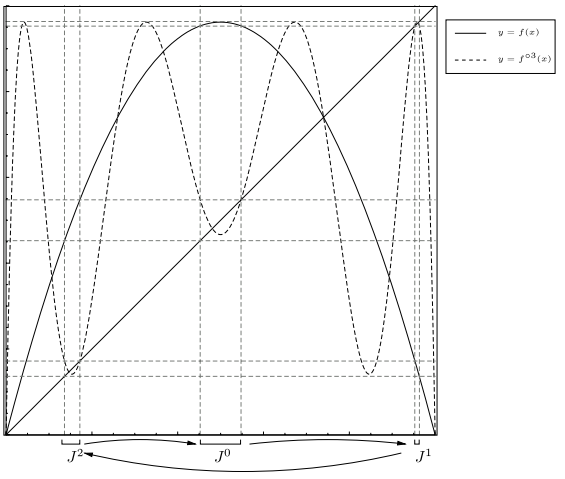


FIGURE 1. The graph of a renormalizable period-three unimodal map f with renormalization interval J^0 and renormalization cycle $\{J^i\}_{i=0,1,2}$. Note that for $p = 3$ there is only one admissible combinatorial type.

Definition 2.3. Let $p > 1$ be an integer and let υ be a unimodal permutation of length p. A map $f \in \mathcal{U}_{\Omega_x,\beta}$ is *renormalizable of combinatorial type* υ if:

(i) f has a renormalization interval of period p which contains the critical point c_0;
(ii) letting J^w denote the connected component of $f^{\circ p-w}(J^0)$ containing $f^{\circ w}(J^0)$, the interiors of the subintervals J^w, $w \in W$, are pairwise disjoint;
(iii) f acts on the set $\{J^0, J^1, \ldots, J^{p-1}\}$, embedded in the line with the standard orientation, as υ acts on the symbols in W. More precisely, if $J', J'' \in \{J^w\}_{w\in W}$ are the ith and jth subintervals from the left endpoint of J respectively, then $f(J')$ lies to the left of $f(J'')$ if and only if $\upsilon(i) < \upsilon(j)$.

In this case the map $\mathcal{R}_{\mathcal{U}} f$ is called the *renormalization of* f and the operator $\mathcal{R}_{\mathcal{U}}$ the *renormalization operator* of combinatorial type υ.

Figure 1 shows, for example, a renormalizable map of combinatorial-type υ, where υ denotes the unique permutation of length three.

Definition 2.4. Let υ be a unimodal permutation of length $p > 1$. Given a renormalizable $f \in \mathcal{U}_{\Omega_x,\beta}$ of combinatorial type υ the renormalization interval J^0 is called the *central interval*. Given J^w, $w \in W$, the *maximal extension* of J^w is the largest open interval J'^w containing J^w such that $f^{\circ p-w}|J'^w$ is a diffeomorphism onto its image.

Remark 2.5. The assumption that $\mathcal{R}_{\mathcal{U}} f$ lies in $\mathcal{U}_{\Omega_x,\beta}$ implies that the boundary of J^0 consists of a p-periodic point and one of its preimages. Moreover, in J^0 there is no other preimage of this point and there is at most one periodic point of period p^n.

Let $\mathcal{U}_{\Omega_x,\beta,\upsilon}$ denote the subspace consisting of unimodal maps $f \in \mathcal{U}_{\Omega_x,\beta}$ which are renormalizable of combinatorial type υ. If $f \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ is infinitely renormalizable there is a nested sequence $\underline{J} = \{J^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ of subintervals such that:

(i) $f(J^{\mathbf{w}}) = J^{1+\mathbf{w}}$ for all $\mathbf{w} \in W^*$;
(ii) $\operatorname{int} J^{\mathbf{w}} \cap \operatorname{int} J^{\tilde{\mathbf{w}}} = \emptyset$ for all $\mathbf{w} \neq \tilde{\mathbf{w}} \in W^*$ of the same length;
(iii) $\bigcup_{w\in W} J^{\mathbf{w}w} \subset J^{\mathbf{w}}$ for each $\mathbf{w} \in W^*$,

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where W^* denotes the totality of all finite words $\mathbf{w} = w_0 \cdots w_n$ over W and $1 + \mathbf{w}$ denotes the adding machine map,

$$1 + w_0 \cdots w_n = \begin{cases} (1 + w_0)w_1 w_2 \cdots w_n & w_0 \neq p - 1 \\ 0^k(1 + w_k)w_{k+1} \cdots & w_0, \ldots, w_{k-1} = p - 1,\ w_k \neq p - 1. \end{cases} \tag{2.2}$$

We will denote by $W^n \subset W^*$ the subset of words of length $n > 0$ and by $\bar{W}$ the set of all infinite words over W. (Note that W^* corresponds to the space of all cylinder sets of $\bar{W}$.) We will call this indexing of $\underline{J}$ the *spatial indexing*.

There is a second indexing of $\underline{J}$ given as follows: any $J' \in \underline{J}$ is the mth preimage of J^{0^n} for some $m, n > 0$ satisfying $0 \leq m < p^n$. We will call this the *dynamical indexing* and the quantity n the *depth* of $J^{\mathbf{w}}$. We note that this indexing is used in [**5**, Ch. VI.3].

For each depth $n \geq 0$ the correspondence is given by the map $\mathbf{q} : W^n \to \{0, 1, \ldots, p^n - 1\}$ where if $\mathbf{w} = w_0 \cdots w_n$ then† $\mathbf{q}(\mathbf{w}) = \sum_{0 \leq i \leq n} p^i(p - w_i)$. More explicitly, if $\mathbf{w} = w_0 \cdots w_n$ then $J^{\mathbf{w}} = f^{\circ \mathbf{r}(\mathbf{w})}(J^{0^n})$ where $\mathbf{r}(\mathbf{w}) = \sum_{0 \leq i \leq n} p^i w_i$ and so, as the first return time of J^{0^n} is p^n, we find that the transfer time of $J^{\mathbf{w}}$ to J^{0^n} is $p^n - \mathbf{r}(\mathbf{w}) = \mathbf{q}(\mathbf{w})$.

Notation 2.6. If $f \in \mathcal{U}_{\Omega_x, \beta, \upsilon}$ is an infinitely renormalizable unimodal map let $f_n = \mathcal{R}^n_{\mathcal{U}} f$. Then all objects associated to f_n will also be given this subscript. For example, we will denote by $\underline{J}_n = \{J^{\mathbf{w}}_n\}_{\mathbf{w} \in W^*}$ the nested collection of intervals constructed for f_n in the same way that $\underline{J}$ was constructed for f.

The following plays a crucial role in the renormalization theory of unimodal maps. (See [**5**] for the proof and more details.)

THEOREM 2.7. (Real C^1-*a priori* bounds) *Let $f \in \mathcal{U}_{\Omega_x, \beta, \upsilon}$ be an infinitely renormalizable unimodal map. Then there exist constants $L(f)$, $K(f) > 1$ and $0 < k_0(f) < k_1(f) < 1$, such that for all $\mathbf{w} \in W^*$, $w, \tilde{w} \in W$ and each $i = 0, 1 \ldots, p^n - \mathbf{q}(\mathbf{w})$ the following properties hold:*

(i) (a) $\mathrm{Dis}(f^{\circ i}; J^{\mathbf{w}}) \leq L(f)$;
(b) *the previous bound is beau: there exists a constant $L > 1$ such that for each f as above $L(\mathcal{R}^n_{\mathcal{U}} f) < L$ for n sufficiently large;*

(ii) (a) $K(f)^{-1} < |J^{\mathbf{w}w}|/|J^{\mathbf{w}\tilde{w}}| < K(f)$;
(b) *the previous bound is beau: there exists a constant $K > 1$ such that for each f as above $K(\mathcal{R}^n_{\mathcal{U}} f) < K$ for n sufficiently large;*

(iii) (a) $k_0(f) < |J^{\mathbf{w}w}|/|J^{\mathbf{w}}| < k_1(f)$;
(b) *the previous bound is beau: there exist constants $0 < k_0 < k_1 < 1$ such that for each f as above $k_0 < k_0(\mathcal{R}^n_{\mathcal{U}} f) < k_1(\mathcal{R}^n_{\mathcal{U}} f) < k_1$ for n sufficiently large.*

The term *beau* for such a property was coined by Sullivan—being an acronym for 'bounded eventually and universally'. We now show some properties of the renormalization operator and renormalizable maps. The proposition below follows directly by observing that the renormalization cycle of a renormalizable unimodal map is determined by a hyperbolic periodic orbit and a collection of its preimages. The proof of injectivity can be found in [**5**, Ch. VI].

† This is because J^w are indexed by images but ϕ^w is constructed from the preimage of J^0 to J^w.

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PROPOSITION 2.8. *Let $p > 1$ be an integer. Let $0 < \gamma < 1$. Let υ be a unimodal permutation of length p. If $f \in \mathcal{U}_{\Omega_x,\beta}$ has renormalization interval J^0 of type p and satisfies the following conditions:*

- *$f^{\circ p}(J^0) \subsetneq J^0$;*
- *f is renormalizable with combinatorics υ;*

then there exists a neighbourhood $\mathcal{N} \subset \mathcal{U}_{\Omega_x,\beta}$ of f such that for any $\tilde{f} \in \mathcal{N}$ the following properties hold:

(i) *$\tilde{f}$ is renormalizable with combinatorics υ;*

(ii) *there exists a constant $C > 0$, depending upon f only, such that*

$$|\mathcal{R}_{\mathcal{U}} f - \mathcal{R}_{\mathcal{U}} \tilde{f}|_{\Omega_x} < C|f - \tilde{f}|_{\Omega_x}; \tag{2.3}$$

(iii) *the operator $\mathcal{R}_{\mathcal{U}}$ is injective.*

2.3. *The fixed point and hyperbolicity.* As was noted in the introduction, real *a priori* bounds were an important component in Sullivan's proof of the following part of the renormalization conjecture. For the proof we refer the reader to [**5**, Ch. VI]. This also contains substantial background material and references.

THEOREM 2.9. (Existence of fixed point) *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, if $\beta > 0$ is sufficiently small there exists an $f_* = f_{*,\upsilon} \in \mathcal{U}_{\Omega_x,\beta,\upsilon}$ such that*

$$\mathcal{R}_{\mathcal{U}} f_* = f_*. \tag{2.4}$$

Notation 2.10. Henceforth we assume that the unimodal permutation υ and the positive real number $\beta > 0$ are fixed, with β small enough to ensure that $\mathcal{U}_{\Omega_x,\beta,\upsilon}$ contains the renormalization fixed point. We therefore drop β from our notation.

Sullivan's result was then strengthened by McMullen. See [**10**].

THEOREM 2.11. (Weak convergence) *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, there exist:*

(i) *a domain $\Omega'_x \Subset \Omega_x$ containing J;*

(ii) *an integer $N > 0$;*

both dependent upon υ and Ω_x, such that for any $n > N$ if $f \in \mathcal{U}_{\Omega_x,\upsilon}$ is n-times renormalizable then

$$|\mathcal{R}^n_{\mathcal{U}} f - f_*|_{\Omega'_x} \leq \tfrac{1}{4}|f - f_*|_{\Omega'_x}. \tag{2.5}$$

Proof of the full renormalization conjecture was then completed by Lyubich in [**9**].

THEOREM 2.12. (Exponential convergence) *Given any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, there exist:*

(i) *a domain $\Omega'_x \Subset \Omega_x$, containing J;*

(ii) *an $\mathcal{R}_{\mathcal{U}}$-invariant subspace; $\mathcal{U}_{\text{adapt}} \subset \mathcal{U}_{\Omega'_x,\upsilon}$;*

(iii) *a metric, d_{adapt}, on $\mathcal{U}_{\text{adapt}}$ which is Lipschitz-equivalent to the sup-norm on $\mathcal{U}_{\Omega'_x,\upsilon}$;*

(iv) *a constant $0 < \rho < 1$;*

such that, for all $f \in \mathcal{U}_{\text{adapt}}$,

$$d_{\text{adapt}}(\mathcal{R}_{\mathcal{U}} f,\ f_*) \leq \rho d_{\text{adapt}}(f,\ f_*). \tag{2.6}$$

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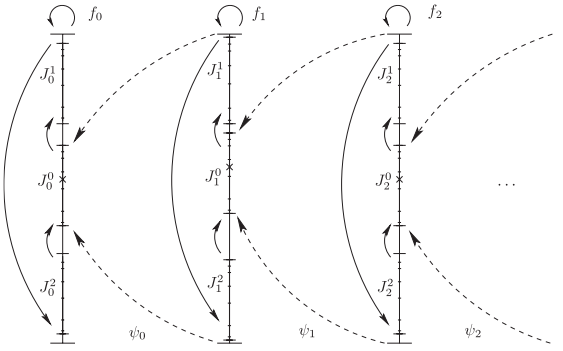


FIGURE 2. The collection of scope maps ψ_n^w for an infinitely renormalizable period-tripling unimodal map. Here f_n denotes the nth renormalization of f.

THEOREM 2.13. (Codimension-one stable manifold) *For any unimodal permutation υ and any domain $\Omega_x \subset \mathbb{C}$ containing J, the renormalization operator $\mathcal{R}_{\mathcal{U}} : \mathcal{U}_{\Omega_x,\upsilon} \to \mathcal{U}_{\Omega_x}$ has a codimension-one stable manifold $\mathcal{W}_\upsilon$ at the renormalization fixed point $f_{*,\upsilon}$.*

COROLLARY 2.14. *Let υ be a unimodal permutation on W. Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$ be an infinitely renormalizable unimodal map. Then the cycle, $\{J_n^w\}_{w\in W}$, of the central interval of f_n converges exponentially, in the Hausdorff topology, to the corresponding cycle, $\{J_*^w\}_{w\in W}$, of the renormalization fixed point f_*.*

2.4. *Scope maps and presentation functions.* We now rephrase the renormalization of unimodal maps in terms of convergence of their *scope maps* defined below. Again the statements here are well known and we make no claim to the originality of their proofs. We merely collect them here for completeness.

Scope maps were studied initially by Rand [**11**], then by Sullivan [**12**], Feigenbaum [**7**] and Birkhoff *et al* [**1**]. Mostly this was in the case of the so-called *presentation function*, which is the scope map of the renormalization fixed point (so called because they permute the presentation of the invariant Cantor set). See also the paper of Jiang *et al* [**8**].

Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$ have cycle $\{J^w\}_{w\in W}$. Consider the functions

$$\begin{aligned} h_0^{-1} : J^0 \to J, \quad w = 0, \\ h_0^{-1} \circ f^{\circ p-w} : J^w \to J, \quad w = 1, \ldots, p-1, \end{aligned} \tag{2.7}$$

where h_0 is the affine map satisfying $\mathcal{R}_{\mathcal{U}} f = h_0^{-1} \circ f^{\circ p} \circ h_0$. The inverses of these maps are called the *scope maps* of f, which we denote by $\psi_f^w : J \to J^w$. For each $w \in W$ we call $\psi_f^w : J \to J^w$ the *w-scope map*. We denote the multi-valued function they form by $\boldsymbol{\psi}_f : J \to \bigcup_{w\in W} J^w$. Similarly, given an n-times renormalizable $f \in \mathcal{U}_{\Omega_x,\upsilon}$ let $\psi_n^w = \psi_{f_n}^w$ denote the wth scope function of f_n and the multi-valued function they form by $\boldsymbol{\psi}_n$ (for an example in the case of period-tripling combinatorics see Figure 2). The multi-valued function $\boldsymbol{\psi}_* = \boldsymbol{\psi}_{f_*}$ associated to the renormalization fixed point f_* is called the *presentation function*.

If $f \in \mathcal{U}_{\Omega_x,\upsilon}$ is infinitely renormalizable we can extend this construction by considering, for each $\mathbf{w} = w_0 \cdots w_n \in W^*$, the function $\psi_f^{\mathbf{w}} = \psi_0^{w_0} \circ \cdots \circ \psi_n^{w_n} : J \to J^{\mathbf{w}}$ and we set $\underline{\psi_f} = \{\psi_f^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$.

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PROPOSITION 2.15. *Let f_* denote the unimodal fixed point of renormalization with presentation function $\boldsymbol{\psi}_*$. Let $m, n > 0$ be integers. Then, for each $\mathbf{w} \in W^m$, the following properties hold:*

(i) $\psi_*^{\mathbf{w}} = f_*^{-\mathbf{q}(\mathbf{w})} \circ \iota_{J \to J^{0^n}}$;

(ii) $\psi_*^{\mathbf{w}}(\bigcup_{\mathbf{w} \in W^n} J_*^{\mathbf{w}}) \subset \bigcup_{\mathbf{w} \in W^{n+m}} J_*^{\mathbf{w}}$.

Proof. We show the first item inductively. Trivially, it is true for $m = 0$. Assume that it holds for $\mathbf{w} \in W^m$, some $m \geq 0$, and consider $w\mathbf{w} \in W^{m+1}$. Since $\mathcal{R}_{\mathcal{U}} f_* = f_*$ implies that $f_*^p \circ \iota_{J \to J_*^0} = \iota_{J \to J_*^0} \circ f_*$, we find that

$$\begin{aligned} \psi_*^{w\mathbf{w}} &= \psi_*^w \circ \psi_*^{\mathbf{w}} \\ &= f_*^{\circ -(p-w)} \circ \iota_{J \to J_*^0} \circ f_*^{\circ -\mathbf{q}(\mathbf{w})} \circ \iota_{J \to J^{0^m}} \\ &= f_*^{\circ -(p-w)} \circ f_*^{\circ -p\mathbf{q}(\mathbf{w})} \circ \iota_{J \to J^0} \circ \iota_{J \to J^{0^m}} \\ &= f_*^{\circ -\mathbf{q}(w\mathbf{w})} \circ \iota_{J \to J^{0^{m+1}}}. \end{aligned} \tag{2.8}$$

This proves the first statement. The second statement then follows from the first since, given $\mathbf{w} \in W^n$ and $\tilde{\boldsymbol{w}} \in W^m$, the image of $J_*^{\mathbf{w}}$ under $\psi_*^{\mathbf{w}}$ can be expressed as a preimage of $J_*^{0^{m+n}}$ under f_*. □

COROLLARY 2.16. *Let f_* denote the unimodal fixed point of renormalization with presentation function $\boldsymbol{\psi}_*$. Let $\mathcal{O}_*$ denote the invariant Cantor set for f_*. Let $\underline{J}_* = \{J_*^{\mathbf{w}}\}_{\mathbf{w} \in W^*}$ denote the collection of all central cycles of all depths. Then, for each $\mathbf{w} \in W$, the following properties hold:*

(i) $\psi_*^{\mathbf{w}}(\underline{J}_*) \subset \underline{J}_*$;

(ii) $\psi_*^{\mathbf{w}}(\mathcal{O}_*) \subset \mathcal{O}_*$.

Proof. Simply take limits in the previous proposition. □

LEMMA 2.17. *Let f_* denote the unimodal fixed point of renormalization with presentation function $\boldsymbol{\psi}_*$. Then, for each $w \in W$, the following properties hold:*

(i) *ψ_*^w has a unique attracting fixed point α;*

(ii) *if $[\psi_*^{w^n}]$ denotes the orientation-preserving affine rescaling of $\psi_*^{w^n}$ to J then $u_*^w = \lim_{n\to\infty}[\psi_*^{w^n}]$ exists, and the convergence is exponential in the sup-norm on some complex domain containing J.*

Proof. It is clear that there exists a fixed point α, as ψ_*^w maps J into itself. It is also unique, since by construction $J_*^{w^{n+1}} = \psi_*^w(J_*^{w^n})$, and all images of J must contain all fixed points. However, Theorem 2.7 implies that $|J_*^{w^n}| \to 0$ as $n \to \infty$, and hence there can be only one fixed point.

Now let us show that α is attracting. Theorem 2.7 tells us, since $J_*^{w^{n+1}} \subset J_*^{w^n}$, that $|J_*^{w^{n+1}}|/|J_*^{w^n}| < k_1 < 1$. By the mean value theorem this implies that there are points $\alpha_n \in J_*^{w^n}$ such that

$$|(\psi_*^w)'(\alpha_n)| = |J_*^{w^{n+1}}|/|J_*^{w^n}| < k_1.$$

Also, since $\alpha \in J_*^{w^n}$ for all $n > 0$ and $|J_*^{w^n}| \to 0$ as $n \to \infty$, we have $\alpha_n \to \alpha$. As ψ_*^w is analytic we must have $|(\psi_*^w)'(\alpha)| < k_1$. Hence if α has multiplier σ_w, then $|\sigma_w| < 1$ and so α is attracting.

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For the second item, let $u_n^w = \iota_{J_*^{w^n} \to J} \circ \psi_*^{w^n} : J \to J$. First we claim that there is a domain $U \subset \mathbb{C}$ containing J on which u_n^w has a univalent extension. This follows as α is an attracting fixed point and ψ_*^w is real analytic on J, so there exists a domain $V \subset \mathbb{C}$ containing α on which ψ_*^w is univalent and $\psi_*^w(V) \subset V$. By Theorem 2.7 there exists an integer $N > 0$ such that $(\psi_*^w)^{\circ n}(J) \subset V$ for all $n \geq N$. Therefore take any domain U containing J such that $(\psi_*^w)^{\circ n}(U)$ is bounded away from the set of the first p^N critical values of f_* for all $n < N$. Then u_n^w will be univalent on U.

Observe that, letting $v_n^w = \mathcal{Z}_{J_*^{w^n}} \psi_*^w$ where $\mathcal{Z}_T$ denotes the zoom operator on the interval T, we can write

$$u_n^w = v_n^w \circ \cdots \circ v_0^w. \tag{2.9}$$

Also observe that the argument above gives a domain W containing J on which each of these composants has a univalent extension. Therefore

$$\begin{aligned} |u_n^w - u_{n+1}^w|_W &= |v_n^w \circ \cdots \circ v_0^w - v_{n+1}^w \circ v_n^w \circ \cdots \circ v_0^w|_W \\ &= |\mathrm{id} - v_{n+1}^w|_{u_n^w(W)}. \end{aligned} \tag{2.10}$$

Theorem 2.7 implies that $|J_*^{w^n}| \to 0$ exponentially as $n \to \infty$. Analyticity of ψ_*^w then implies that $\mathrm{Dis}(\psi_*^w; J_*^{w^n}) \to 0$ exponentially as well. Moreover, also by analyticity, this holds on a subdomain W^n of W containing $J_*^{w^n}$. Hence, by the mean value theorem,

$$\left| 1 - \frac{|J_*^{w^n}|}{|J_*^{w^{n+1}}|} (\psi_*^w)' \right|_{J^{w^n}} \to 0 \tag{2.11}$$

exponentially, and this will also hold on W^n if $n > 0$ is sufficiently large. Integrating then gives us

$$|\mathrm{id} - v_n^w|_{u_n^w(W)} \to 0 \tag{2.12}$$

exponentially, and hence $|u_n^w - u_{n+1}^w|_W \to 0$ exponentially. This implies that the limit u_*^w exists and is univalent on W. □

Remark 2.18. In the period-doubling case more precise information was given by Birkhoff *et al* [**1**]. Since f_* is convex in this case (see [**2**]), we know that $f_*|J_*^1$ is expanding and hence ψ_*^1 is contracting. This simplified the construction of the renormalization Cantor set for a strongly dissipative non-degenerate Hénon-like map given by de Carvalho *et al* [**3**].

Remark 2.19. For (ii) of the above it is clear this can be extended from words w^∞ to words $\mathbf{w} = w_0 w_1 \cdots$ which are eventually periodic. For arbitrary words a different strategy must be used; see [**1**].

PROPOSITION 2.20. *Let υ be a unimodal permutation and let $\upsilon(n)$ be the unimodal permutation satisfying $\mathcal{R}_{\mathcal{U},\upsilon}^n = \mathcal{R}_{\mathcal{U},\upsilon(n)}$. Given an n-times renormalizable $f \in \mathcal{U}_{\Omega_x,\upsilon}$, let*

$$f_{\upsilon,i} = \mathcal{R}_{\mathcal{U},\upsilon}^i f \quad \text{and} \quad f_{\upsilon(n)} = \mathcal{R}_{\mathcal{U},\upsilon(n)} f \tag{2.13}$$

for all $i = 0, 1, \ldots, n$. Let $\boldsymbol{\psi}_{\upsilon,i}$ denote the presentation function for $f_{\upsilon,i}$ with respect to $\mathcal{R}_{\mathcal{U},\upsilon}$ and let $\boldsymbol{\psi}_{\upsilon(n)}$ denote the presentation function for $f_{\upsilon(n)}$ with respect to $\mathcal{R}_{\mathcal{U},\upsilon(n)}$. Then

$$\boldsymbol{\psi}_{\upsilon(n)} = \{\psi_{\upsilon,0}^{w_0} \circ \cdots \circ \psi_{\upsilon,n}^{w_n}\}_{w_0,\ldots,w_n \in W}. \tag{2.14}$$

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Proof. This follows from the fact that $\mathcal{R}_{\mathcal{U},\upsilon} f_{\upsilon,i} = f_{\upsilon,i+1}$ implies that $f_{\upsilon,i}^p \circ \iota_{J \to J_i^0} = \iota_{J \to J_i^0} \circ f_{\upsilon,i+1}$ and the fact that the central interval of f under $\mathcal{R}_{\mathcal{U},\upsilon(n)}$ is equal to J^{0^n}. □

Since the renormalization interval J^0 of f is determined by a p-periodic point and its preimage and the scope maps ψ^w are compositions of preimages of f, and all of these vary continuously with f, we get the following proposition.

PROPOSITION 2.21. *Let υ be a unimodal permutation. There exists a constant $C > 0$ such that for any $f_0, f_1 \in \mathcal{U}_{\Omega_x,\upsilon}$, and any $w \in W$,*

$$|\psi_{f_0}^w - \psi_{f_1}^w|_{\Omega_x} \leq C|f_0 - f_1|_{\Omega_x}. \tag{2.15}$$

The following corollary follows directly from the above Proposition 2.21 and convergence of renormalization (Theorem 2.12).

COROLLARY 2.22. *There exist constants $C > 0$ and $0 < \rho < 1$ such that the following holds: given any infinitely renormalizable $f \in \mathcal{U}_{\text{adapt}}$,*

$$|\psi_n^w - \psi_*^w|_{\Omega_x} \leq C\rho^n. \tag{2.16}$$

2.5. *A reinterpretation of the operator.* Let us now consider $\mathcal{H}_\Omega(0)$, defined to be the space of maps $F \in C^\omega(B, B)$ of the form $F = (f \circ \pi_x, \pi_x)$ where $f \in \mathcal{U}_{\Omega_x}$. Let us also consider the subspace $\mathcal{H}_{\Omega,\upsilon}(0)$ of maps $F = (f \circ \pi_x, \pi_x)$ where $f \in \mathcal{U}_{\Omega_x,\upsilon}$. These will be called the space of *degenerate Hénon-like maps* and the space of *renormalizable degenerate Hénon-like maps of type υ*, respectively. The reasons for this will become apparent in §3 when we introduce non-degenerate Hénon-like maps. Observe that there is an embedding $\underline{\mathrm{i}} : \mathcal{U}_{\Omega_x} \to \mathcal{H}_\Omega(0)$, given by $\underline{\mathrm{i}}(f) = (f \circ \pi_x, \pi_x)$, which restricts to an embedding $\underline{\mathrm{i}} : \mathcal{U}_{\Omega_x,\upsilon} \to \mathcal{H}_{\Omega,\upsilon}(0)$. We will construct an operator $\mathcal{R}$, defined on $\mathcal{H}_\Omega(0)$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}_{\Omega_x,\upsilon} & \xrightarrow{\mathcal{R}_{\mathcal{U}}} & \mathcal{U}_{\Omega_x} \\ \downarrow{\scriptstyle \underline{\mathrm{i}}} & & \downarrow{\scriptstyle \underline{\mathrm{i}}} \\ \mathcal{H}_{\Omega,\upsilon}(0) & \xrightarrow[\mathcal{R}]{} & \mathcal{H}_\Omega(0) \end{array} \tag{2.17}$$

Let $f \in \mathcal{U}_{\Omega_x,\upsilon}$, let $\{J^w\}_{w \in W}$ be its renormalization cycle and let $\{J'^w\}_{w \in W}$ be the set of corresponding maximal extensions. Let $F = \underline{\mathrm{i}}(f)$ be the corresponding degenerate Hénon-like map, let

$$B^w = J^{w+1} \times J^w, \quad B'^w = J'^{w+1} \times J'^w \tag{2.18}$$

and let

$$B_{\text{diag}}^w = J^w \times J^w, \quad B_{\text{diag}}'^w = J'^w \times J'^w, \tag{2.19}$$

where $w \in W$ is taken modulo p. Observe that B^w is invariant under $F^{\circ p}$ for each $w \in W$.

Consider the map $H : B \to B$ defined by $H = (f^{\circ p-1}, \pi_y)$. Since H preserves horizontal lines, if it is not injective Rolles' theorem implies that between any two points with the same image there exists a solution to $(f^{\circ p-1})' = 0$. By the inverse function theorem this will be locally invertible on any connected open set bounded away from the set of points for which DH is singular. This set coincides with the *critical locus* $\mathcal{C}^{p-1} = \{(x, y) : (f^{\circ p-1})'(x) = 0\}$. Hence H is a diffeomorphism onto its image on any connected

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open set bounded away from $\mathcal{C}^{p-1}$. In particular, since the box B^0 is bounded away from $\mathcal{C}^{p-1}$ whenever the maximal extensions are proper extensions, the map H will be a diffeomorphism there. We call B^0 the *central box*. We call the map H the *horizontal diffeomorphism*. Observe that $B^0_{\text{diag}} = H(B^0)$. Define $\bar{H} : B^0_{\text{diag}} \to B^0$ to be the inverse of H restricted to B^0_{diag}. For a diagram in the case of period-tripling combinatorics see Figure 3.

Remark 2.23. More generally, given any map T which we do not iterate but which is related to the dynamics of a map $F : B \to B$, if it is invertible we denote its inverse by $\bar{T}$. We use this unconventional notation for the following reason: later we consider maps of the form $F = (\phi, \pi_x) : B \to B$ and we define $\phi^w : B \to J$ by $F^{\circ w}(x, y) = (\phi^w(x, y), \phi^{w-1}(x, y))$ for all $w > 0$ and for $w < 0$ whenever they exist. Plain superscripts index objects related to the number of iterates of F and plain subscripts index objects related to the number of renormalizations performed. However, we define $H = (\phi^{p-1}, \pi_y)$, whose inverse is related to the $(p-1)$th preimage of F, not the first preimage.

Since $\bar{H}$ is well defined on B^0_{diag}, the map

$$G = H \circ F^{\circ p} \circ \bar{H} : B^0_{\text{diag}} \to B^0_{\text{diag}}, \tag{2.20}$$

is also well defined. We call this map the *pre-renormalization* of F around B^0. Let I denote the affine bijection from B^0_{diag} onto B such that the map

$$\mathcal{R}F = I \circ G \circ \bar{I} : B \to B \tag{2.21}$$

is again a degenerate Hénon-like map where $\bar{I}$ denotes the inverse of I. Then $\mathcal{R}F$ is called the *Hénon renormalization* of F around B^0 and the operator $\mathcal{R}$ is called the *renormalization operator* on $\mathcal{H}_{\Omega_x,\upsilon}(0)$. Observe that $\mathcal{R}F = (\mathcal{R}_{\mathcal{U}} f \circ \pi_x, \pi_x)$.

Remark 2.24. By the same argument as above, H will be a diffeomorphism onto its image when restricted to any B^w. Since $B^w_{\text{diag}} = H(B^w)$ and B^w is invariant under $F^{\circ p}$ by construction, the maps

$$G^w = H \circ F^{\circ p} \circ \bar{H} : B^w_{\text{diag}} \to B^w_{\text{diag}} \tag{2.22}$$

are well defined. We call G^w the wth *pre-renormalization*. There are affine bijections I^w from B^w_{diag} to B such that

$$\mathcal{R}_w F = I^w \circ G^w \circ \bar{I}^w : B \to B \tag{2.23}$$

is again a degenerate Hénon-like map (where, as above, $\bar{I}^w$ denotes the inverse of I^w). Then the map $\mathcal{R}_w F$ is called the *Hénon renormalization* of F around B^w and the operator $\mathcal{R}_w$ is called the wth *renormalization operator* on $\mathcal{H}_{\Omega_x,\upsilon}(0)$. Observe that $\mathcal{R}_w F = (\mathcal{R}_{\mathcal{U},w} f \circ \pi_x, \pi_x)$, where $\mathcal{R}_{\mathcal{U},w}$ denotes the renormalization around J^w.

Remark 2.25. The affine bijections I^w in the remark above map squares to squares. Hence the linear part of I^w has the form

$$\pm \begin{pmatrix} \sigma^{[w]} & 0 \\ 0 & \pm\sigma^{[w]} \end{pmatrix} \tag{2.24}$$

for some $\sigma^{[w]} > 0$. Here the sign depends upon the combinatorial type of υ only. We call the quantity $\sigma^{[w]}$ the wth *scaling ratio* of F.

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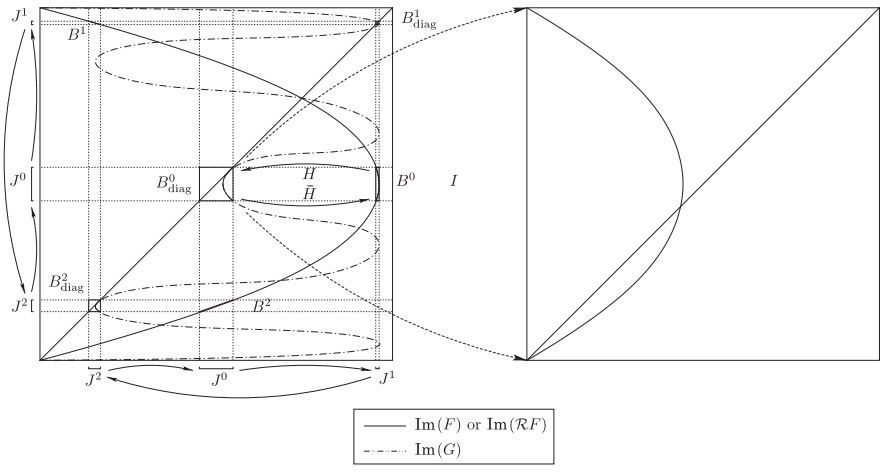


FIGURE 3. A period-three renormalizable unimodal map considered as a degenerate Hénon-like map. In this case the period is three. Observe that the image of the pre-renormalization lies on the smooth curve $(f^{\circ 3}(x), x)$.

Remark 2.26. Since $\underline{\mathrm{i}}$ is an embedding preserving the actions of $\mathcal{R}_{\mathcal{U}}$ and $\mathcal{R}$ it is clear that $\mathcal{R}$ also has a fixed point F_*. It must have the form $F_* = (f_* \circ \pi_x, \pi_x)$ where f_* is the fixed point of $\mathcal{R}_{\mathcal{U}}$. Then F_* also has a codimension-one stable manifold and dimension-one local unstable manifold in $\mathcal{H}_{\Omega_x}(0)$.

Given $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$, let $\Psi = \bar{H} \circ \bar{I} : B \to B^0$ and $\Psi^w = F^{\circ w} \circ \Psi : B \to B^w$. Then Ψ^w is called the *wth scope map* of F. The reason for this terminology is given by the following proposition.

PROPOSITION 2.27. *Let $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$. Then*

$$\Psi_F^w(x, y) = \begin{cases} (\psi_f^0(x), \psi^{p-1}) & w = p-1 \\ (\psi_f^{w+1}(x), \psi_f^w(x)) & 0 < w < p-1 \\ (\psi_f^1(x), \psi_f^0(y)) & w = 0, \end{cases} \tag{2.25}$$

where ψ_f^w denotes the wth scope function for f.

Proof. Observe that $\bar{H}(x, y) = (f^{\circ -p+1}(x), y)$ and

$$F^{\circ w}(x, y) = \begin{cases} (f^{\circ w}(x), f^{\circ w-1}(x)) & w > 0 \\ (x, y) & w = 0, \end{cases} \tag{2.26}$$

which implies that

$$F^{\circ w} \circ \bar{H}(x, y) = \begin{cases} (f^{\circ -p+w+1}(x), f^{\circ -p+w}(x)) & w > 0 \\ (f^{\circ -p+1}(x), y) & w = 0, \end{cases} \tag{2.27}$$

where appropriate branches of $f^{\circ -p+w+1}$ and $f^{\circ -p+w}$ are chosen. Also observe that $\bar{I}(x, y) = (\iota_{J \to J^0}(x), \iota_{J \to J^0}(y))$. Composing these gives us the result. □

Remark 2.28. Only the zeroth scope map $\Psi = \Psi^0$ is a diffeomorphism onto its image.

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Now assume that $F \in \mathcal{H}_{\Omega,\upsilon}$ is n-times renormalizable and denote its nth renormalization $\mathcal{R}^n F$ by F_n. For each F_n we can construct the wth scope map $\Psi_n^w = \Psi^w(F_n)$: $\mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_n)$, where $\mathrm{Dom}(F_n) = B$ denotes the domain of F_n. For $\mathbf{w} = w_0 \cdots w_n \in W^*$ the function

$$\Psi^{\mathbf{w}} = \Psi_0^{w_0} \circ \cdots \circ \Psi_n^{w_n} : \mathrm{Dom}(F_{n+1}) \to \mathrm{Dom}(F_0) \tag{2.28}$$

is called the $\mathbf{w}$-*scope map*. Let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}$ denote the collection of all scope maps. The following corollary is an immediate consequence of the above proposition.

COROLLARY 2.29. *Let $F = \underline{\mathrm{i}}(f) \in \mathcal{H}_{\Omega,\upsilon}(0)$ be an n-times renormalizable degenerate Hénon-like map. Then given a word $\mathbf{w} = w_0 \cdots w_{n-1} \in W^n$,*

$$\Psi_f^{\mathbf{w}}(x, y) = \begin{cases} (\psi_f^{\mathbf{w}+1^n}(x), \psi_f^{\mathbf{w}}(x)) & \mathbf{w} \neq 0 \\ (\psi_f^{\mathbf{w}+1^n}(x), \psi_f^{\mathbf{w}}(y)) & \mathbf{w} = 0, \end{cases} \tag{2.29}$$

where $\psi_f^{\mathbf{w}}$ denotes the $\mathbf{w}$th scope map for f.

In particular, we may do this for F_*, the renormalization fixed point. This gives

$$\Psi_*^w(x, y) = \begin{cases} (\psi_*^{w+1}(x), \psi_*^w(x)) & w > 0 \\ (\psi_*^{w+1}(x), \psi_*^w(y)) & w = 0, \end{cases} \tag{2.30}$$

where ψ_*^w are the branches of the presentation function. We will denote the family of scope maps for F_* by $\underline{\Psi}_* = \{\Psi_*^{\mathbf{w}}\}_{\mathbf{w} \in W^*}$ where $\Psi_*^{\mathbf{w}} : B \to B_*^{\mathbf{w}}$ is constructed as above.

3. *Hénon-like maps*

In this section we construct a space of Hénon-like maps and a renormalization operator acting on it which coincides with the renormalization operator on degenerate maps. We show that the standard unimodal renormalization picture can be extended to the space of such maps if the Hénon-like maps are sufficiently dissipative. We then examine the dynamics of infinitely renormalizable maps in more detail.

3.1. *The space of Hénon-like maps.*

Let $\bar{\varepsilon} > 0$. Let $\mathcal{T}_\Omega(\bar{\varepsilon})$ denote the space of maps $\varepsilon \in C^\omega(B, \mathbb{R})$, which satisfy:

(i) $\varepsilon(x, 0) = 0$ for all $x \in J$;
(ii) $\varepsilon(x, y) \geq 0$ for all $(x, y) \in B$;
(iii) ε admits a holomorphic extension to Ω;
(iv) $|\varepsilon|_\Omega \leq \bar{\varepsilon}$, where $|\cdot|_\Omega$ denotes the sup-norm on Ω.

Such maps will be called *thickenings* or *$\bar{\varepsilon}$-thickenings*. Let $B' = J' \times J' \subset \mathbb{R}^2$ for some closed interval $J' \subset \mathbb{R}$. Given $\varepsilon' \in C^\omega(B', \mathbb{R})$, let $E'(x, y) = (x, \varepsilon'(x, y))$. If there is an affine bijection $I : B' \to B$ such that $E(x, y) = I \circ E' \circ I(x, y) = (x, \varepsilon(x, y))$ where ε is a thickening, then we say that ε' is a *thickening on B'*.

Let $f \in \mathcal{U}_{\Omega_x}$ be a unimodal map and let $\bar{\varepsilon} > 0$ be a constant. Let

$$\mathcal{H}_\Omega^-(f, \bar{\varepsilon}) = \{F \in \mathrm{Emb}^\omega(B, \mathbb{R}^2) : F(x, y) = (f(x) + \varepsilon(x, y), x), \varepsilon \in \mathcal{T}_\Omega(\bar{\varepsilon})\} \tag{3.1}$$

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and

$$\mathcal{H}^+_\Omega(f, \bar{\varepsilon}) = \{F \in \mathrm{Emb}^\omega(B, \mathbb{R}^2) : F(x, y) = (f(x) - \varepsilon(x, y), x), \varepsilon \in \mathcal{T}_\Omega(\bar{\varepsilon})\}. \quad (3.2)$$

Note that such maps do not necessarily leave B invariant. Typically we take extensions of these F to some rectangular domain $B' \subset \Omega$ containing B on which F is invariant. Also note that $\mathcal{H}^-_\Omega(f, \bar{\varepsilon})$ consists of orientation-reversing maps and $\mathcal{H}^+_\Omega(f, \bar{\varepsilon})$ consists of orientation-preserving maps.

Given these spaces, let

$$\mathcal{H}^\pm_\Omega(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega_x}} \mathcal{H}^\pm_\Omega(f, \bar{\varepsilon}) \quad (3.3)$$

and

$$\mathcal{H}^\pm_\Omega = \bigcup_{\bar{\varepsilon}>0} \mathcal{H}^\pm_\Omega(\bar{\varepsilon}). \quad (3.4)$$

and finally set $\mathcal{H}_\Omega = \mathcal{H}^+_\Omega \cup \mathcal{H}^-_\Omega$. Observe that the condition $\varepsilon(x, 0) = 0$ ensures that each Hénon-like map F has a unique representation as either $F = (f - \varepsilon, \pi_x)$ or $F = (f + \varepsilon, \pi_x)$. We will call this representation the *parametrization* of F. We will write $F = (\phi, \pi_x)$ when the parametrization is not explicit. Observe that the degenerate Hénon-like maps considered in §2 lie in a subset of the boundary of $\mathcal{H}_\Omega(\bar{\varepsilon})$ for all $\bar{\varepsilon} > 0$. Given a square $B' \subset \mathbb{R}^2$, a map $F \in \mathrm{Emb}^\omega(B', B')$ is *Hénon-like on* B' if there exists an affine bijection $I : B' \to B$ such that $I \circ F \circ \bar{I} : B \to B$ is a Hénon-like map.

Given a Hénon-like map $F = (\phi, \pi_x) : B \to F(B)$, its inverse will have the form $F^{\circ -1} = (\pi_y, \phi^{-1}) : F(B) \to B$ where $\phi^{-1} : F(B) \to J$ satisfies

$$\pi_y = \phi^{-1}(\phi, \pi_x), \quad \pi_x = \phi(\pi_y, \phi^{-1}). \quad (3.5)$$

More generally, given an integer $w > 0$, let us denote the wth iterate of F by $F^{\circ w} : B \to B$, and the wth preimage by $F^{\circ -w} : F^{\circ w}(B) \to B$. Observe that they have the respective forms $F^{\circ w} = (\phi^w, \phi^{w-1})$ and $F^{\circ -w} = (\phi^{-w+1}, \phi^{-w})$ for some functions $\phi^w : B \to J$ and $\phi^{-w} : F^{\circ w}(B) \to J$. We then define the *wth critical curve* or *critical locus* to be the set $\mathcal{C}^w = \mathcal{C}^w(F) = \{\partial_x \phi^w(x, y) = 0\}$.

3.2. *The renormalization operator.* Let us consider the operators $\mathcal{R}_\mathcal{U}$ and $\mathcal{R}$ from §2.5. Observe that $\mathcal{R}_\mathcal{U}$ is constructed as some iterate under an affine coordinate change whereas $\mathcal{R}$ uses non-affine coordinate changes. That they are equivalent is a coincidence that we now exploit.

Our starting point is that non-trivial iterates of non-degenerate $F \in \mathcal{H}_\Omega$ will most likely not have the form $(f \circ \pi_x \pm \varepsilon, \pi_x)$ after affine rescaling. Therefore, unlike the one-dimensional case, we will need to perform a 'straightening' via a non-affine change of coordinates.

Definition 3.1. Let $p > 1$ be an integer. A map $F \in \mathcal{H}_\Omega$ is *pre-renormalizable with period* p if there exists a closed topological disk $B^0 \subset B$ with $F^{\circ p}(B^0) \subset B^0$. The domain B^0 is called the *central box*. The topological disks $B^w = F^{\circ w}(B^0)$, $w \in W$, will be called the *boxes* and the collection $\underline{B} = \{B^w\}_{w \in W}$ will be called the *cycle*.

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Definition 3.2. Let $p > 1$ be an integer. A map $F \in \mathcal{H}_\Omega$ is *renormalizable with period p* if the following properties hold:

(i) F is pre-renormalizable with period p;

(ii) there exists a diffeomorphism onto its image $H : B^0 \to B^0_{\text{diag}}$, where B^0 is the pre-renormalization domain of F and B^0_{diag} is a square symmetric about the diagonal, such that

$$G = H \circ F^{\circ p} \circ \bar{H} : B^0_{\text{diag}} \to B^0_{\text{diag}} \tag{3.6}$$

is Hénon-like on B^0_{diag}.

The map G is called the *pre-renormalization* of F with respect to H. By definition, there exists an affine map $I : B^0_{\text{diag}} \to B$ such that

$$\mathcal{R}F = I \circ G \circ \bar{I} : B \to B \tag{3.7}$$

is an element of $\mathcal{H}_\Omega$, where G denotes the pre-renormalization of F. The map $\mathcal{R}F$ is called the *Hénon renormalization* of F. We denote the space of all renormalizable maps by $\mathcal{H}_{\Omega,p}$. The operator $\mathcal{R} : \mathcal{H}_{\Omega,p} \to \mathcal{H}_\Omega$ is called the *Hénon renormalization operator* or simply the *renormalization operator* on $\mathcal{H}_\Omega$. See Figure 4 in the case of period-tripling combinatorics (compare also with Figure 3). The absolute value of the eigenvalues of the linear part of $\bar{I}$ (which coincide as it maps a square box to a square box) is called the *scaling ratio* of F.

Remark 3.3. We denote the subspace of $\mathcal{H}_{\Omega,p}$ consisting of renormalizable maps expressible as $F = (f \pm \varepsilon, \pi_x)$, where $|\varepsilon|_\Omega < \bar{\varepsilon}$, by $\mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$ and let

$$\mathcal{H}_{\Omega,p}(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega_x}} \mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$$

denote their union.

There are, *a priori*, many coordinate changes H satisfying these properties. However, we now choose one canonically which has sufficient dynamical meaning. By analogy with the degenerate case, consider the map $H = (\phi^{p-1}, \pi_y)$. As H preserves horizontal lines between two preimages of the same point there lies a solution to $\partial_x \phi^{p-1} = 0$. The inverse function theorem then tells us this is a local diffeomorphism away from the *critical locus* $\mathcal{C}^{p-1} = \{(x, y) \in B : \partial_x \phi^{p-1}(x, y) = 0\}$. Also, since it maps horizontal lines to horizontal lines, it must be a diffeomorphism onto its image. Hence, in a slight abuse of terminology, we will call this map the *horizontal diffeomorphism* associated to F.

Also consider the map $V = F^{\circ p-1} \circ \bar{H} : H(B^0) \to B^{p-1}$. Since $F^{\circ p-1}$ is a diffeomorphism onto its image everywhere and H is a diffeomorphism onto its image when restricted to B^0 we find that V is also a diffeomorphism onto its image. We will call V the *vertical diffeomorphism*. The reason for considering the maps H and V is given by the following proposition.

PROPOSITION 3.4. *Let $F = (\phi, \pi_x) \in \mathcal{H}_\Omega$. Assume that, for some integer $p > 1$, the following properties hold:*

(i) *$B^0 \subset B$ is a subdomain on which $F^{\circ p}$ is invariant;*

(ii) *the horizontal diffeomorphism $H = (\phi^{p-1}, \pi_y)$ is a diffeomorphism onto its image when restricted to B^0.*

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Then $H \circ F^{\circ p} \circ \bar{H} : H(B^0) \to H(B^0)$ *has the form*

$$H \circ F^{\circ p} \circ \bar{H}(x, y) = (\phi^p \circ V(x, y), x) \tag{3.8}$$

where V *is the vertical diffeomorphism described above. Moreover, the vertical diffeomorphism has the form* $V(x, y) = (x, v(x, y))$ *for some* $v \in C^\omega(B, J)$.

Proof. Observe that $\bar{H}$ has the form $\bar{H} = (\bar{\phi}^{p-1}, \pi_y)$ for some $\bar{\phi}^{p-1} : H(B^0) \to \mathbb{R}$. Equating $F^{\circ p-1} \circ F$ with $F^{\circ p}$ implies that $\phi^{p-1}(\phi, \pi_x) = \phi^p$. Equating $H \circ \bar{H}$ and $\bar{H} \circ H$ with the identity implies that

$$\pi_x = \phi^{p-1}(\bar{\phi}^{p-1}, \pi_y) = \bar{\phi}^{p-1}(\phi^{p-1}, \pi_y). \tag{3.9}$$

Hence, by definition of H and V,

$$H \circ F = (\phi^{p-1}(\phi, \pi_x), \pi_x) = (\phi^p, \pi_x) \tag{3.10}$$

and

$$V = F^{\circ p-1} \circ \bar{H} = (\phi^{p-1}(\bar{\phi}^{p-1}, \pi_y), \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)) = (\pi_x, \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)). \tag{3.11}$$

Therefore, if we set $v(x, y) = \phi^{p-2}(\bar{\phi}^{p-1}, \pi_y)$ the result is shown. □

We now show that maps satisfying the hypotheses of the above proposition exist, are numerous and in fact renormalizable in the sense described above. More precisely, we show that $\mathcal{R}$ is defined on a tubular neighbourhood of $\mathcal{H}_{\Omega,\upsilon}(0)$ in the closure of $\mathcal{H}_\Omega$. This is essentially a perturbative result. To do this we need the following.

PROPOSITION 3.5. (Variational formula of the first order) *Let* $F \in \mathcal{H}_\Omega$ *be expressible as* $F = (\phi, \pi_x)$ *where* $\phi(x, y) = f(x) + \varepsilon(x, y)$. *Then, for all* $w \in W$,

$$\phi^w(x, y) = f^{\circ w}(x) + L^w(x) + \varepsilon(x, y)(f^{\circ w})'(x) + \mathrm{O}(\bar{\varepsilon}^2) \tag{3.12}$$

where

$$\begin{aligned} L^w(x) = {} & \varepsilon(f^{\circ w-1}(x), f^{\circ w-2}(x)) + \varepsilon(f^{\circ w-2}(x), f^{\circ w-3}(x)) f'(f^{\circ w-1}(x)) \\ & + \cdots + \varepsilon(f(x), x) \prod_{i=1}^{w-1} f'(f^{\circ i}(x)). \end{aligned} \tag{3.13}$$

Proof. We proceed by induction. Assume that this holds for all integers $0 < i < w$ and let $L^w(x)$ be as above. Then

$$\begin{aligned} \phi^w(x, y) &= \phi(\phi^{w-1}(x, y), \phi^{w-2}(x, y)) \\ &= f(\phi^{w-1}(x, y)) + \varepsilon(\phi^{w-1}(x, y), \phi^{w-2}(x, y)) \end{aligned} \tag{3.14}$$

but observe that, by Taylors' theorem,

$$\begin{aligned} f(\phi^{w-1}(x, y)) &= f(f^{\circ w-1}(x) + L^{w-1}(x) + \varepsilon(x, y)(f^{\circ w-1})'(x) + \mathrm{O}(\bar{\varepsilon}^2)) \\ &= f^{\circ w}(x) + f'(f^{\circ w-1}(x)) L^{w-1}(x) + \varepsilon(x, y)(f^{\circ w})'(x) + \mathrm{O}(\bar{\varepsilon}^2) \end{aligned} \tag{3.15}$$

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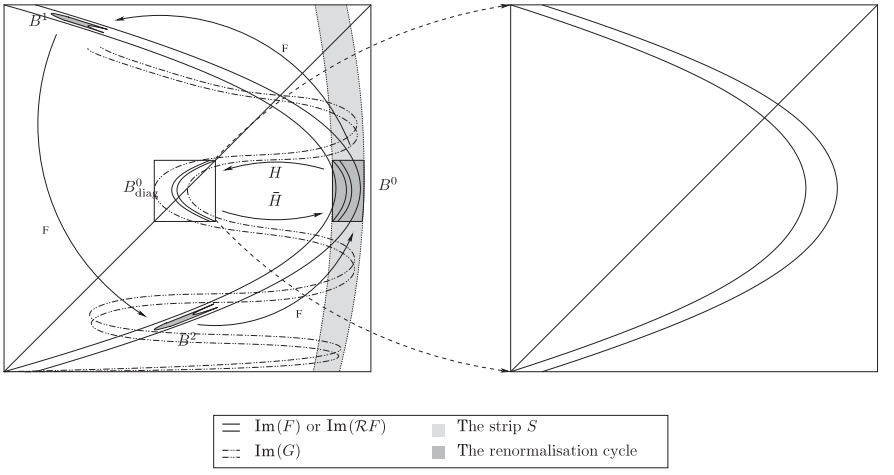


FIGURE 4. A renormalizable Hénon-like map which is a small perturbation of a degenerate Hénon-like map. In this case the combinatorial type is period-tripling. Here the lightly shaded region is the preimage of the vertical strip through B^0_{diag}. The dashed lines represent the image of the square B under the pre-renormalization G. (If the order of the critical points of $f^{\circ 2}$ are all equal then G extends to B.)

and

$$\begin{aligned}\varepsilon(\phi^{w-1}(x, y), \phi^{w-2}(x, y)) &= \varepsilon((f^{\circ w-1}(x), f^{\circ w-2}(x)) + \mathrm{O}(\bar{\varepsilon})) \\ &= \varepsilon(f^{\circ w-1}(x), f^{\circ w-2}(x)) + \mathrm{O}(\bar{\varepsilon}^2), \qquad (3.16)\end{aligned}$$

where, since ε is analytic, we have used the fact that all derivatives of ε are of the order $\bar{\varepsilon}$. Combining these gives us the result. □

PROPOSITION 3.6. *Let $p > 1$ be an integer. Let $F \in \mathcal{H}_\Omega$, let $B^0 \subset B$ be a pre-renormalization domain of period p and let G be its pre-renormalization. Assume that:*

- $\pi_x G(B^0_{\text{diag}}) \subsetneq \pi_x(B^0_{\text{diag}})$;
- *G is Hénon-like on B^0_{diag}.*

Then there exists a neighbourhood $\mathcal{N} \subset \mathcal{H}_\Omega$ of F such that $\tilde{F} \in \mathcal{N}$ implies that:

(i) *$\tilde{F}$ has a pre-renormalization domain with the same properties;*

(ii) *there exists a constant $C > 0$, depending upon f only, such that*

$$\mathrm{dist}_{\text{Haus}}(B^0_{\text{diag}}, \tilde{B}^0_{\text{diag}}) < C|F - \tilde{F}|_\Omega; \qquad (3.17)$$

and

$$\mathrm{dist}_{\text{Haus}}(\Omega^0_{\text{diag}}, \tilde{\Omega}^0_{\text{diag}}) < C|F - \tilde{F}|_\Omega. \qquad (3.18)$$

Proof. Given $F = (\phi, \pi_x)$ satisfying our hypotheses, let H denote its horizontal diffeomorphism and V its vertical diffeomorphism. Let $G = (\varphi, \pi_x)$ denote its pre-renormalization. Let $B^0_{\text{diag}} = J^0 \times J^0$. Let $g_\pm(x) = \varphi(x, \partial^\pm J^0)$ be the two bounding curves of the image of G.

Similarly, given $\tilde{F} = (\tilde{\phi}, \pi_x) \in \mathcal{H}_\Omega$, let $\tilde{H}$ denote its horizontal diffeomorphism and $\tilde{V}$ its vertical diffeomorphism. Let $\tilde{G} = (\tilde{\varphi}, \pi_x)$ denote its pre-renormalization. These all vary continuously with $\tilde{F}$.

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Observe that G has a fixed point $(\alpha, \alpha) \in \partial B^0_{\text{diag}}$. Observe also that $\alpha \in \partial J^0$ is a fixed point for g_- which we can assume is expanding. Let $\beta \in \partial J^0$ be the other boundary component. Then β is a preimage of α under g_- and we may assume has non-zero derivative. The image of the horizontal line through (α, α) intersects the diagonal $\{x = y\}$ transversely at (α, α). These properties are all open conditions. Hence there exists a neighbourhood $\mathcal{N}_0 \subset \mathcal{H}_\Omega$ of F such that $\tilde{F} \in \mathcal{N}_0$ implies that $\tilde{F}$ also has these properties once we set $\tilde{g}_-(x) = \tilde{\varphi}(x, \tilde{\alpha})$. If we let $\tilde{J}^0 = [\tilde{\alpha}, \tilde{\beta}]$ then it is clear that $\tilde{g}_-$ is unimodal on $\tilde{J}^0$.

Now let $\tilde{B}^0_{\text{diag}} = \tilde{J}^0 \times \tilde{J}^0$ and $g_+(x) = \tilde{\varphi}(x, \tilde{\beta})$. Since $\pi_x(G(B^0_{\text{diag}})) \subsetneq \pi_x(B^0_{\text{diag}})$, the critical value of g_+ lies in $\text{int}(J^0)$. Since the critical value of g_+ and ∂J^0 depend continuously on F, there exists a neighbourhood $\mathcal{N}_1 \subset \mathcal{N}_0$ such that $\tilde{F} \in \mathcal{N}_1$ implies that the critical value of $\tilde{g}_+$ lies in $\text{int}(J^0)$. Hence $\tilde{B}^0_{\text{diag}}$ is $\tilde{G}$-invariant and $\pi_x(\tilde{G}(\tilde{B}^0_{\text{diag}})) \subset \pi_x(\tilde{B}^0_{\text{diag}})$.

For the second statement observe that the horizontal diffeomorphisms H and $\tilde{H}$ will map diffeomorphically onto B^0_{diag} and $\tilde{B}^0_{\text{diag}}$. Hence the Hausdorff distance will only depend on the distance between B^0 and $\tilde{B}^0$ and on the distance between H and $\tilde{H}$. Both of these in turn depend on $|F - \tilde{F}|_\Omega$. Finally, the existence of the affine bijection $\tilde{I} : \tilde{B}^0_{\text{diag}} \to B$ is clear. □

PROPOSITION 3.7. *Let $p > 1$ be an integer. Let $F \in \mathcal{H}_{\Omega,p}$ be renormalizable of combinatorial type p. Let $B^0 \subset B$ be the pre-renormalization domain of period p and let G be its pre-renormalization. Assume that:*

- *$\pi_x G(B^0_{\text{diag}}) \subsetneq \pi_x(B^0_{\text{diag}})$;*
- *G is Hénon-like on B^0_{diag}.*

Then there exists a neighbourhood $\mathcal{N} \subset \mathcal{H}_\Omega$ of F and a constant $C > 0$, depending upon F only, such that $F \in \mathcal{N}$ implies that:

(i) *$\tilde{F}$ is p-renormalizable with the same properties;*

(ii) *there exists a constant $C > 0$, depending upon f only, such that*

$$|\mathcal{R}F - \mathcal{R}\tilde{F}|_\Omega < C|F - \tilde{F}|_\Omega. \tag{3.19}$$

Proof. Given $F \in \mathcal{H}_{\Omega,p}$, let B^0 denote its pre-renormalization domain and H denote its horizontal diffeomorphism. For each $w \in W$ let $B^w = F^{\circ w}(B^0)$. Let $\mathcal{N}_0$ denote the neighbourhood of F from Proposition 3.6. Given $\tilde{F} \in \mathcal{N}_0$, let $\tilde{B}^0$ denote its pre-renormalization domain, let $\tilde{H}$ denote the horizontal diffeomorphism and for each $w \in W$ let $\tilde{B}^w = \tilde{F}^{\circ w}(\tilde{H}(\tilde{B}^0))$. Let $\tilde{\mathcal{C}}^{p-1}$ denote its critical curve.

For the first assertion observe that the set $\tilde{\mathcal{C}}^{p-1}$ and the domain $\tilde{B}^0$ vary continuously with $\tilde{F}$. As $\mathcal{C}^{p-1}$ and the domain B^0_{diag} are separated by some distance γ, there is a neighbourhood $\mathcal{N}_1 \subset \mathcal{N}_0$ of F such that $\tilde{F} \in \mathcal{N}_1$ implies that $\tilde{\mathcal{C}}^{p-1}$ and $\tilde{B}^0_{\text{diag}}$ are separated by a distance of $\gamma/2$ or greater.

For the second assertion observe that $\tilde{H}$ and $\tilde{B}^0$ vary continuously with $\tilde{F}$. Hence $\tilde{B}^0_{\text{diag}}$, and therefore $\tilde{I}$, will also vary continuously with $\tilde{F}$. As B^0 is bounded away from $\tilde{\mathcal{C}}^{p-1}$ for $\bar{\varepsilon} > 0$ sufficiently small the result follows. □

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The previous two results are quite general, in that they deal with perturbations of any renormalizable Hénon-like map, not just perturbations of the degenerate maps. However, we now turn our attention to this particular case.

COROLLARY 3.8. *Let υ be a unimodal permutation of length $p > 1$. Let $F = \underline{i}(f) \in \mathcal{H}_{\Omega_x,\upsilon}(0)$. Then there exist constants $C, \bar{\varepsilon}_f > 0$ and a domain $\Omega' \subset \mathbb{C}^2$ such that for any $0 < \bar{\varepsilon} < \bar{\varepsilon}_f$, $\tilde{F} \in \mathcal{H}_\Omega(f, \bar{\varepsilon})$ implies that:*
(i) $\tilde{F} \in \mathcal{H}_{\Omega,p}(f, \bar{\varepsilon})$;
(ii) $\mathcal{R}\tilde{F} \in \mathcal{H}_{\Omega'}(C\bar{\varepsilon}^p)$.

Notation 3.9. Given a unimodal permutation υ of length $p > 1$, let

$$\mathcal{H}_{\Omega,\upsilon} = \bigcup_{f \in \mathcal{U}_{\Omega_x,\upsilon}} \mathcal{H}_{\Omega,p}(f, \bar{\varepsilon}_f). \tag{3.20}$$

When restricting to $\bar{\varepsilon}$-thickenings we will use the notation $\mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon})$.

Proof. The first property follows from Proposition 3.7. We now show the second property. Let $F \in \mathcal{H}_{\Omega,p}(0)$ be as above and let $\tilde{F} \in \mathcal{H}_{\Omega,\upsilon}$ denote the thickening of F by $\varepsilon \in \mathcal{T}_\Omega(\bar{\varepsilon})$. Let H and $\tilde{H}$ denote their respective horizontal diffeomorphisms. Let

$$G(x, y) = (g(x) \pm \delta(x, y), x), \quad \tilde{G}(x, y) = (\tilde{g}(x) \pm \tilde{\delta}(x, y), x) \tag{3.21}$$

denote their respective pre-renormalizations. To simplify notation we only consider the orientation-preserving case. The orientation-reversing case is the same. Observe that

$$\partial_y \delta(x, y) = \mathrm{Jac}_{(x,y)} G = \mathrm{Jac}_{\bar{H}(x,y)} F^{\circ p} \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H} \tag{3.22}$$

and

$$\partial_y \tilde{\delta}(x, y) = \mathrm{Jac}_{(x,y)} \tilde{G} = \mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{F}^{\circ p} \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}}. \tag{3.23}$$

Now observe that

$$|\mathrm{Jac}_{\bar{H}(x,y)} F^{\circ p}|_\Omega = 0 \quad \text{and} \quad |\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{F}^{\circ p}|_\Omega \le |\varepsilon|_\Omega^p.$$

Next recall that $\mathrm{Jac}_{(x,y)} H = \partial_x \phi^{p-1}(x, y)$, so by the variational formula in Proposition 3.5 there is a constant $C_0 > 0$ such that, for $|\varepsilon|_\Omega$ sufficiently small,

$$\left| \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H} - \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}} \right| \le C_0 |\varepsilon|_\Omega. \tag{3.24}$$

Since f is renormalizable, $f^{\circ p-1}$ is a diffeomorphism on J^1. Therefore

$$\left| \frac{\mathrm{Jac}_{F^{\circ p}(\bar{H}(x,y))} H}{\mathrm{Jac}_{\bar{H}(x,y)} H} \right| \le \exp(\mathrm{Dis}(H;\, B^0_{\mathrm{diag}})) \le \exp(\mathrm{Dis}(f^{\circ p-1};\, J^1)) \tag{3.25}$$

is bounded and we find that there exists a constant $C_1 > 0$ such that, for $|\varepsilon|_\Omega$ sufficiently small,

$$\left| \frac{\mathrm{Jac}_{\tilde{F}^{\circ p}(\bar{\tilde{H}}(x,y))} \tilde{H}}{\mathrm{Jac}_{\bar{\tilde{H}}(x,y)} \tilde{H}} \right| < C_1. \tag{3.26}$$

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Hence $|\partial_y \tilde{\delta}(x, y)| < C_1|\varepsilon|_\Omega^p$. By construction the renormalization, $\tilde{F}_1$, of $\tilde{F}$ has parametrization $(\tilde{f}_1, \tilde{\varepsilon}_1)$ which is an affine rescaling of $(\tilde{g}, \tilde{\delta})$. There exists a constant $C_2 > 0$ such that the affine rescaling has scaling ratio $\sigma + C_2|\varepsilon|_\Omega$, where σ is the scaling ratio for F. This implies that there exists a constant $C_3 > 0$ such that $|\partial_y \tilde{\varepsilon}_1|_{\Omega'} \le C_3|\varepsilon|_\Omega^p$. Moreover, $\tilde{\varepsilon}_1$ satisfies $\tilde{\varepsilon}_1(x, 0) = 0$ by construction. Therefore $|\tilde{\varepsilon}_1|_{\Omega'} \le C_3|\varepsilon|_\Omega^p$ and the result is shown. □

THEOREM 3.10. *Let υ be a unimodal permutation of length $p > 1$. Then there are constants $C, \bar{\varepsilon}_0 > 0$ and a domain $\Omega' \subset \mathbb{C}$, depending upon υ and Ω, such that the following holds: for any $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ there is a subspace $\mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \subset \mathcal{H}_\Omega(\bar{\varepsilon})$ containing $\mathcal{H}_{\Omega,\upsilon}(0)$ and a dynamically defined continuous operator,*

$$\mathcal{R} : \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_{\tilde{\Omega}}(C\bar{\varepsilon}^p), \tag{3.27}$$

which extends continuously to $\mathcal{R}$ on $\mathcal{H}_{\Omega,\upsilon}(0)$. Moreover, $\bar{\varepsilon}_0 > 0$ can be chosen so that

$$\mathcal{R} : \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_{\tilde{\Omega}}(\bar{\varepsilon}). \tag{3.28}$$

Proof. By Corollary 3.8, for each $f \in \mathcal{U}_{\Omega,\upsilon}$ there exist an $\bar{\varepsilon}_f > 0$ and a $C_f > 0$ such that $\mathcal{R}$ has an extension $\mathcal{R} : \mathcal{H}_\Omega(f, \bar{\varepsilon}_f) \to \mathcal{H}_\Omega(C_f\bar{\varepsilon}_f^p)$. By compactness of $\mathcal{U}_{\Omega,\upsilon}$ these constants can be chosen uniformly, so, setting

$$\mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) = \bigcup_{f \in \mathcal{U}_{\Omega,\upsilon}} \mathcal{H}_\Omega(f, \bar{\varepsilon}_f), \tag{3.29}$$

we find that $\mathcal{R} : \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon}) \to \mathcal{H}_\Omega(C\bar{\varepsilon}^p)$. Choosing $\bar{\varepsilon}_0 > 0$ sufficiently small so that $\bar{\varepsilon} < C\bar{\varepsilon}^p$ for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$ gives the final claim. □

3.3. *The fixed point and hyperbolicity.* We now consider Hénon-like maps that are infinitely renormalizable. Throughout the rest of this section we fix a unimodal permutation υ of length p. Denote by $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ the subspace of $\mathcal{H}_\Omega(\bar{\varepsilon})$ consisting of infinitely renormalizable Hénon-like maps, where each renormalization has the same combinatorial type υ. We call $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ the space of infinitely renormalizable Hénon-like maps with *stationary combinatorics* υ. Given any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, we write $F_n = \mathcal{R}^n F$. We will use subscripts to denote quantities associated with the nth Hénon renormalization. (For example, $\phi_n = \phi(F_n)$ will denote the function satisfying $F_n = (\phi_n, \pi_x)$.)

THEOREM 3.11. *Let υ be a unimodal permutation of length $p > 1$. Let $\Omega = \Omega_x \times \Omega_y \Subset \mathbb{C}^2$ be a polydisk containing B. Let $f_* \in \mathcal{U}_{\Omega_x,\upsilon}$ denote the unimodal renormalization fixed point of type υ, and let $F_* = (f_* \circ \pi_x, \pi_x)$ denote the associated degenerate Hénon-like map. Then F_* is a hyperbolic fixed point of $\mathcal{R} : \mathcal{H}_{\Omega_x,\upsilon} \to \mathcal{H}_{\Omega_x}$ with codimension-one stable manifold and dimension-one unstable manifold.*

Proof. As was noted in Remark 2.26, we can canonically embed $\mathcal{U}_{\Omega_x,\upsilon}$ into $\mathcal{H}_{\Omega_x,\upsilon}(0)$. By definition, $\mathcal{R}$ restricted to $\mathcal{H}_{\Omega_x,\upsilon}(0)$ is given by $\mathcal{R}(f \circ \pi_x, \pi_x) = (\mathcal{R}_\mathcal{U} f \circ \pi_x, \pi_x)$. Therefore it is clear that the fixed point of $\mathcal{R}_\mathcal{U}$ induces a fixed point of $\mathcal{R}$. That is, if f_* denotes the fixed point of $\mathcal{R}_\mathcal{U}$ then the point $F_* = (f_* \circ \pi_x, \pi_x)$ in $\mathcal{H}_{\Omega_x,\upsilon}(0)$ is a fixed point of $\mathcal{R}$. It is also clear that, when restricted to $\mathcal{H}_\Omega(0)$, the fixed point is unique and hyperbolic, with codimension-one stable manifold and dimension-one unstable manifold.

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Next observe that, by Theorem 3.10, $\mathcal{R} : \mathcal{H}_{\Omega_x,\upsilon} \cap B_{\bar{\varepsilon}}(F_*) \to \mathcal{H}_{\Omega_x}(C\bar{\varepsilon}^p)$ is super-exponentially contracting in the ε-direction if $\bar{\varepsilon} = \bar{\varepsilon}(\Omega_x, \upsilon) > 0$ is sufficiently small. This implies that $D_{F_*}\mathcal{R}$ is hyperbolic in the ε-direction, and moreover has zero spectrum.

Combining the conclusions of each of the two preceding paragraphs, we find that F_* is hyperbolic, with a unique expanding direction. Therefore the stable manifold theorem in [**4**] implies that there exist a codimension-one stable manifold and dimension-one unstable manifold. (The stable manifold theorem in [**4**] is stated for diffeomorphisms but the argument holds for endomorphisms as well.) □

3.4. *Scope maps and the renormalization Cantor set.* We now recast the renormalization theory we have just developed for Hénon-like maps in terms of scope maps and presentation functions (defined below) in a way analogous to that in §2.4. We show using these that, similar to the unimodal case, infinitely renormalizable Hénon-like maps also possess an invariant Cantor set on which the Hénon-like map acts as the adding machine.

Throughout this section, υ will be a fixed unimodal permutation of length $p > 1$, $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalization for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

If $F \in \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\{B^w\}_{w\in W}$ denote its cycle. Let $H : B^0 \to B^0_{\text{diag}}$ denote its horizontal diffeomorphism and $G : B^0_{\text{diag}} \to B^0_{\text{diag}}$ its pre-renormalization. Let $I : B^0_{\text{diag}} \to B$ denote the affine rescaling such that $\mathcal{R}F = I \circ G \circ \bar{I}$. Then we call the coordinate change $\Psi = \Psi(F) : B \to B^0$, given by $\Psi = \bar{H} \circ \bar{I}$, the *scope map* of F. More generally, for $w \in W$ we will call the map $\Psi^w = F^{\circ w} \circ \Psi : B \to B^w$ the *w-scope map* of F.

If F is n-times renormalizable we denote the nth renormalization $\mathcal{R}^n F$ by F_n. For $w \in W$ let $\Psi_n^w = \Psi^w(F_n) : \text{Dom}(F_{n+1}) \to \text{Dom}(F_n)$ be the w-scope map for F_n. Then, if $\mathbf{w} = w_0 \cdots w_n \in W^*$, the function

$$\Psi^{\mathbf{w}} = \Psi_0^{w_0} \circ \cdots \circ \Psi_n^{w_n} : \text{Dom}(F_{n+1}) \to \text{Dom}(F_0) \tag{3.30}$$

is called the $\mathbf{w}$-*scope map* for F. Let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{\mathbf{w}\in W^n}$ denote the collection of scope maps for F.

Similarly, if $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{w\in W^*}$ denote the family of scope maps associated to F. See Figure 5 for the case of period-tripling combinatorics (compare also with Figure 2). Let $\underline{\Psi_n} = \{\Psi_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ denote the family of scope maps associated with F_n. For any $n \geq 0$, let $B_n^{\mathbf{w}} = \Psi_n^{\mathbf{w}}(B)$. These are closed simply-connected domains which we call the *pieces* for F_n. Finally, let $B_*^{\mathbf{w}} = \Psi_*^{\mathbf{w}}(B)$.

PROPOSITION 3.12. *Let $f_n \in \mathcal{U}_{\Omega_x,\upsilon}$ be a sequence of renormalizable unimodal maps and let $\boldsymbol{\psi}_n = \{\psi_n^w\}_{w\in W}$ denote the presentation function of f_n. Assume that:*

(i) *the central cycle $\{J_n^w\}_{w\in W}$ has uniformly bounded geometry for all $n > 0$;*

(ii) $\text{Dis}(\psi_n^w; z)$ *is uniformly bounded;*

(iii) *there exists an integer $N > 0$ such that the Schwarzian derivative $\text{S}_{\psi_n^w} > 0$ for all integers $n > N$ and $w \in W$.*

Then

$$\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \psi^{\mathbf{w}}(J) \tag{3.31}$$

is a Cantor set.

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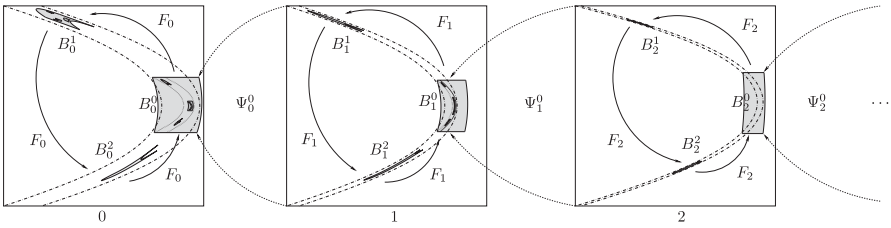


FIGURE 5. The sequence of scope maps for a period-three infinitely renormalizable Hénon-like map. In this case the maps have stationary combinatorics of period-tripling type. Here the dashed line represents the bounding arcs of the image of the square B under consecutive renormalizations F_n.

Proof. Given closed intervals $J \subset T$, with J properly contained in T, consider their cross-ratio

$$D(J, T) = \frac{|J||T|}{|L||R|} \tag{3.32}$$

where L and R are the left and right connected components of $T \setminus J$, respectively. We recall the following properties:

(i) maps with positive Schwarzian derivative contract the cross-ratio;

(ii) for all $K > 0$ there exists a $0 < K' < 1$ such that $D(J, T) < K$ implies that $|J|/|T| < K'$.

The first assumption implies that there is some $K > 0$ such that $D(J_n^w, J) < K$, for all $w \in W$ and $n \in \mathbb{N}$. The third assumption implies that the intervals $J_N^{w_N \cdots w_n} = \psi_N^{w_N} \circ \cdots \circ \psi_n^{w_n}(J)$ are images of J_n^w under positive Schwarzian maps for all $n > N$. Hence the first property of the cross-ratio implies that $D(J_N^{w_N \cdots w_n}, J_N^{w_N \cdots w_{n-1}}) < K$ for all $n > N$. By the second property of the cross-ratio this implies that $|J_N^{w_N \cdots w_n}|/|J_N^{w_N \cdots w_{n-1}}| < K' < 1$. The same argument applies to the images of the gaps between the J_n^w. Therefore

$$\mathcal{O}_N = \bigcap_{n \geq N} \bigcup_{\mathbf{w} \in W^n} \psi_N^{w_N \cdots w_n}(J) \tag{3.33}$$

is a Cantor set. By the second assumption $\psi_0^{w_0 \cdots w_{N-1}}$ has bounded distortion for all $w_0 \cdots w_{N-1} \in W^N$. The image of a Cantor set under a map with bounded distortion is still a Cantor set. Hence

$$\mathcal{O} = \bigcap_{n \geq 0} \bigcup_{\mathbf{w} \in W^n} \psi^{\mathbf{w}}(J) \tag{3.34}$$

is a Cantor set and the result is shown. □

We rephrase the above proposition in terms of scope maps for degenerate Hénon-like maps in the following.

COROLLARY 3.13. *Let $F_n = \underline{\mathrm{i}}(f_n) \in \mathcal{H}_{\Omega,\upsilon}$ be a sequence of renormalizable degenerate Hénon-like maps, let $\mathbf{\Psi}_n = \{\Psi_n^w\}_{w \in W}$ denote the scope map of F_n and let $\boldsymbol{\psi}_n = \{\psi_n^w\}_{w \in W}$ denote the scope function for f_n. Assume that:*

(i) *the central cycle $\{B_n^w\}_{w \in W}$ has uniformly bounded geometry;*

(ii) $\mathrm{Dis}(\Psi_n^w; z)$ *is uniformly bounded;*

(iii) *there exists an integer $N > 0$ such that $\mathrm{S}_{\psi_n^w} > 0$, for all $n > N$ and $w \in W$.*

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Then

$$\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \Psi^{\mathbf{w}}(B) \tag{3.35}$$

is a Cantor set.

We now present the main result of this section. It states that, under suitable conditions, a perturbation of a family of scope maps whose limit set is a Cantor set will also have a limit set which is a Cantor set.

PROPOSITION 3.14. *Let $F_n \in \mathcal{H}_{\Omega,\upsilon}$ be a sequence of renormalizable Hénon-like maps and let $\mathbf{\Psi}_n = \{\Psi_n^w\}_{w\in W}$ denote the presentation function of F_n. Assume that:*

(i) *the set $\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \Psi^{\mathbf{w}}(B)$ is a Cantor set;*

(ii) *for $\mathbf{w} = w_0 w_1 \cdots \in W^*$ the cylinder sets $\Psi^{w_0\cdots w_n}(B)$ 'nest down exponentially', that is, there exists a constant $0 < \delta < 1$ such that $\operatorname{diam}(\Psi^{w_0\cdots w_n}(B)) < \delta \operatorname{diam}(\Psi^{w_0\cdots w_{n-1}}(B))$ for all $n > 0$;*

(iii) *$\|\mathrm{D}_z \Psi_n^w\| < K$ for all $z \in \Omega$, $w \in W$ and $n > 0$.*

Then there exists an $\bar{\varepsilon} > 0$ such that for any sequence $\tilde{F}_n \in \mathcal{H}_{\Omega,\upsilon}$ of renormalizable Hénon-like maps with presentation functions $\mathbf{\Psi}_n = \{\Psi_n^w\}_{w\in W}$ satisfying $|F_n - \tilde{F}_n|_\Omega < C\bar{\varepsilon}^{p^n}$, the set

$$\tilde{\mathcal{O}} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W^n} \tilde{\Psi}^{\mathbf{w}}(B) \tag{3.36}$$

is also a Cantor set.

Proof. It is clear that $\tilde{\mathcal{O}}$ is closed and non-empty, hence we are just required to show that it is totally disconnected and contains no isolated points. Let us first introduce some notation. Define functions E_n by $\tilde{F}_n = F_n + E_n$. By hypothesis, $|E_n|_\Omega \leq C_0\bar{\varepsilon}^{p^n}$. This implies that we can write $\tilde{\Psi}_n^w = \Psi_n^w + \Lambda_n^w$ where $|\Lambda_n^w|_\Omega \leq C_1\bar{\varepsilon}^{p^n}$ for some constant $C_1 > 0$. For $\mathbf{w} = w_0 \cdots w_n \in W^*$ let

$$\Psi^{w_0\cdots w_n} = \Psi_0^{w_0} \circ \cdots \circ \Psi_n^{w_n}, \qquad \tilde{\Psi}^{w_0\cdots w_n} = \tilde{\Psi}_0^{w_0} \circ \cdots \circ \tilde{\Psi}_n^{w_n}. \tag{3.37}$$

Then define $\Lambda^{w_0\cdots w_n}$ to be the function satisfying $\tilde{\Psi}^{w_0\cdots w_n} = \Psi^{w_0\cdots w_n} + \Lambda^{w_0\cdots w_n}$. From Proposition B.2 we find for each $z \in B$, if we set $z_i = \Psi^{w_i\cdots w_n}(z)$ and $\tilde{\Psi}^{\emptyset} = \mathrm{id}$, that

$$\Lambda^{w_0\cdots w_n}(z) = \sum_{i\geq 1} \mathrm{D}_{z_i} \Psi^{w_0\cdots w_{i-1}}(\Lambda^{w_i}(z_{i+1})) + \mathrm{O}(|\mathrm{D}\Lambda^{w_i}||\Lambda^{w_j}|, |\Lambda^{w_i}|^2). \tag{3.38}$$

Now fix $\mathbf{w} = w_0 w_1 \cdots \in \bar{W}$ and let $z, z' \in B$ be any distinct fixed pair of points. Observe that for any $0 < m < n$,

$$|\tilde{\Psi}^{w_0\cdots w_n}(z) - \tilde{\Psi}^{w_0\cdots w_n}(z')| \leq \sup_{\xi\in B} \|\mathrm{D}_\xi \tilde{\Psi}^{w_0\cdots w_{m-1}}\| |\tilde{\Psi}^{w_m\cdots w_n}(z) - \tilde{\Psi}^{w_m\cdots w_n}(z')|. \tag{3.39}$$

Since the derivatives of Ψ_n^w are uniformly bounded and $|\tilde{F}_n - F_n|_\Omega$ decreases super-exponentially, there is a constant $\bar{\varepsilon} > 0$ such that $\|D_\xi \Psi_n^w\| < 2K$ for all $\xi \in B$. Hence

$$\|D_\xi \tilde{\Psi}^{w_0\cdots w_m}\| \leq \prod_0^{m-1} \|D_{\xi_i} \tilde{\Psi}_i^{w_i}\| \leq (2K)^m. \tag{3.40}$$

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Now we consider the splitting

$$|\tilde{\Psi}^{w_m\cdots w_n}(z) - \tilde{\Psi}^{w_m\cdots w_n}(z')| \\ \leq |\Psi^{w_m\cdots w_n}(z) - \Psi^{w_m\cdots w_n}(z')| + |\Lambda^{w_m\cdots w_n}(z) - \Lambda^{w_m\cdots w_n}(z')|. \tag{3.41}$$

By hypothesis, $|\Psi^{w_m\cdots w_n}(z) - \Psi^{w_m\cdots w_n}(z')| \leq \delta^{n-m}$, while Proposition B.2 in the appendix implies that there exists a constant $C > 0$ such that

$$|\Lambda^{w_m\cdots w_n}(z) - \Lambda^{w_m\cdots w_n}(z')| \leq C(2K)^{n-m}\bar{\varepsilon}^{p^m}. \tag{3.42}$$

Therefore

$$|\tilde{\Psi}^{w_0\cdots w_n}(z) - \tilde{\Psi}^{w_0\cdots w_n}(z')| \leq (2K)^m(\delta^{n-m} + C(2K)^{n-m}\bar{\varepsilon}^{p^m}). \tag{3.43}$$

By Proposition A.4 this can be made arbitrarily small by taking $m, n/m > 0$ sufficiently large.

Next we show that $\mathcal{O}$ does not have any isolated points. Assume that there is a word $\mathbf{w} = w_0 w_1 \cdots \in \bar{W}$ for which the associated cylinder set $B^{\mathbf{w}}$ is isolated. Then for any other word $\tilde{\mathbf{w}} \in \bar{W}$ we must have $\operatorname{dist}(B^{\mathbf{w}}, B^{\tilde{\mathbf{w}}}) > \rho$ for some $\rho > 0$ which we may assume satisfies $\rho < 1$. We know that for any $0 < \rho < 1$ there is an integer $N > 0$ such that for all $n > N$, $\operatorname{diam}(B^{w_0\cdots w_n}) < \rho$. In particular, $\operatorname{dist}(B^{w_0\cdots w_n w_{n+1}}, B^{w_0\cdots w_n\tilde{w}}) < \rho$ for any $\tilde{w} \in W$, which is a contradiction. Hence $\mathcal{O}$ does not have any isolated points. □

Using these last two results, we can now prove the following proposition.

PROPOSITION 3.15. *Let υ be a unimodal permutation of length $p > 1$. There exists a constant $\bar{\varepsilon}_0 > 0$, depending upon υ, for which the following holds: given any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$, let $\underline{\Psi} = \{\Psi^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ denote its family of scope maps. Then the set*

$$\mathcal{O} = \bigcap_{n\geq 0} \bigcup_{\mathbf{w}\in W_p^n} \Psi^{\mathbf{w}}(B) \tag{3.44}$$

has the following properties:

(i) *it is an F-invariant Cantor set;*

(ii) *F acts as the adding machine upon $\mathcal{O}$, that is, there exists a map $h : \overline{W}_p \to \mathcal{O}$ such that the following diagram commutes;*

$$\begin{array}{ccc} \overline{W}_p & \xrightarrow{\mathbf{w}\mapsto 1+\mathbf{w}} & \overline{W}_p \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{O} & \xrightarrow[F]{} & \mathcal{O} \end{array} \tag{3.45}$$

(iii) *there is a unique F-invariant measure, μ, with support on $\mathcal{O}$.*

The set $\mathcal{O}$ will be called the *renormalization Cantor set* for F, or simply the *Cantor set* for F when there is no risk of confusion.

Proof. The only thing that needs to be shown is that the limit set is actually a Cantor set. The rest follows by standard arguments.

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Let $F \in \mathcal{I}_{\Omega,\upsilon}$ and let F_n denote its nth renormalization. Then for $n > 0$ sufficiently large the unimodal part f_n of F_n will be renormalizable and they will converge exponentially to f_*. Therefore the corresponding degenerate maps $\underline{i}(f_n)$ will satisfy the conditions of Corollary 3.13 and hence the limit set of their scope maps will be a Cantor set that nests down exponentially. Applying the proposition above shows the limit set of the scope maps for the F_n is also a Cantor set. □

Remark 3.16. Let us denote the cylinder sets of $\mathcal{O}$ under the action of F by $\mathcal{O}^{\mathbf{w}}$. That is, $\mathcal{O}^{\mathbf{w}} = \mathcal{O} \cap \Psi^{\mathbf{w}}(B)$. Then the collection $\underline{\mathcal{O}} = \{\mathcal{O}^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ has the following structure:
(i) $F(\mathcal{O}^{\mathbf{w}}) = \mathcal{O}^{1+\mathbf{w}}$ for all $\mathbf{w} \in W^*$;
(ii) $\mathcal{O}^{\mathbf{w}}$ and $\mathcal{O}^{\tilde{\mathbf{w}}}$ are disjoint for all $\mathbf{w} \neq \tilde{\mathbf{w}}$ of the same length;
(iii) the disjoint union of the $\mathcal{O}^{\mathbf{w}w}$ is equal to $\mathcal{O}^{\mathbf{w}}$, for all $\mathbf{w} \in W^*$, $w \in W$;
(iii) $\mathcal{O} = \bigcup_{\mathbf{w}\in W^n} \mathcal{O}^{\mathbf{w}}$ for each $n \geq 1$.
This will play an important role in studying the geometry of the Cantor set $\mathcal{O}$.

Remark 3.17. For any integer $n > 0$ we can construct the functions $\Psi_n^{\mathbf{w}} = \Psi^{\mathbf{w}}(F_n)$ and the sets $\mathcal{O}_n^{\mathbf{w}} = \mathcal{O}^{\mathbf{w}}(F_n)$ in the same way as above. Let $\underline{\Psi}_n = \{\Psi_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ and $\underline{\mathcal{O}}_n = \{\mathcal{O}_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$.

The number n is called the *height* of $\Psi_n^{\mathbf{w}}$ and $\mathcal{O}_n^{\mathbf{w}}$ and the length of $\mathbf{w}$ is called the *depth*. We use the terms 'height' and 'depth' to reflect a kind of duality in our construction. We will also use these adjectives for all associated objects.

COROLLARY 3.18. *Let υ be a unimodal permutation of length $p > 1$. There exist constants $C > 0$ and $0 < \rho < 1$ such that the following holds. Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathbf{w} \in \bar{W}$ be an arbitrary infinite word. Then the points $\mathcal{O}_n^{\mathbf{w}}$ and $\mathcal{O}_*^{\mathbf{w}}$ satisfy*

$$|\mathcal{O}_n^{\mathbf{w}} - \mathcal{O}_*^{\mathbf{w}}| < C\rho^n. \tag{3.46}$$

Definition 3.19. Let $F \in \mathcal{I}_{\Omega\upsilon}(\bar{\varepsilon})$ be an infinitely renormalizable Hénon-like map with renormalization Cantor set $\mathcal{O}$ with F-invariant measure μ. Then the *average Jacobian* of F is defined by

$$b_F = \exp \int_{\mathcal{O}} \log |\mathrm{Jac}_z F| \, d\mu(z). \tag{3.47}$$

The remainder of this work can be considered as a study of this quantity. The following result was given in [**3**] for period doubling, but the proof is valid for any υ. We state it here without proof.

LEMMA 3.20. (Distortion Lemma) *Let υ be a unimodal permutation of length $p > 1$. Then there exist constants $C > 0$ and $0 \leq \rho < 1$ such that the following holds. Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $B^{\mathbf{w}}$ denote the piece associated to the word $\mathbf{w} \in W^*$. Then for any $B^{\mathbf{w}}$, where $\mathbf{w} \in W^n$, and any $z_0, z_1 \in B^{\mathbf{w}}$,*

$$\log \left| \frac{\mathrm{Jac}_{z_0} F^{\circ m}}{\mathrm{Jac}_{z_1} F^{\circ m}} \right| \leq C\rho^n \tag{3.48}$$

for all $m = 1, p, \ldots, p^n$.

COROLLARY 3.21. *Let υ be a unimodal permutation of length $p > 1$. Then there exists a universal constant $0 < \rho < 1$ such that the following holds. Given $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$,*

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let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$. Then for any integer $n \geq 0$, any $\mathbf{w} \in W^n$, and any $z \in B^{\mathbf{w}}$,

$$\operatorname{Jac}_z F^{\circ p^n} = b_F^{p^n}(1 + \mathrm{O}(\rho^n)). \tag{3.49}$$

Remark 3.22. The constant ρ may be taken as the universal constant from Theorem 3.11.

Proof. Observe that, as μ has support on $\mathcal{O}$,

$$\begin{aligned}\int_{B^{\mathbf{w}}} \log|\operatorname{Jac}_z F^{\circ p^n}|\, d\mu(z) &= \int_{\mathcal{O}^{\mathbf{w}}} \log|\operatorname{Jac}_z F^{\circ p^n}|\, d\mu(z)\\ &= \int_{\mathcal{O}} \log|\operatorname{Jac}_z F|\, d\mu(z)\\ &= \log b_F.\end{aligned} \tag{3.50}$$

Therefore, there is a $\xi \in B^{\mathbf{w}}$ such that

$$\log|\operatorname{Jac}_\xi F^{\circ p^n}| = \frac{\log b_F}{\mu(B^{\mathbf{w}})} = p^n \log b_F \tag{3.51}$$

so the result follows from the Lemma 3.20. □

PROPOSITION 3.23. (Monotonicity) *Let υ be a unimodal permutation of length $p > 1$. Let $F_t \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ be a one-parameter family of infinitely renormalizable Hénon-like maps such that the average Jacobian $b_t = b(F_t)$ depends strictly monotonically on t. Let $\tilde{F}_t = \mathcal{R}F_t$ and let $\tilde{b}_t = b(\tilde{F}_t)$. Then $\tilde{b}_t$ is also strictly monotone in t.*

Proof. Let $\tilde{F}_t = \mathcal{R}F_t$, $\tilde{\mathcal{O}}_t = \mathcal{O}(\tilde{F}_t)$ and $\tilde{\mu}_t = \mu(\tilde{F}_t)$. Recall that, by definition,

$$\log b_t = \int_{\mathcal{O}_t} \log|\operatorname{Jac}_z F_t|\, d\mu_t(z), \quad \log \tilde{b}_t = \int_{\tilde{\mathcal{O}}_t} \log|\operatorname{Jac}_z \tilde{F}_t|\, d\tilde{\mu}_t(z). \tag{3.52}$$

Then by construction $\tilde{F}_t = \Psi_t^{-1} F_t^{\circ p} \Psi_t$ and $\tilde{\mathcal{O}}_t = \bar{\Psi}_t(\mathcal{O}_t^0)$, where $\mathcal{O}_t^0 = \mathcal{O}_t \cap B_t^0$. Since μ_t and $\tilde{\mu}_t$ are determined by the adding machine actions on $\mathcal{O}_t$ and $\tilde{\mathcal{O}}_t$ respectively, we also have $\tilde{\mu}_t = p\mu_t \circ \Psi_t$. Therefore

$$\begin{aligned}&\int_{\tilde{\mathcal{O}}_t} \log|\operatorname{Jac}_z \tilde{F}_t|\, d\tilde{\mu}_t(z)\\ &\quad = p\int_{\bar{\Psi}_t(\mathcal{O}_t^0)} \log\left(|\operatorname{Jac}_{\Psi_t(z)} F_t^{\circ p}| \left|\frac{\operatorname{Jac}_z \Psi_t}{\operatorname{Jac}_{\tilde{F}_t(z)} \Psi_t}\right|\right) d(\mu_t \circ \Psi_t)(z)\end{aligned} \tag{3.53}$$

hence†

$$\begin{aligned}\int_{\tilde{\mathcal{O}}_t} \log|\operatorname{Jac}_z \tilde{F}_t|\, d\tilde{\mu}_t(z) &= p\int_{\mathcal{O}_t^0} \log\left(|\operatorname{Jac}_z F_t^{\circ p}| \left|\frac{\operatorname{Jac}_{\bar{\Psi}_t(z)} \Psi_t}{\operatorname{Jac}_{\bar{\Psi}_t F_t^{\circ p}(z)} \Psi_t}\right|\right) d\mu_t(z)\\ &= p\int_{\mathcal{O}_t^0} \sum_{i=0}^{p-1} \log|\operatorname{Jac}_{F_t^{\circ i}(z)} F_t|\, d\mu_t(z)\\ &\quad + p\int_{\mathcal{O}_t^0} \log\left(\frac{\operatorname{Jac}_{\bar{\Psi}_t(z)} \Psi_t}{\operatorname{Jac}_{\bar{\Psi} F_t^{\circ p}(z)} \Psi_t}\right) d\mu_t(z).\end{aligned} \tag{3.54}$$

† Here we use the integral substitution formula, namely if $(X, \mathcal{B})$, $(X', \mathcal{B}')$ are measurable spaces, μ is a measure on X, and $T : X \to Y$ is surjective, then for all $\mu \circ T^{-1}$-measurable ϕ on Y,

$$\int_X \phi \circ T\, d\mu = \int_Y \phi\, d(\mu \circ T^{-1}).$$

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Now observe that, by definition of μ_t,

$$\int_{\mathcal{O}_t^0} \sum_{i=0}^{p-1} \log |\mathrm{Jac}_{F_t^{\circ i}(z)} F_t| \, d\mu_t(z) = \int_{\mathcal{O}_t} \log |\mathrm{Jac}_z F_t| \, d\mu_t(z) \tag{3.55}$$

and

$$\int_{\mathcal{O}_t^0} \log |\mathrm{Jac}_{\bar{\Psi} F_t^{\circ p}(z)} \Psi_t| \, d\mu_t(z) = \int_{\mathcal{O}_t^0} \log |\mathrm{Jac}_{\bar{\Psi}(z)} \Psi_t| \, d\mu_t(z). \tag{3.56}$$

Together these imply that

$$\log \tilde{b}_t = \int_{\tilde{\mathcal{O}}_t} \log |\mathrm{Jac}_z \tilde{F}_t| \, d\tilde{\mu}_t(z) = p \int_{\mathcal{O}_t} \log |\mathrm{Jac}_z F_t| \, d\mu_t = p \log b_t \tag{3.57}$$

which depends monotonically on t if $\log b_t$ depends monotonically. Since the logarithm function is monotone the proof is complete. □

4. *Asymptotics of scope functions*

We study affine rescalings of scope functions and their compositions. We only consider the case where $w_i = 0$ for all $i > 0$ as this is the simplest to deal with and the most relevant in the next sections. However, we believe that a large portion of the results below can be extended to the more general case. As before, unless otherwise stated, throughout this section υ will be a fixed unimodal permutation of length $p > 1$, $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalization for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

PROPOSITION 4.1. *Let υ be a unimodal permutation of length $p > 1$. Then there exists a constant $\bar{\varepsilon}_0 > 0$ such that the following holds. Given $0 \le \bar{\varepsilon} < \bar{\varepsilon}_0$, for any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\Psi_n : B \to B$ denote its nth scope map. Explicitly, for any $(x, y) \in B$, let*

$$F_n(x, y) = (\phi_n(x, y), x), \quad \Psi_n(x, y) = (\psi_n^1(x, y), \psi_n^0(x, y)). \tag{4.1}$$

Then there is a constant $C > 0$, depending upon F only, such that

$$|\partial_{x^i} \psi_n^1(x, y)| < C, \quad |\partial_{x^i y^j} \psi_n^1(x, y)| < C \bar{\varepsilon}^{p^n} \tag{4.2}$$

for any $(x, y) \in B$ and any integers $i, j \ge 1$.

Proof. By Theorem 3.11 we know there exist a constant $C_0 > 0$ and, for each integer $n > 0$, a degenerate $\tilde{F}_n \in \mathcal{H}_{\Omega,\upsilon}(0)$ such that $|F_n - \tilde{F}_n|_\Omega \le C_0 \bar{\varepsilon}^{p^n}$ and $\tilde{F}_n$ converges exponentially to F_*. Let $\tilde{\Psi}_n$ denote the scope map for $\tilde{F}_n$. Then this implies that there exists a constant $C_1 > 0$ such that $|\Psi_n - \tilde{\Psi}_n|_\Omega \le C_1 \bar{\varepsilon}^{p^n}$ and $\tilde{\Psi}_n$ converges exponentially to Ψ_*. Since Ψ_* is analytic there exists a constant $C_2 > 0$ such that $|\partial_{x^i} \psi_*^1| < C_2$ and, as F_* is degenerate, $\partial_{x^i y^j} \psi_*^1 = 0$ for $j > 0$. Hence the result follows. □

The next lemma is a simple application of Taylor's theorem.

LEMMA 4.2. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ let $\Psi : B \to B^0$ denote its scope map. Explicitly, for $(x, y) \in B$ let*

$$F(x, y) = (\phi(x, y), x), \quad \Psi(x, y) = (\psi^1(x, y), \psi^0(x, y)). \tag{4.3}$$

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Then, for $z_0 \in B$ and $z_1 \in \mathbb{R}^2$ satisfying $z_0 + z_1 \in B$, Ψ can be expressed as

$$\Psi(z_0 + z_1) = \Psi(z_0) + \mathrm{D}_{z_0}\Psi \circ (\mathrm{id} + \mathrm{R}_{z_0}\Psi)(z_1) \tag{4.4}$$

where $\mathrm{D}_{z_0}\Psi$ denotes the derivative of Ψ at z_0 and $\mathrm{R}_{z_0}\Psi$ is a nonlinear remainder term. The maps $\mathrm{D}_{z_0}\Psi$ and $\mathrm{R}_{z_0}\Psi$ take the form

$$\mathrm{D}_{z_0}\Psi = \sigma \begin{pmatrix} s(z_0) & t(z_0) \\ 0 & 1 \end{pmatrix}, \quad \mathrm{R}_{z_0}\Psi(z_1) = \begin{pmatrix} r(z_0)(z_1) \\ 0 \end{pmatrix} \tag{4.5}$$

for some functions $r(z_0)(z_1)$, $s(z_0)$ and $t(z_0)$. Here σ denotes the scaling ratio of Ψ.

Remark 4.3. There are two related quantities that will henceforth play an important role. The first is the scaling ratio of F_*, defined to be the unique eigenvalue, of multiplicity two, of the affine factor of $\Psi_* = \Psi_*^0$. The second is the derivative of Ψ_*^0 at the tip τ_* of F_*. By Lemma 2.17 the derivative of ψ_*^1 at its fixed point is also this scaling ratio (up to sign, which depends on the combinatorics), but the fixed point of ψ^1 is the critical value, which is the projection onto the x-axis of τ_*. Hence these two quantities coincide and shall be denoted by σ.

Definition 4.4. The functions $s(z)$ and $t(z)$ given by Lemma 4.2 above are called the *squeeze* and the *tilt* of Ψ at z, respectively.

PROPOSITION 4.5. *Let $F \in \mathcal{H}_{\Omega,\upsilon}(\bar{\varepsilon})$ and $\tilde{F} \in \mathcal{H}_{\Omega,\upsilon}(0)$ satisfy $|F - \tilde{F}|_\Omega < \bar{\varepsilon}$. Let $\Psi = (\psi^1, \psi^0)$ and $\tilde{\Psi}_f = (\tilde{\psi}^1, \tilde{\psi}^0)$ denote their respective scope maps. Assume that there is a constant $C > 1$ such that, for all $i > 0$,*

$$C^{-1} < |\partial_x \psi^1|_\Omega \quad \text{and} \quad \left| \frac{\partial_{x^i} \psi^1}{\partial_x \psi^1} \right|_\Omega < C. \tag{4.6}$$

Then there is a constant $K > 0$ such that, if $R(z_0)(z_1) = \mathrm{R}_{z_0}\Psi(z_1)$ is defined as above,

$$|\partial_x r(z_0)(z_1)|, |\partial_{xx} r(z_0)(z_1)| < K(1 + |F - F_*| + \bar{\varepsilon}) \tag{4.7}$$

and

$$|\partial_y r(z_0)(z_1)|, |\partial_{xy} r(z_0)(z_1)|, |\partial_{yy} r(z_0)(z_1)| < K\bar{\varepsilon} \tag{4.8}$$

for any $z_0 \in B$ and $z_1 \in \mathbb{R}^2$ satisfying $z_0 + z_1 \in B$.

Proof. Let $z_i = (x_i, y_i)$ for $i = 0, 1$. Expanding Ψ as a power series around z_0 and equating it with the above representation gives

$$r(z_0)(z_1) = \sum_{i,j \geq 0; i+j \geq 2} \binom{i+j}{j} x_1^i y_1^j \frac{\partial_{x^i y^j} \psi^1(z_0)}{\partial_x \psi^1(z_0)} \tag{4.9}$$

and we get a similar expression for $\tilde{r}(z_0)(z_1)$ and $r_*(z_0)(z_1)$. We may write

$$r(z_0)(z_1) = A_0(x_1) + y_1 A_1(x_1) + y_1^2 A_2(x_1, y_1)$$

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where

$$A_0(x_1) = \sum_{i\geq 2; j=0} x_1^i \frac{\partial_{x^i}\psi^1(z_0)}{\partial_x \psi^1(z_0)}, \tag{4.10}$$

$$A_1(x_1) = \sum_{i\geq 1; j=1} \binom{i+1}{1} x_1^i \frac{\partial_{x^i y}\psi^1(z_0)}{\partial_x \psi^1(z_0)}, \tag{4.11}$$

$$A_2(x_1, y_1) = \sum_{i\geq 0; j\geq 2} \binom{i+j}{j} x_1^i y_1^j \frac{\partial_{x^i y^j}\psi^1(z_0)}{\partial_x \psi^1(z_0)}. \tag{4.12}$$

Define $\tilde{A}_0(x_1)$, $\tilde{A}_1(x_1)$ and $\tilde{A}_2(x_1, y_1)$ for $\tilde{r}(z_0)(z_1)$ and $A_{*,0}(x_1)$, $A_{*,1}(x_1)$ and $A_{*,2}(x_1, y_1)$ for $r_*(z_0)(z_1)$ similarly. We claim there exists a constant $K_0 > 0$ such that

$$|A_0'(x_1)|, |A_0''(x_1)| \leq K_0 \frac{|x_1|^2}{1-|x_1|}(1 + |\tilde{F} - F_*|_\Omega + |\tilde{F} - F|_\Omega). \tag{4.13}$$

First, as F_* is fixed we may assume, without loss of generality, that the constant $C > 0$ satisfies $|\partial_{x^i}\psi_*^1/\partial_x\psi_*^1| < C$. Also observe that $C^{-1} < |\partial_x\psi^1|$ implies that $|\partial_x\tilde{\psi}^1|^{-1} < C(1 + \kappa\bar{\varepsilon})$ for some $\kappa > 0$. Therefore, by Lemma A.5,

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi_*^1}{\partial_x\psi_*^1}\right|_\Omega \leq C(1+\kappa\bar{\varepsilon})\max(1, C)\max_i(|\partial_{x^i}\tilde{\psi}^1 - \partial_{x^i}\psi_*^1|_\Omega). \tag{4.14}$$

The same argument, this time using the assumption that $|\partial_{x^i}\psi^1/\partial_x\psi^1| < C$, also implies that

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi^1}{\partial_x\psi^1}\right|_\Omega \leq C(1+\kappa\bar{\varepsilon})\max(1, C)\max_i(|\partial_{x^i}\tilde{\psi}^1 - \partial_{x^i}\psi^1|_\Omega), \tag{4.15}$$

so analyticity of F, $\tilde{F}$ and F_* implies that there is a constant $C_0 > 0$ such that

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi_*^1}{\partial_x\psi_*^1}\right|_\Omega \leq C_0|\tilde{F} - F_*|_\Omega \tag{4.16}$$

and

$$\left|\frac{\partial_{x^i}\tilde{\psi}^1}{\partial_x\tilde{\psi}^1} - \frac{\partial_{x^i}\psi^1}{\partial_x\psi^1}\right|_\Omega \leq C_0|\tilde{F} - F|_\Omega. \tag{4.17}$$

Hence, by the summation formula for a geometric progression,

$$\begin{aligned}|\tilde{A}_0(x_1) - A_{*,0}(x_1)| &\leq \sum_{i\geq 2}|x_1|^i\left|\frac{\partial_{x^i}\tilde{\psi}^1(x_0)}{\partial_x\tilde{\psi}^1(x_0)} - \frac{\partial_{x^i}\psi_*^1(x_0)}{\partial_x\psi_*^1(x_0)}\right| \\ &\leq \frac{C_0|x_1|^2}{1-|x_1|}|\tilde{F} - F_*|_\Omega \end{aligned} \tag{4.18}$$

and similarly

$$|\tilde{A}_0(x_1) - A_0(x_1)| \leq \sum_{i\geq 2}|x_1|^i\left|\frac{\partial_{x^i}\tilde{\psi}^1(x_0)}{\partial_x\tilde{\psi}^1(x_0)} - \frac{\partial_{x^i}\psi_*^1(x_0)}{\partial_x\psi_*^1(x_0)}\right| \leq \frac{C_0|x_1|^2}{1-|x_1|}|\tilde{F} - F|_\Omega. \tag{4.19}$$

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Secondly, observe that analyticity and degeneracy of F_* imply that there exists a constant $C_1 > 0$ such that $|A_{*,0}(x_1)| < C_1|x_1|^2/(1-|x_1|)$. Therefore there exists a $K_0 > 0$ such that

$$\begin{aligned} |A_0(x_1)| &\le |A_{*,0}(x_1)| + |\tilde{A}_0(x_1) - A_{*,0}(x_1)| + |A_0(x_1) - \tilde{A}_0(x_1)| \\ &\le \frac{K_0|x_1|^2}{1-|x_1|}(1 + |\tilde{F} - F_*|_\Omega + |\tilde{F} - F|_\Omega) \end{aligned} \tag{4.20}$$

and, by analyticity of A_0, this implies the bound on its derivatives. Next we claim that there are constants C_2, $C_3 > 0$ such that

$$|A_1(z_1)|, |A_1'(z_1)|, |A_1''(z_1)| \le C_2\bar{\varepsilon}|z_1| \tag{4.21}$$

and

$$|\partial_x A_2(z_1)|, |\partial_y A_2(z_1)|, |\partial_{xx} A_2(z_1)|, |\partial_{xy} A_2(z_1)|, |\partial_{yy} A_2(z_1)| \le C_3\bar{\varepsilon}|z_1|. \tag{4.22}$$

This can be seen by observing that all the coefficients of $A_1(z_1)$ and $A_2(z_1)$ are of the form $\partial_{x^i y^j}\psi^{p-1}(z_0)/\partial_x\psi^{p-1}(z_0)$, but from the variational formula there exists a constant $C_4 > 0$ such that

$$C_4^{-1} < |\partial_x \psi^1| < C_4, \quad |\partial_{x^i y^j}\psi^1| < C_4\bar{\varepsilon}, \tag{4.23}$$

hence all coefficients are bounded by $C_4^2\bar{\varepsilon}$ in absolute value. Therefore, assuming that $|z_1| \le \gamma < 1$, the above estimates must hold by setting $C_3 = C_4^2/(1-\gamma)$.

Now differentiating $r(\Psi; z_0)$ and applying the above estimates, we find that there exists a $C > 0$ such that, for $|z_1| \le \gamma < 1$,

$$\begin{aligned} |\partial_x r(\Psi; z_0)(z_1)| &\le |A_0'(x_1)| + |y_1||A_1'(x_1)| + |y_1|^2|\partial_x A_2(x_1, y_1)| \\ &\le C(1 + |f - f_*| + \bar{\varepsilon}), \end{aligned} \tag{4.24}$$

$$\begin{aligned} |\partial_y r(\Psi; z_0)(z_1)| &\le |A_1(x_1)| + 2|y_1||A_2(x_1, y_1)| + |y_1|^2|\partial_y A_2(x_1, y_1)| \\ &\le C\bar{\varepsilon}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} |\partial_{xx} r(\Psi; z_0)(z_1)| &\le |A_0''(x_1)| + |y_1||A_1''(x_1)| + |y_1|^2|\partial_{xx} A_2(x_1, y_1)| \\ &\le C(1 + |f - f_*| + \bar{\varepsilon}), \end{aligned} \tag{4.26}$$

$$\begin{aligned} |\partial_{xy} r(\Psi; z_0)(z_1)| &\le |A_1'(x_1)| + 2|y_1||\partial_x A_2(x_1, y_1)| + |y_1|^2|\partial_{xy} A_2(x_1, y_1)| \\ &\le C\bar{\varepsilon}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} |\partial_{yy} r(\Psi; z_0)(z_1)| &\le 2|A_2(x_1, y_2)| + 4|y_1||\partial_y A_2(x_1, y_1)| + |y_1|^2|\partial_{yy} A_2(x_1, y_1)| \\ &\le C\bar{\varepsilon}, \end{aligned} \tag{4.28}$$

and hence the result is proved. □

5. *Asymptotics around the tip*

As before, unless otherwise stated, throughout this section υ will be a fixed unimodal permutation of length $p > 1$, $\bar{\varepsilon}_0 > 0$ will be a constant and $\Omega \subset \mathbb{C}^2$ will be a complex polydisk containing the square B in its interior such that $\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ is invariant under renormalization for all $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$.

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For a given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ we now wish to study the Cantor set $\mathcal{O}$, and the behaviour of F around it, in more detail. We will do this locally around a pre-assigned point. Let

$$\tau = \tau(F) = \bigcap_{n\geq 0} B^{0^n}. \tag{5.1}$$

We call this point the *tip*. The study of the orbit of this point is analogous to studying the critical orbit for a unimodal map. The remainder of our work can be viewed as the study of the behaviour of F around τ.

For $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, as usual, let F_n denote the nth renormalization and let $\Psi_n : B \to B_n^0$ denote the scope map for F_n. Explicitly,

$$F_n(x, y) = (\phi_n(x, y), x) \quad \text{and} \quad \Psi_n(x, y) = (\psi_n^1(x, y), \psi_n^0(x, y)).$$

Now let $\Psi_{m,n} = \Psi_m \circ \cdots \circ \Psi_n$. Then

$$\Psi_{m,n}(x, y) = (\psi_{m,n}^1(x, y), \psi_{m,n}^0(x, y))$$

from height $n+1$ to height m. By this convention we let $\Psi_{n,n} = \Psi_n$. Observe that $\psi_{m,n}^0$ is affine and depends upon y only. Let us define points τ_n inductively by $\tau_0 = \tau$ and $\tau_{n+1} = \Psi_n^{-1}(\tau_n)$. We will call τ_n the *tip at height n*. We wish to use the decompositions

$$\begin{aligned}\Psi_n(\tau_{n+1} + z) &= \Psi_n(\tau_{n+1}) + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z) \\ &= \tau_n + \mathrm{D}_{\tau_{n+1}}\Psi_n \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z)\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\Psi_{m,n}(\tau_{n+1} + z) &= \Psi_{m,n}(\tau_{n+1}) + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z_1) \\ &= \tau_m + \mathrm{D}_{\tau_{n+1}}\Psi_{m,n} \circ (\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z_1),\end{aligned} \tag{5.3}$$

whenever $\tau_{n+1} + z$ is in $\mathrm{Dom}(\Psi_n)$ or $\mathrm{Dom}(\Psi_{m,n})$ respectively. For notational simplicity let us denote the derivatives $\mathrm{D}_{\tau_{n+1}}\Psi_n$, $\mathrm{D}_{\tau_{n+1}}\Psi_{m,n}$ and remainder terms $\mathrm{R}_{\tau_{n+1}}\Psi_n$, $\mathrm{R}_{\tau_{n+1}}\Psi_{m,n}$, by D_n, $D_{m,n}$ and R_n, $R_{m,n}$ respectively.

It will turn out to be fruitful to change to coordinates in which the tips are situated at the origin. Therefore let $\mathbf{T}_n(z) = z - \tau_n$ and consider the maps $\hat{\Psi}_n = \mathbf{T}_n \circ \Psi_n \circ \mathbf{T}_{n+1}^{-1}$ and their composites

$$\hat{\Psi}_{m,n} = \hat{\Psi}_m \circ \cdots \circ \hat{\Psi}_n = \mathbf{T}_m \Psi_{m,n} \mathbf{T}_{n+1}^{-1}. \tag{5.4}$$

From Proposition B.1 we know, since $\mathbf{T}_n$ is a translation, that $\mathrm{R}_{z_0}\hat{\Psi}_n = \mathrm{R}_{\mathbf{T}_{n+1}^{-1}z_0}\Psi_n$. Therefore, using the same decomposition as above, we find that

$$\begin{aligned}\hat{\Psi}_n(z) &= \hat{\Psi}_n(0 + z) \\ &= \hat{\Psi}_n(0) + \mathrm{D}_0\hat{\Psi}_n(\mathrm{id} + \mathrm{R}_0\hat{\Psi}_n)(z) \\ &= \mathrm{D}_{\tau_{n+1}}\Psi_n(\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_n)(z),\end{aligned} \tag{5.5}$$

and similarly,

$$\begin{aligned}\hat{\Psi}_{m,n}(z) &= \hat{\Psi}_{m,n}(0 + z) \\ &= \hat{\Psi}_{m,n}(0) + \mathrm{D}_0\hat{\Psi}_{m,n}(\mathrm{id} + \mathrm{R}_0\hat{\Psi}_{m,n})(z) \\ &= \mathrm{D}_{\tau_{n+1}}\Psi_{m,n}(\mathrm{id} + \mathrm{R}_{\tau_{n+1}}\Psi_{m,n})(z).\end{aligned} \tag{5.6}$$

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For notational simplicity let us denote the quantities $\mathrm{D}_0\hat{\Psi}_n$, $\mathrm{D}_0\hat{\Psi}_{m,n}$, $\mathrm{R}_0\hat{\Psi}_n$ and $\mathrm{R}_0\hat{\Psi}_{m,n}$, by $\hat{D}_n$, $\hat{D}_{m,n}$, $\hat{R}_n$ and $\hat{R}_{m,n}$ respectively. Observe that, because our coordinate changes were translations, these quantities are equal to D_n, $D_{m,n}$, R_n and $R_{m,n}$ respectively. The next lemma follows directly from Lemma 4.2.

LEMMA 5.1. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let the linear map D_n and the function $R_n(z)$ be as above. Then D_n and $R_n(z)$ have the respective forms*

$$D_n = \sigma_n \begin{pmatrix} s_n & t_n \\ 0 & 1 \end{pmatrix}, \quad R_n(z) = \begin{pmatrix} r_n(z) \\ 0 \end{pmatrix}. \tag{5.7}$$

Definition 5.2. The quantities s_n and t_n from the preceding lemma will respectively be called the *squeeze* and *tilt* of Ψ_n at τ_{n+1}.

PROPOSITION 5.3. *For $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $B_n^{\mathbf{w}}$ denote the box of height n with word $\mathbf{w} \in W^*$. Then:*

(1) *for each $\mathbf{w} \in W^*$, $\mathrm{dist}_{\mathrm{Haus}}(B_n^{\mathbf{w}}, B_*^{\mathbf{w}}) \to 0$ exponentially;*
(2) *for each $\mathbf{w} \in \overline{W}$, $\mathrm{dist}_{\mathrm{Haus}}(\mathcal{O}_n^{\mathbf{w}}, \mathcal{O}_*^{\mathbf{w}}) \to 0$ exponentially.*

PROPOSITION 5.4. *There exist constants $C > 1$ and $0 < \rho < 1$ such that the following holds: given $0 < \bar{\varepsilon} < \bar{\varepsilon}_0$, let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, and for each integer $n > 0$ let σ_n, s_n, t_n be the constants and $r_n(z)$ the function defined above. Then for any $z \in B$,*

$$\sigma(1 - C\rho^n) < |\sigma_n| < \sigma(1 + C\rho^n), \tag{5.8}$$

$$\sigma(1 - C\rho^n) < |s_n| < \sigma(1 + C\rho^n), \tag{5.9}$$

$$C^{-1}\bar{\varepsilon}^{p^n} < |t_n| < C\bar{\varepsilon}^{p^n}, \tag{5.10}$$

$$|\partial_x r_n(z)| < C|z|, \quad |\partial_y r_n(z)| < C\bar{\varepsilon}^{p^n}|z|, \tag{5.11}$$

$$|\partial_{xx} r_n(z)| < C|z|, \quad |\partial_{xy} r_n(z)| < C\bar{\varepsilon}^{p^n}|z|, \quad |\partial_{yy} r_n(z)| < C\bar{\varepsilon}^{p^n}|z|. \tag{5.12}$$

Proof. Observe that σ_n is the eigenvalue of DI_n^{-1}, the affine bijection between $B_{n,\mathrm{diag}}^0$ and B. By Proposition 5.3 there exists a constant $C_0 > 0$ such that $\mathrm{dist}_{\mathrm{Haus}}(B_n^0, B_*^0) < C_0\rho^n$, and we see that $|\sigma_n - \sigma_*| < C_0\rho^n$. Next observe that $s_n = \partial_x \psi_n^1(\tau_{n+1})$ and, by Lemma 2.17, $\sigma = \partial_x \psi_{f_*}^1(\tau_*)$ which implies that

$$\begin{aligned} |s_n - \sigma| &\le |\partial_x \psi_n^1(\tau_{n+1}) - \partial_x \psi_*^1(\tau_{n+1})| + |\partial_x \psi_*^1(\tau_{n+1}) - \partial_x \psi_*^1(\tau_*)| \\ &\le |\psi_n^1 - \psi_*^1|_\Omega + |\partial_{xx} \psi_*^1|_\Omega |\pi_x(\tau_{n+1}) - \pi_x(\tau_*)|. \end{aligned} \tag{5.13}$$

Again by Proposition 5.3, $|\tau_n - \tau_*| < C_0\rho^n$. Also, a consequence of Theorem 3.11 is that there exists a constant $C_1 > 0$ such that $|\psi_n^1 - \psi_*^1|_\Omega < C_1\rho^n$. Since fixing the combinatorial type fixes the map ψ_*^1, we may assume that $|\partial_{xx}\psi_*^1|_\Omega < C_2$ for some constant $C_2 > 0$. Therefore

$$|s_n - \sigma| \le C_1\rho^n + C_0C_2\rho^n = (C_1 + C_0C_2)\rho^n. \tag{5.14}$$

Now for each $n > 0$ choose a $\tilde{F}_n \in \mathcal{H}_{\Omega,\upsilon}(0)$ such that $|F_n - \tilde{F}_n|_\Omega < C_3\bar{\varepsilon}^{p^n}$, where $C_3 > 0$ is the constant from Theorem 3.10. A consequence of convergence of renormalization, Theorem 3.11, is that there exists a constant $C_4 > 0$ such that

$$|\partial_y \psi_n^1| = |\partial_y \psi_n^1 - \partial_y \psi_{f_n}^1| < C_4\bar{\varepsilon}^{p^n}.$$

This implies (5.10) and concludes the proof of inequalities (5.8)–(5.10). For the inequalities (5.11)–(5.12) we apply Proposition 4.5. □

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LEMMA 5.5. *For any $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ let the linear map $D_{m,n}$ and the function $R_{m,n}(z)$ be as above. Then $D_{m,n}$ and the function $R_{m,n}(z)$ have the respective forms*

$$D_{m,n} = \sigma_{m,n}\begin{pmatrix} s_{m,n} & t_{m,n} \\ 0 & 1 \end{pmatrix}, \quad R_{m,n}(z) = \begin{pmatrix} r_{m,n}(z) \\ 0 \end{pmatrix}, \tag{5.15}$$

and so if $\tau_m = (\xi_m, \eta_n)$,

$$\Psi_{m,n}(z) = \tau_m + \sigma_{m,n}\begin{pmatrix} s_{m,n}((x-\xi_m) + r_{m,n}(z-\tau_m)) + t_{m,n}(y-\eta_m) \\ y - \eta_m \end{pmatrix}. \tag{5.16}$$

Moreover,

$$\sigma_{m,n} = \prod_{i=m}^{n} \sigma_i, \quad s_{m,n} = \prod_{i=m}^{n} s_i, \quad t_{m,n} = \sum_{i=m}^{n} s_{m,i-1} t_i. \tag{5.17}$$

Proof. From Lemma 5.1 we know this holds for $m = n$. For $m < n$ the chain rule $D_{m,n} = D_{m,n-1} D_n$ implies that $D_{m,n}$ is again upper triangular and

$$\sigma_{m,n} = \sigma_{m,n-1}\sigma_n, \quad s_{m,n} = s_{m,n-1}s_n, \quad t_{m,n} = s_{n-1}t_n + t_{m,n-1}, \tag{5.18}$$

from which the lemma immediately follows by induction. □

PROPOSITION 5.6. *There exist constants $C > 0$, and $0 < \rho < 1$ such that the following holds: for $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $\sigma_{m,n}, s_{m,n}, t_{m,n}$ be the constants and $r_{m,n}(z)$ the function defined above. Then*

$$\sigma^{n-m}(1 - C\rho^m) < |\sigma_{m,n}| < \sigma^{n-m}(1 + C\rho^m), \tag{5.19}$$

$$\sigma^{n-m}(1 - C\rho^m) < |s_{m,n}| < \sigma^{n-m}(1 + C\rho^m), \tag{5.20}$$

$$|t_{m,n}| < C\bar{\varepsilon}^{p^m}, \tag{5.21}$$

$$|\partial_x r_{m,n}(z)| < C|z|, \quad |\partial_y r_{m,n}(z)| < C\bar{\varepsilon}^{p^{m-1}}|z|, \tag{5.22}$$

$$|\partial_{xx} r_{m,n}(z)| < C|z|, \quad |\partial_{xy} r_{m,n}(z)| < C\sigma^{2(n-m)}\bar{\varepsilon}^{p^m}|z|, \quad |\partial_{yy} r_{m,n}(z)| < C\bar{\varepsilon}^{p^m}|z|. \tag{5.23}$$

Proof. Throughout the proof $C_0 > 0$ will denote the constant from Proposition 5.4. From Lemma 5.5, Proposition 5.4 and Proposition A.1 respectively, we find there exists a constant $C_1 > 0$ such that

$$|\sigma_{m,n}| = \prod_{i=m}^{n} |\sigma_i| \le \sigma^{n-m} \prod_{i=m}^{n} (1 + C_0\rho^i) \le \sigma^{n-m}(1 + C_1\rho^m), \tag{5.24}$$

and similarly

$$|s_{m,n}| = \prod_{i=m}^{n} |s_i| \le \sigma^{n-m} \prod_{i=m}^{n} (1 + C_0\rho^i) \le \sigma^{n-m}(1 + C_1\rho^m). \tag{5.25}$$

Again by Lemma 5.5 and Proposition 5.4 above we find, for $i > m$, that

$$\left|\frac{t_i}{t_m}\right| = \left|\frac{\partial_y \phi_i^{p-1}(\tau_{i+1})}{\partial_y \phi_m^{p-1}(\tau_{m+1})}\right| \left|\frac{\partial_x \phi_m^{p-1}(\tau_{m+1})}{\partial_x \phi_i^{p-1}(\tau_{i+1})}\right| \le C_0^4 \bar{\varepsilon}^{p^{i+1}-p^m}. \tag{5.26}$$

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Therefore, by Lemma A.3 there exists a constant $C_2 > 0$ such that

$$\begin{aligned} |t_{m,n}| &\le |t_m| \sum_{i=m}^{n} |s_{m,i-1}| \left| \frac{t_i}{t_m} \right| \\ &\le C_0^2 \bar{\varepsilon}^{p^m} \sum_{i=m}^{n} \sigma^{i-m-1} \bar{\varepsilon}^{p^{i+1}-p^m} (1 + C_1 \rho^i). \\ &\le C_2 \bar{\varepsilon}^{p^m}. \end{aligned} \tag{5.27}$$

This concludes the proof of the first three inequalities (5.19)–(5.21). For the inequalities (5.22)–(5.23) we will proceed by induction. The case when $m = n$ is shown in Proposition 5.4 so, for $m + 1 \le n$, assume that the inequalities hold for $r_{m+1,n}$ and consider $r_{m,n}$. Choose $z = (x, y) \in \mathbb{R}^2$ such that $\tau_{n+1} + z \in \mathrm{Dom}(\Psi_{m,n})$. Then since $\Psi_{m,n} = \Psi_m \circ \Psi_{m+1,n}$, decomposing the left-hand side gives

$$\Psi_{m,n}(\tau_{n+1} + z) = \tau_m + D_{m,n}(\mathrm{id} + R_{m,n})(z) \tag{5.28}$$

and decomposing the right-hand side and applying Proposition B.1 gives

$$\begin{aligned} &\Psi_m(\Psi_{m+1,n}(\tau_{n+1} + z)) \\ &\quad = \Psi_n(\tau_{m+1} + D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)) \\ &\quad = \tau_m + D_m(\mathrm{id} + R_m)(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)) \\ &\quad = \tau_m + D_{m,n}(\mathrm{id} + R_{m+1,n})(z) + D_m(R_m(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z))). \end{aligned} \tag{5.29}$$

Equating these and making appropriate cancellations then gives

$$R_{m,n}(z) = R_{m+1,n}(z) + D_{m+1,n}^{-1}(R_m(D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z))). \tag{5.30}$$

By definition, $R_{m,n}(z) = (r_{m,n}(z), 0)$, $R_{m+1,n}(z) = (r_{m+1,n}(z), 0)$ and $R_m(z) = (r_m(z), 0)$. Therefore setting $z' = (x', y') = D_{m+1,n}(\mathrm{id} + R_{m+1,n})(z)$, that is,

$$(x', y') = (\sigma_{m+1,n} s_{m+1,n}(x + r_{m+1,n}(x, y)) + \sigma_{m+1,n} t_{m+1,n} y, \sigma_{m+1,n} y), \tag{5.31}$$

we find that

$$r_{m,n}(x, y) = r_{m+1,n}(x, y) + \sigma_{m+1,n}^{-1} s_{m+1,n}^{-1} r_m(x', y'). \tag{5.32}$$

Differentiating this with respect to x and y gives

$$\partial_x r_{m,n}(x, y) = \partial_x r_{m+1,n}(x, y) + (1 + \partial_x r_m(x, y)) \partial_x r_m(x', y'), \tag{5.33}$$

$$\partial_y r_{m,n}(x, y) = \partial_y r_{m+1,n}(x, y) + s_{m+1,n}^{-1}(t_{m+1,n} \partial_x r_m(x', y') + \partial_y r_m(x', y')). \tag{5.34}$$

Now let $C_4 > 1$ be the maximum of the constant from Proposition 5.4 and the constant from the first item above which ensures that

$$|s_{m+1,n}| > C_4^{-1} \sigma^{n-m-1}, \quad |t_{m+1,n}| < C_4 \bar{\varepsilon}^{p^{m+1}}, \tag{5.35}$$

and

$$|\partial_x r_m(z)| < C_4 |z|, \quad |\partial_y r_m(z)| < C_4 \bar{\varepsilon}^{p^m} |z|, \quad |\partial_{xy} r_m(z)| < C_4 \bar{\varepsilon}^{p^m} |z|. \tag{5.36}$$

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As a consequence of our induction hypothesis, there exists a constant $C_5 > 0$ such that $|z'| < C_5\sigma^{n-m-1}|z|$. Together these imply the existence of a constant $C_6 > 0$ such that

$$\begin{aligned}|\partial_x r_{m,n}(z)| &\le |\partial_x r_{m+1,n}(z)| + |\partial_x r_m(z')|(1 + |\partial_x r_m(z)|)\\ &\le |\partial_x r_{m+1,n}(z)| + C_4|z'|(1 + C_4|z|)\\ &\le |\partial_x r_{m+1,n}(z)| + C_4C_5\sigma^{n-m-1}|z|(1 + C_4|z|)\\ &\le |\partial_x r_{m+1,n}(z)| + C_6\sigma^{n-m-1}|z| \end{aligned} \tag{5.37}$$

and a constant $C_7 > 0$ such that

$$\begin{aligned}|\partial_y r_{m,n}(z)| &\le |\partial_y r_{m+1,n}(z)| + |s_{m+1,n}|^{-1}(|\partial_x r_m(z')||t_{m+1,n}| + |\partial_y r_m(z')|).\\ &\le |\partial_y r_{m+1,n}(z)| + C_4\sigma^{-(n-m-1)}(C_4^2\bar{\varepsilon}^{p^{m+1}}|z'| + C_4\bar{\varepsilon}^{p^m}|z'|)\\ &\le |\partial_y r_{m+1,n}(z)| + C_4^2C_5(C_4\bar{\varepsilon}^{p^{m+1}} + \bar{\varepsilon}^{p^m})|z|\\ &\le |\partial_y r_{m+1,n}(z)| + C_7\bar{\varepsilon}^{p^m}|z|. \end{aligned} \tag{5.38}$$

Next we consider the second-order derivatives. As all functions are analytic the estimates for $\partial_{xx} r_{m,n}$ and $\partial_{yy} r_{m,n}$ follow from those of $\partial_x r_{m,n}$ and $\partial_y r_{m,n}$ respectively. Therefore we only need to consider the mixed second-order partial derivative. This is given by

$$\partial_{xy} r_{m,n}(z) = \partial_{xy} r_{m+1,n}(z) + \sigma_{m+1,n}\partial_{xy} r_m(z)(\partial_{xx} r_m(z')t_{m+1,n} + \partial_{xy} r_m(z')) \tag{5.39}$$

and hence, using the above estimates, there exists a constant $C_8 > 0$ such that

$$\begin{aligned}|\partial_{xy} r_{m,n}(z)| &\le |\partial_{xy} r_{m+1,n}(z)| + |\sigma_{m+1,n}||\partial_{xy} r_m(z)|(|\partial_{xx} r_m(z')||t_{m+1,n}| + |\partial_{xy} r_m(z')|)\\ &\le |\partial_{xy} r_{m+1,n}(z)| + C_4^2\sigma^{n-m-1}\bar{\varepsilon}^{p^m}|z|(C_4^2\bar{\varepsilon}^{p^{m+1}}|z'| + C_4\bar{\varepsilon}^{p^m}|z'|)\\ &\le |\partial_{xy} r_{m+1,n}(z)| + C_8\sigma^{2(n-m)}\bar{\varepsilon}^{2p^m}|z|. \end{aligned} \tag{5.40}$$

Therefore invoking the induction hypothesis and setting $C = \max_i C_i$, we achieve the desired result. □

PROPOSITION 5.7. *There exists a constant $0 < \rho < 1$ such that that following holds. For $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $r_{m,n}(x, y)$ denote the function constructed above for integers $0 < m < n$. Then there exists a constant $C > 0$ such that for any $(x, y) \in B$,*

$$|[x + r_{m,n}(x, y)] - v_*(x)| < C(\bar{\varepsilon}^{p^m} y + \rho^{n-m}) \tag{5.41}$$

and

$$|[1 + \partial_x r_{m,n}(x, y)] - \partial_x v_*(x)| < C\rho^{n-m} \tag{5.42}$$

where $v_*(x)$ is the affine rescaling of the universal function u_* so that its fixed point lies at the origin with multiplier 1.

Proof. Given $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, let $F_n : B \to B$ denote the nth renormalization and let $\Psi_n : B \to B$ denote the nth scope function. Let $\hat{F}_n : \hat{B}_{n+1} \to \hat{B}_n$ and $\hat{\Psi}_n : \hat{B}_{n+1} \to \hat{B}_n$ denote these maps under the translational change of coordinates described above.

First, let us consider the functions $\hat{\Psi}_m : \hat{B}_{m+1} \to \hat{B}_m$. By construction these preserve the x-axis, since they preserve the family of horizontal lines and the origin is a fixed point

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for each of them. This implies that there exists a function $\hat{\psi}_m : \hat{J}_{m+1} \to \hat{J}_m$ such that $\hat{\Psi}_m(x, 0) = (\hat{\psi}_m(x), 0)$. Next observe that there is a constant $C_0 > 0$ such that for each $n \geq 0$ there exists $f_n \in \mathcal{U}_{\Omega_x, \upsilon}$ satisfying $|F_n - (f_n \circ \pi_x, \pi_x)|_\Omega < C_0 \bar{\varepsilon}^{p^n}$. Let $\hat{f}_n : \hat{J}_n \to \hat{J}_n$ denote f_n under the translational change of coordinates and let $\hat{\psi}_n^1 : \hat{J}_n \to \hat{J}_n^1$ be the branch of its presentation function corresponding to the interval $\hat{J}_n^1$. Proposition 2.21 implies that there is a constant $C_1 > 0$ such that $|\hat{\psi}_n^1 - \hat{\psi}_n|_{C^2} < C_1 \bar{\varepsilon}^{p^n}$. Proposition 2.21 and Theorem 3.11 imply that there is a constant $C_2 > 0$ such that $|\hat{\psi}_n^1 - \hat{\psi}_*^1|_{C^2} < C_2 \rho^n$. Combining these, we find that there is a constant $C_3 > 0$ such that

$$|\hat{\psi}_n - \hat{\psi}_*^1|_{C^2} < C_3 \rho^n. \tag{5.43}$$

Now observe that there exist functions $\hat{\psi}_{m,n} : \hat{J}_{n+1} \to \hat{J}_{m,n}^0 \subset \hat{J}_m$, where $\hat{J}_{m,n}^0 = \hat{\psi}_{m,n}(\hat{J}_{n+1})$, such that $\hat{\Psi}_{m,n}(x, 0) = (\hat{\psi}_{m,n}(x), 0)$. Moreover, since $\hat{\Psi}_{m,n} = \hat{\Psi}_m \circ \cdots \circ \hat{\Psi}_n$ we must have $\hat{\psi}_{m,n} = \hat{\psi}_m \circ \cdots \circ \hat{\psi}_n$. Also observe that, since $\hat{\Psi}_{m,n} = \mathbf{T}_m \circ \Psi_{m,n} \circ \mathbf{T}_{n+1}^{-1}$, there are translations $\mathtt{T}_m$ such that $\hat{\psi}_{m,n} = \mathtt{T}_m \circ \psi_{m,n} \circ \mathtt{T}_{n+1}^{-1}$.

Now let $[\psi_{m,n}]$ and $[\psi_{*,m,n}]$ denote, respectively, the orientation-preserving affine rescalings of the maps $\hat{\psi}_m \circ \cdots \circ \hat{\psi}_n$ and $\hat{\psi}_* \circ \cdots \circ \hat{\psi}_*$ to the interval J. Here the composition of $\hat{\psi}_*$ with itself is taken $n - m$ times. Then [**3**, Lemma 7.3] implies that there exists a constant $C_4 > 0$ such that $|\psi_{m,n} - \psi_{*,m,n}|_{C^1} < C_4 \rho^{n-m}$. This then implies, together with the second part of Lemma 2.17, that there is a constant $C_5 > 0$ such that

$$\begin{aligned} |[\psi_{m,n}] - u_*|_{C^1} &\leq |[\psi_{m,n}] - [\psi_{*,m,n}]|_{C^1} + |[\psi_{*,m,n}] - u_*|_{C^1} \\ &\leq C_5 \rho^{n-m} \end{aligned} \tag{5.44}$$

where u_* is the universal function from that lemma. Next we perform a translational change of coordinates on $[\psi_{m,n}]$ and u_* so that the fixed point lies at the origin. Observe that these coordinate changes also converge exponentially. Therefore, if $[\hat{\psi}_{m,n}]$, and $\hat{u}_*$ denote these functions in the new coordinates, there exists a constant $C_6 > 0$ such that

$$|[\hat{\psi}_{m,n}] - \hat{u}_*|_{C^1} < C_6 \rho^{n-m}. \tag{5.45}$$

Now observe that, since multipliers of fixed points have uniform Lipschitz-type dependence (by the Cauchy estimates), this implies that the difference between the multiplier $\mu_{m,n}$ of the fixed point 0 for $[\hat{\psi}_{m,n}]$ and the multiplier μ_* of the fixed point 0 for $\hat{u}_*$ decreases exponentially in $n - m$ at the same rate. This implies that there exists a constant $C_7 > 0$ such that

$$|\mu_{m,n}^{-1}[\hat{\psi}_{m,n}] - \mu_*^{-1}\hat{u}_*|_{C^1} < C_7 \rho^{n-m}. \tag{5.46}$$

Now we claim that $\mu_{m,n}^{-1}[\hat{\psi}_{m,n}] = x + r_{m,n}(x, 0)$. Both come from affinely rescaling $\Psi_{m,n}$ so that the origin is fixed, the horizontal line $\{y = 0\}$ is fixed and their derivatives in the x-direction are 1. Hence they are equal. Also, by definition, $\mu_*^{-1}\hat{u}_* = v_*$. This then implies, by the above and Proposition 5.6, that there is a constant $C > 0$ such that

$$\begin{aligned} |[x + r_{m,n}(x, y)] - v_*(x)| &\leq |[x + r_{m,n}(x, y)] - [x + r_{m,n}(x, 0)]| \\ &\quad + |[x + r_{m,n}(x, 0)] - v_*(x)| \\ &\leq |\partial_y r_{m,n}||y| + |\mu_{m,n}^{-1}[\hat{\psi}_{m,n}] - \mu_*^{-1}\hat{u}_*|_{C^0} \\ &\leq C(\bar{\varepsilon}^{p^{m-1}}|y| + \rho^{n-m}) \end{aligned} \tag{5.47}$$

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which gives the first bound while

$$\begin{aligned}|[1+\partial_x r_{m,n}(x,\,y)]-\partial_x v_*(x)| &\le |[1+\partial_x r_{m,n}(x,\,y)]-[1+\partial_x r_{m,n}(x,\,0)]| \\ &\quad + |[1+\partial_x r_{m,n}(x,\,0)]-\partial_x v_*(x)| \\ &\le |\partial_{xy} r_{m,n}||y| + |\mu_{m,n}^{-1}[\hat{\psi}_{m,n}]-\mu_*^{-1}\hat{u}_*|_{C^1} \\ &\le C(\sigma^{n-m}\bar{\varepsilon}^{p^m}|y|+\rho^{n-m}) \end{aligned} \tag{5.48}$$

which, since z lies in a bounded domain and $\bar{\varepsilon}^{p^m}$ is bounded from above, gives us the bound for the derivative. □

PROPOSITION 5.8. *There exist constants $C>0$ and $0<\rho<1$ such that the following holds: given $F\in\mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$, for each integer $m>0$ there exists a constant $\kappa_{(m)}=\kappa_{(m)}(F)\in\mathbb{R}$, satisfying $|\kappa_{(m)}|<C\bar{\varepsilon}^{p^m}$, such that*

$$|[x+r_{m,n}(x,\,y)]-[v_*(x)+\kappa_{(m)}y^2]|<C\rho^{n-m}. \tag{5.49}$$

Proof. Observe that, since $v_*(0)=0$, Proposition 5.7 tells us that there exist a constant $C_0>0$ and a point $\xi_{0,x}\in[0,\,x]$ such that

$$\begin{aligned}&|[x+r_{m,n}(x,\,y)]-[v_*(x)+r_{m,n}(0,\,y)]| \\ &\quad=|[x+r_{m,n}(x,\,y)-v_*(x)]-[0+r_{m,n}(0,\,y)-v_*(0)]| \\ &\quad\le|1+\partial_x r_{m,n}(\xi_{0,x},\,y)-\partial_x v_*(\xi_{0,x})||x| \\ &\quad\le C_0\rho^{n-m}|x|. \end{aligned} \tag{5.50}$$

We now claim that there exists a constant $\kappa_{(m)}$ such that $|\kappa_{(m)}|<C\bar{\varepsilon}^{p^m}$ and

$$|r_{m,n}(0,\,y)-\kappa_{(m)}y^2|<C_1\rho^n. \tag{5.51}$$

To show this we use induction. Recall that $\Psi_{m,n}(z)=\Psi_{m,n-1}\circ\Psi_n(z)$ for $z\in B$. This implies that

$$R_{m,n}(z)=R_n(z)+D_n^{-1}(R_{m,n-1}(D_n(\mathrm{id}+R_n(z)))). \tag{5.52}$$

Since $R_{m,n}$, R_n and D_n have the forms given by Lemmas 5.1 and 5.5, we find that, setting $z'=\Psi_n(z)$,

$$(x',\,y')=(\sigma_n s_n(x+r_n(x,\,y))+\sigma_n t_n y,\,\sigma_n y), \tag{5.53}$$

where we write $(x',\,y')$ for z'. This then gives

$$r_{m,n}(x,\,y)=r_n(x,\,y)+\sigma_n^{-1}s_n^{-1}r_{m,n-1}(x',\,y'). \tag{5.54}$$

Let $\omega_n(y)=\sigma_n(s_n r_n(0,\,y)+t_n y)$. Then this, together with the mean value theorem, implies that there exists a $\xi\in[0,\,\omega_n(y)]$ such that

$$\begin{aligned}r_{m,n}(0,\,y)&=r_n(0,\,y)+\sigma_n^{-1}s_n^{-1}r_{m,n-1}(\omega_n(y),\,\sigma_n y) \\ &=r_n(0,\,y)+\sigma_n^{-1}s_n^{-1}(r_{m,n-1}(0,\,\sigma_n y)+\partial_x r_{m,n-1}(\xi,\,\sigma_n y)\omega_n(y)). \end{aligned} \tag{5.55}$$

Next observe that, by construction, $r_n(x,\,y)$ consists of terms of degree two or higher. Therefore, by the above equation, so too must $r_{m,n}(x,\,y)$. Thus, we may write $r_n(0,\,y)$ and $r_{m,n}(0,\,y)$ in the forms

$$r_n(0,\,y)=\kappa_n y^2+K_n(y),\quad r_{m,n}(0,\,y)=\kappa_{m,n}y^2+K_{m,n}(y), \tag{5.56}$$

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where κ_n, $\kappa_{m,n}$ are real constants and $K_n(y)$, $K_{m,n}(y)$ are functions of the third order in y. This implies, together with equation (5.55), that

$$\kappa_{m,n}y^2 + K_{m,n}(y) = \kappa_n y^2 + K_n(y) + \sigma_n^{-1}s_n^{-1}(\kappa_{m,n-1}y^2 + K_{m,n-1}(y) + \partial_x r_{m,n-1}(\xi, \sigma_n y)\omega_n(y)). \tag{5.57}$$

By Proposition 5.4 there exists a constant $C_1 > 0$ such that $|\partial_y r_n(z)| < C_1\bar{\varepsilon}^{p^n}|z|$ for all suitable z. Therefore κ_n satisfies $|\kappa_n| < C_1\bar{\varepsilon}^{p^n}$ and K_n satisfies $|K_n(y)| < C_1\bar{\varepsilon}^{p^n}|y|^3$. Proposition 5.4 also implies that there exists a constant $C_2 > 0$ such that $|\omega(y)| < C_2\bar{\varepsilon}^{p^n}|y|$. Proposition 5.6 implies that there exists a constant $C_3 > 0$ such that $|\partial_x r_{m,n-1}(x, y)| < C_3$. These imply that there is a constant $C_4 > 0$ such that

$$\begin{aligned} |\kappa_{m,n}| &\le |\kappa_n| + |\sigma_n s_n^{-1}||\kappa_{m,n-1}| + C_4\bar{\varepsilon}^{p^n} \\ &\le 2C_4\bar{\varepsilon}^{p^n} + (1 + C_4\rho^n)|\kappa_{m,n-1}|, \qquad (5.58) \\ |K_{m,n}(y)| &\le |K_n(y)| + |\sigma_n^2 s_n^{-1}||K_{m,n-1}(y)| + C_4\bar{\varepsilon}^{p^n} \\ &\le \sigma(1 + C_4\rho^n)|K_{m,n-1}(y)| + 2C_4\bar{\varepsilon}^{p^n}, \qquad (5.59) \end{aligned}$$

which implies that $\kappa_{m,n}$ converges as n tends to infinity and $K_{m,n}(y)$ decreases exponentially if n is sufficiently large. Moreover, by Proposition A.3, $\kappa_{(m)} = \lim_{n\to\infty}\kappa_{m,n}$ satisfies $|\kappa_{(m)}| \le C_5\bar{\varepsilon}^{p^n}$ for some constant $C_5 > 0$. Hence the proposition is shown. □

6. *Three applications*

We extend three results in [**3**] to the case of arbitrary combinatorics using the results of the previous section. First we will show that universality holds at the tip. By this we mean that the rate of convergence to the renormalization fixed point is controlled by a universal quantity. In the unimodal case this is a positive real number, but here the quantity is a real-valued real analytic function. This universality is then used to show two other results, namely the non-existence of continuous invariant line fields on the renormalization Cantor set and the non-rigidity of these Cantor sets.

6.1. *Universality at the tip.*

THEOREM 6.1. *There exist a constant $\bar{\varepsilon}_0 > 0$, a universal constant $0 < \rho < 1$ and a universal function $a \in C^\omega(J, \mathbb{R})$ such that the following holds. Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ and let the sequence of renormalizations be denoted by F_n. Then*

$$F_n(x, y) = (f_n(x) + b^{p^n}a(x)y(1 + \mathrm{O}(\rho^n)), y) \tag{6.1}$$

where $b = b(F)$ denotes the average Jacobian of F and f_n are unimodal maps converging exponentially to f_*.

Proof. Let $F_n = (\phi_n, \pi_x)$ denote the nth renormalization of F. Let τ_n denote the tip of height n and let $\varsigma \in \mathrm{Dom}(F_n)$ be any other point. Applying the chain rule to $F_n = \Psi_{0,n-1}^{-1} \circ F^{\circ p^n} \circ \Psi_{0,n-1}$ at the point ς gives

$$\partial_y\phi_n(\varsigma) = \mathrm{Jac}_\varsigma F_n = \mathrm{Jac}_{\Psi_{0,n-1}(\varsigma)}F^{\circ p^n}\frac{\mathrm{Jac}_\varsigma \Psi_{0,n-1}}{\mathrm{Jac}_{F_n(\varsigma)}\Psi_{0,n-1}}. \tag{6.2}$$

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By the Distortion Lemma (Lemma 3.20), since $\Psi_{0,n-1}(\varsigma) \in B^{0^n}$, there exists a constant $C_0 > 0$ such that

$$|\mathrm{Jac}_{\Psi_{0,n-1}(\varsigma)} F^{\circ p^n}| \leq b^{p^n}(1 + C_0\rho^n). \tag{6.3}$$

It is clear from the decomposition in Lemma 5.5 that

$$\mathrm{Jac}_{\varsigma} \Psi_{0,n-1} = \mathrm{Jac}_{\tau_n} \Psi_{0,n-1} \mathrm{Jac}_{\varsigma-\tau_n}(\mathrm{id} + R_{0,n-1}) \tag{6.4}$$

and

$$\mathrm{Jac}_{F_n(\varsigma)} \Psi_{0,n-1} = \mathrm{Jac}_{\tau_n} \Psi_{0,n-1} \mathrm{Jac}_{F_n(\varsigma)-\tau_n}(\mathrm{id} + R_{0,n-1}). \tag{6.5}$$

Let $\delta_n^0 = \varsigma - \tau_n$ and $\delta_n^1 = F_n(\varsigma) - \tau_n$. Observe that, by Theorem 3.11 and Corollary 3.18, there exists a constant $C_1 > 0$ such that

$$|\tau_n - \tau_*|, |F_n - F_*|_\Omega < C_1\rho^n.$$

Therefore there exists a constant $C_2 > 0$ such that, if $\varsigma_* = \tau_* + (\varsigma - \tau_n)$, $\delta_*^0 = \varsigma_* - \tau_*$ and $\delta_*^1 = F_*(\varsigma_*) - \tau_*$,

$$|\delta_n^0 - \delta_*^0| = |[\varsigma - \tau_n] - [\varsigma_* - \tau_*]| = 0 \tag{6.6}$$

and

$$|\delta_n^1 - \delta_*^1| = |[F_n(\varsigma) - \tau_n] - [F_*(\varsigma_*) - \tau_*]| < C_2\rho^n. \tag{6.7}$$

By Proposition 5.7 there is a constant $C_3 > 0$ such that

$$|1 + \partial_x r_{0,n-1} - v_*'|_{C^0} < C_3\rho^n. \tag{6.8}$$

Combining these and observing that v_* has bounded derivatives and δ_n^0 and δ_n^1 both lie in a bounded domain gives us a constant $C_4 > 0$ satisfying

$$\begin{aligned}
&|\mathrm{Jac}_{\delta_n^0}(\mathrm{id} + R_{0,n-1}) - v_*'(\pi_x(\delta_*^0))| \\
&\quad \leq |\mathrm{Jac}_{\delta_n^0}(\mathrm{id} + R_{0,n-1}) - v_*'(\pi_x(\delta_n^0))| + |v_*'(\delta_n^0) - v_*'(\delta_*^0)| \\
&\quad \leq |1 + \partial_x r_{0,n-1}(\delta_n^0) - v_*'(\pi_x(\delta_n^0))| + |v_*''|_{C^0}|\tau_n - \tau_*| \\
&\quad \leq C_3\rho^n + C_2|v_*|_{C^2}\rho^n \\
&\quad \leq C_4\rho^n
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
&|\mathrm{Jac}_{\delta_n^1}(\mathrm{id} + R_{0,n-1}) - v_*'(\pi_x(\delta_*^1))| \\
&\quad \leq |\mathrm{Jac}_{\delta_n^1}(\mathrm{id} + R_{0,n-1}) - v_*'(\pi_x(\delta_n^1))| + |v_*'(\pi_x(\delta_n^1)) - v_*'(\pi_x(\delta_*^1))| \\
&\quad \leq |1 + \partial_x r_{0,n-1}(\delta_n^1) - v_*'(\pi_x(\delta_n^1))| + |v_*''|_{C^0}|\delta_n^1 - \delta_*^1| \\
&\quad \leq C_4\rho^n.
\end{aligned} \tag{6.10}$$

Observe that there exists a constant $C_5 > 0$ such that $|v_*'(x)| \geq C_5 > 0$, as v_* is a rescaling of a diffeomorphism onto its image. Observe also that there exists an $N > 0$ such that

$$|1 + \partial_x r_{0,n}|_{C^0} \geq \tfrac{1}{2} \inf |v_*'(x)| \geq C_5 \quad \text{for all } n > N.$$

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Therefore there exists a constant $C_6 > 1$ such that for all $n > N$,

$$\max\left(1, \left|\frac{v'_*(\pi_x(\delta_*^0))}{v'_*(\pi_x(\delta_*^1))}\right|\right) < C_6, \quad C_6^{-1} < |\mathrm{Jac}_{\delta_n^1}\Psi_{0,n}|. \tag{6.11}$$

Therefore, applying Lemma A.5, we find

$$\begin{aligned}\left|\frac{\mathrm{Jac}_{\delta_n^0}\Psi_{0,n-1}}{\mathrm{Jac}_{\delta_n^1}\Psi_{0,n-1}} - \frac{v'_*(\pi_x(\delta_*^0))}{v'_*(\pi_x(\delta_*^1))}\right| &= \left|\frac{\mathrm{Jac}_{\delta_n^0}(\mathrm{id}+R_{0,n-1})}{\mathrm{Jac}_{\delta_n^1}(\mathrm{id}+R_{0,n-1})} - \frac{v'_*(\pi_x(\delta_*^0))}{v'_*(\pi_x(\delta_*^1))}\right| \\ &\le C_6^2 \max_{i=0,1}(|1+\partial_x r_{0,n-1}(\delta_n^i) - v'_*(\pi_x(\delta_*^i))|) \\ &\le C_4 C_6^2 \rho^n. \end{aligned} \tag{6.12}$$

Together with equations (6.2) and (6.3), this implies that

$$\partial_y \phi_n(\varsigma) = b^{p^n} a(\xi)(1+\mathrm{O}(\rho^n)) \tag{6.13}$$

where $\varsigma = (\xi, \eta)$ and

$$a(\xi) = \frac{v'_*(\xi - \pi_x(\tau_*))}{v'_*(f_*(\xi) - \pi_x(\tau_*))}. \tag{6.14}$$

This implies that, if $z = (x, y) \in B$, upon integrating with respect to the y-variable we find that

$$\phi_n(x, y) = g_n(x) + y b^{p^n} a(x)(1+\mathrm{O}(\rho^n)), \tag{6.15}$$

for some function g_n independent of y. But now let (f_n, ε_n) be any parametrization of F_n such that $|\varepsilon_n| \le C_7 \bar{\varepsilon}^{p^n}$ and $|f_n - f_*| < C_8 \rho^n$. Here $C_7 > 0$ is the constant from Theorem 3.10 and $C_8 > 0$ is the constant from Theorem 3.11. Then there is a constant $C_9 > 0$ such that $|g_n - f_n| = |\varepsilon_n - b^{p^n}\pi_y a| \le C_9 \rho^n$. Therefore, for $n > 0$ sufficiently large g_n will also be unimodal and

$$|g_n - f_*| \le |g_n - f_n| + |f_n - f_*| \le (C_9 + C_8)\rho^n.$$

Hence we may absorb their difference into the $\mathrm{O}(\rho^n)$ term. □

The following is an immediate consequence of the proof of the above theorem.

PROPOSITION 6.2. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\Psi_{m,n}$ denote the scope function from height $n+1$ to height m. Let $t_{m,n}$ denote the tilt of $\Psi_{m,n}$ and let τ_{m+1} denote the tip at height $m+1$. Let $a = a(\tau_*)$ where $a(x)$ is the universal function from Theorem 6.1 above. Then there exist constants $C > 0$ and $0 < \rho < 1$ such that for all $0 < m < n$ sufficiently large,*

$$ab^{p^m}(1 - C\rho^m) < |t_m(\tau_{m+1})| < ab^{p^m}(1 + C\rho^m) \tag{6.16}$$

$$ab^{p^m}(1 - C\rho^m) < |t_{m,n}(\tau_{n+1})| < ab^{p^m}(1 + C\rho^m). \tag{6.17}$$

Moreover, $t_{m,*} = \lim_{n\to\infty} t_{m,n}(\tau_{n+1})$ exists and the convergence is exponential.

Proof. Let $\tau_m = (\xi_m, \eta_m)$. Recall that

$$t_m = \pm \frac{\partial_y \phi_m^{p-1}(\tau_m)}{\partial_x \phi_m^{p-1}(\tau_m)}, \tag{6.18}$$

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but by the variational formula 3.5 we know that

$$\phi_m^{p-1}(\xi_m, \eta_m) = f_m^{\circ p-1}(\xi_m) + L_m^{p-1}(\xi_m) + \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})'(\xi_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}) \quad (6.19)$$

which implies that

$$\begin{aligned}\partial_x \phi_m^{p-1}(\xi_m, \eta_m) &= (f_m^{\circ p-1})'(\xi_m) + (L_m^{p-1})'(\xi_m) + \partial_x \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})'(\xi_m) \\ &\quad + \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})''(\xi_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}), \quad (6.20)\\ \partial_y \phi_m^{p-1}(\xi_m, \eta_m) &= \partial_y \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})'(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}). \quad (6.21)\end{aligned}$$

Therefore, by the fact that $(f_m^{\circ p-1})'(\xi_m)$ is uniformly bounded from zero, if n is sufficiently large we find

$$\begin{aligned}\frac{\partial_y \phi_m^{p-1}(\xi_m, \eta_m)}{\partial_x \phi_m^{p-1}(\xi_m, \eta_m)} &= \frac{\partial_y \varepsilon_m(\xi_m, \eta_m)(f_m^{\circ p-1})'(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m})}{(f_m^{\circ p-1})'(\xi_m) + \mathrm{O}(\bar{\varepsilon}^{p^n})} \\ &= (\partial_y \varepsilon_m(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}))(1 + \mathrm{O}(\bar{\varepsilon}^{p^m})) \\ &= \partial_y \varepsilon_m(\xi_m, \eta_m) + \mathrm{O}(\bar{\varepsilon}^{2p^m}). \quad (6.22)\end{aligned}$$

Theorem 6.1 above and observing that the $\mathrm{O}(\bar{\varepsilon}^{2p^m})$ term can be absorbed into the $\mathrm{O}(\rho^m)$ term then tell us that

$$|t_m(\tau_{m+1})| = a(\xi_m) b^{p^m}(1 + \mathrm{O}(\rho^m)), \quad (6.23)$$

but by Proposition 5.3 we know that ξ_m converges to ξ_* exponentially and so analyticity of a implies that $a(\xi_m) = a(\xi_*)(1 + \mathrm{O}(\rho^m))$. Hence we get the first claim. Secondly, observe that by Lemma 5.5,

$$\begin{aligned}t_{m,n-1}(\tau_n) &= \sum_{i=m}^{n-1} s_{m,i-1}(\tau_i) t_i(\tau_{i+1}) \\ &= t_m(\tau_{m+1}) \sum_{i=m}^{n-1} s_{m,i-1}(\tau_i) \left(\frac{\partial_x \phi_m^{p-1}(\tau_m)}{\partial_x \phi_i^{p-1}(\tau_i)} \right) \left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)} \right) \\ &= t_m(\tau_{m+1}) \sum_{i=m}^{n-1} s_{m+1,i}(\tau_{i+1}) \left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)} \right). \quad (6.24)\end{aligned}$$

Therefore we can write $t_{m,n-1}(\tau_n) = t_m(\tau_{m+1})(1 + K_{m,n-1}(\tau_{n+1}))$ where

$$K_{m,n-1} \sum_{i=m+1}^{n-1} s_{m+1,i}(\tau_{i+1}) \left(\frac{\partial_y \phi_i^{p-1}(\tau_i)}{\partial_y \phi_m^{p-1}(\tau_m)} \right). \quad (6.25)$$

By Proposition 5.6 and the variational formula 3.5, there exists a constant $C_7 > 0$ such that $|K_{m,n-1}(\tau_{n+1})| \geq C_7 \bar{\varepsilon}^{p^{m+1} - p^m}$. Absorbing this error into the $\mathrm{O}(\rho^m)$ term gives us the second claim. The third claim follows as the terms in $K_{m,n}$ decrease super-exponentially as n tends to infinity, but τ_m only converges exponentially to τ_*. □

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PROPOSITION 6.3. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ be as above, let τ_n denote the tip of F_n and let $\varsigma_n = F_n^{\circ p}(\tau_n)$. Then there exists a constant $C > 1$ such that, for all $0 < m < n$,*

$$C^{-1}|s_{m,n-1}(\tau_n)| \le |s_{m,n-1}(\varsigma_n)| \le C|s_{m,n-1}(\tau_n)|, \tag{6.26}$$

$$C^{-1}|t_{m,n-1}(\tau_n)| \le |t_{m,n-1}(\varsigma_n)| \le C|t_{m,n-1}(\tau_n)|, \tag{6.27}$$

$$|s_{m,n-1}(\varsigma_n) - s_{m,n-1}(\tau_n)| > C^{-1}|\varsigma_n - \tau_n|, \tag{6.28}$$

$$|t_{m,n-1}(\varsigma_n) - t_{m,n-1}(\tau_n)| > C^{-1}|\varsigma_n - \tau_n|. \tag{6.29}$$

Proof. These follow from the estimates on the second-order terms (that is, the functions $r_{m,n}$) given by Proposition 5.6 and the observation that $\tau_n,\ \varsigma_n \in B_n^0$ implies, for n sufficiently large, that the derivatives of $s_{m,n-1}$, $t_{m,n-1}$ in the rectangle spanned by τ_n, ς_n will be uniformly bounded. □

6.2. *Invariant line fields.* Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalization Cantor set. We will now consider the space of F-invariant line fields on $\mathcal{O}$. As we are considering line fields, let us projectivize all the transformations under consideration. Let us take the projection onto the line $\{y = 1\}$, and let us denote the projected coordinate by X. Then the maps $D(\Psi_{m,n}; z)$ and $D(F_n^{\circ p}; z)$ induce the transformations

$$\widetilde{\mathrm{D}}_z \Psi_{m,n}(X) = s_{m,n}(z)X + t_{m,n}(z), \tag{6.30}$$

$$\widetilde{\mathrm{D}}_z F_n^{\circ p}(X) = \zeta_n(z)\frac{X + \eta_n(z)}{X + \theta_n(z)}, \tag{6.31}$$

where $s_{m,n}(z)$, $t_{m,n}(z)$ are as in §3.4 and $\zeta_n(z)$, $\eta_n(z)$, $\theta_n(z)$ are given by

$$\zeta_n(z) = \frac{\partial_x \phi_n^p(z)}{\partial_x \phi_n^{p-1}(z)}, \quad \eta_n(z) = \frac{\partial_y \phi_n^p(z)}{\partial_x \phi_n^p(z)}, \quad \theta_n(z) = \frac{\partial_y \phi_n^{p-1}(z)}{\partial_x \phi_n^{p-1}(z)}. \tag{6.32}$$

PROPOSITION 6.4. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ be as above. Then there exists a constant $C > 1$ such that for all $n > 0$*

$$C^{-1} < |\zeta_n(\tau_n)| < C, \tag{6.33}$$

$$|\eta_n(\tau_n)| < C\bar{\varepsilon}^{p^{n+1}}, \tag{6.34}$$

$$|\theta_n(\tau_n)| < C\bar{\varepsilon}^{p^n}. \tag{6.35}$$

Proof. Let (f_n, ε_n) be a parametrization for F_n. Let v_n denote the critical value of f_n. Observe, by convergence of renormalization (Theorem 3.11), that v_n and $\pi_x \tau_n$ are exponentially close and so there is a constant $C_0 > 0$ such that

$$|(f_n^{\circ p-1})'(v_n)|,\ |(f_n^{\circ p-1})'(\pi_x \tau_n)| > C_0, \tag{6.36}$$

if $n > 0$ is sufficiently large. Therefore by the variational formula, there is a constant $C_1 > 0$ such that

$$\left| \frac{\partial_x \phi_n^p(\tau_n)}{\partial_x \phi_n^{p-1}(\tau_n)} - \frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)} \right| \le C_1 \bar{\varepsilon}^{p^n}. \tag{6.37}$$

Now observe, by Theorem 3.11, that there is a $C_2 > 0$ such that

$$\left| \frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)} - \frac{(f_*^{\circ p})'(v_*)}{(f_*^{\circ p-1})'(v_*)} \right| < C_2 \rho^n. \tag{6.38}$$

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Therefore there exists a $C_3 > 0$ such that

$$\begin{aligned}|\zeta_n(\tau_n) - f_*'(f_*^{\circ p}(v_*))| &\le \left|\frac{\partial_x \phi_n^p(z)}{\partial_x \phi_n^{p-1}(z)} - \frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)}\right| \\ &\quad + \left|\frac{(f_n^{\circ p})'(v_n)}{(f_n^{\circ p-1})'(v_n)} - \frac{(f_*^{\circ p})'(v_*)}{(f_*^{\circ p-1})'(v_*)}\right| \\ &\le C_3\rho^n. \end{aligned} \tag{6.39}$$

Since $f_*'(f_*^{\circ p}(v_*)) \neq 0$ (infinitely renormalizable maps are never postcritically finite), this implies, for $n > 0$ sufficiently large, that the first item is true.

For the second item, taking the Jacobian of $F_n^{\circ p}$ at τ_n, applying Proposition 5.4 and making the same observation regarding $f_*'(f_*^{\circ p}(v_*)) \neq 0$ as above, gives us the result.

The third item follows directly from Proposition 5.4. □

THEOREM 6.5. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalization Cantor set. Then there do not exist any continuous invariant line fields on $\mathcal{O}$. More precisely, if X is an invariant line field then it must be discontinuous at the tip, τ, of F.*

Proof. Let X be a continuous invariant line field on $\mathcal{O}$. Let τ_n denote the tip of F_n and let $\varsigma_n = F_n^{\circ p}(\tau_n)$ denote its first return, under F_n, to B_n^0.

Before we begin let us define some constants that will help our exposition. Let $C_0 > 0$ satisfy $|\theta_n(\tau_n)| < C_0\bar{\varepsilon}^{p^n}$ and $|\eta_n(\tau_n)| < C_0\bar{\varepsilon}^{p^{n+1}}$ for all $n > 0$. Such a constant exists by Proposition 6.4. Let $C_1 > 1$ satisfy $C_1^{-1} < |\zeta_n(\tau_n)| < C_1$ for all $n > 0$. Such a constant exists by Proposition 6.4. Let $C_2 > 0$ satisfy $|t_m(\tau_{m+1})|, |t_{m,n-1}(\tau_n)| < C_2\bar{\varepsilon}^{p^m}$ for all $0 < m < n$. Such a constant exists by Propositions 5.4 and 5.6. Let $C_3 > 0$ satisfy $|s_{m,n-1}(\tau_n)| > C_3\sigma^{n-m-1}$ for all $0 < m < n$. Finally, let $C_4 > 1$ satisfy

$$C_4^{-1}|s_{m,n-1}(\tau_n)| \le |s_{m,n-1}(\varsigma_n)| \le C_4|s_{m,n-1}(\tau_n)|, \tag{6.40}$$

$$C_4^{-1}|t_{m,n-1}(\tau_n)| \le |t_{m,n-1}(\varsigma_n)| \le C_4|t_{m,n-1}(\tau_n)|, \tag{6.41}$$

$$|s_{m,n-1}(\varsigma_m) - s_{m,n-1}(\tau_m)| > C_4^{-1}|\varsigma_m - \tau_m|, \tag{6.42}$$

$$|t_{m,n-1}(\varsigma_m) - t_{m,n-1}(\tau_m)| > C_4^{-1}|\varsigma_m - \tau_m|, \tag{6.43}$$

for all $0 < m < n$. Such a constant exists by Proposition 6.3 above.

Observe that X induces continuous invariant line fields X_n for F_n on $\mathcal{O}_n$, the induced Cantor sets. Thus

$$X_m(\tau_m) = \widetilde{\mathrm{D}}_\tau \Psi_{0,m}^{-1} X(\tau) = (X(\tau) - t_{0,m}(\tau_m))/s_{0,m}(\tau_m). \tag{6.44}$$

There are two possibilities: either

$$X(\tau) = t_{0,*}(\tau_*) = \lim t_{0,m-1}(\tau_m),$$

and so $X_m(\tau_m)$ converges to zero (since $t_{0,m}$ converges super-exponentially to $t_{0,*}$ but $s_{0,m}$ converges only exponentially to 0); or $X(\tau) \neq t_{0,*}(\tau_*)$, and so $X_m(\tau_m)$ tends to infinity.

First, let us show that the second case cannot occur. Let $K, \kappa > 0$ be constants. Choose $M > 0$ such that $|X_m(\tau_m)| > K$ for all $m > M$. Fix such an $m > M$. By the continuity of X_m there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|X_m(x) - X_m(y)| < \kappa$ for

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any $x, y \in \mathcal{O}_m$. Choose $N > m$ such that, for all $n > N$, $|\Psi_{m,n-1}(\tau_n) - \Psi_{m,n-1}(\varsigma_n)| < \delta$. This then implies that

$$|X_m(\Psi_{m,n-1}(\tau_n)) - X_m(\Psi_{m,n-1}(\varsigma_n))| < \kappa.$$

By invariance of the X_n,

$$|X_n(\varsigma_n)| = |\widetilde{\mathrm{D}}_{\tau_n} F_n^{\circ p}(X_n(\tau_n))| = |\zeta_n(\tau_n)| \left| \frac{X_n(\tau_n) + \eta_n(\tau_n)}{X_n(\tau_n) + \theta_n(\tau_n)} \right|. \tag{6.45}$$

By our hypotheses above we know that $|\theta_n(\tau_n)|, |\eta_n(\tau_n)| < C_0 \bar{\varepsilon}^{p^n}$. Since $n > m$, we also know that $|X_n(\tau_n)| > K$. Therefore

$$\begin{aligned} \left| \frac{X_n(\tau_n) + \eta_n(\tau_n)}{X_n(\tau_n) + \theta_n(\tau_n)} \right| &\leq \frac{1 + |\eta_n(\tau_n)/X_n(\tau_n)|}{1 - |\theta_n(\tau_n)/X_n(\tau_n)|} \\ &\leq \frac{1 + C_0 \bar{\varepsilon}^{p^n}/K}{1 - C_0 \bar{\varepsilon}^{p^n}/K}. \end{aligned} \tag{6.46}$$

Therefore, combining this with the above equation (6.45) and the definition of $C_1 > 1$ above we find that

$$|X_n(\varsigma_n)| \leq C_1 \left(\frac{1 + C_0 \bar{\varepsilon}^{p^n}/K}{1 - C_0 \bar{\varepsilon}^{p^n}/K} \right). \tag{6.47}$$

Now we apply $\widetilde{\mathrm{D}}_{\varsigma_n} \Psi_{m,n-1}$. Then by the definition of the constant $C_4 > 0$ above and Proposition 5.6,

$$\begin{aligned} |X_m(\Psi_{m,n-1}(\varsigma_n))| &= |s_{m,n-1}(\varsigma_n) X_n(\varsigma_n) + t_{m,n-1}(\varsigma_n)| \\ &\leq |s_{m,n-1}(\varsigma_n)||X_n(\varsigma_n)| + |t_{m,n-1}(\varsigma_n)| \\ &\leq C_4(|s_{m,n-1}(\tau_n)||X_n(\varsigma_n)| + |t_{m,n-1}(\tau_n)|) \\ &\leq C_4 \sigma^{n-m-1}(1 + |X_n(\varsigma_n)|) \end{aligned} \tag{6.48}$$

and hence

$$\begin{aligned} &|X_m(\Psi_{m,n-1}(\tau_n)) - X_m(\Psi_{m,n-1}(\varsigma_n))| \\ &\quad \geq ||X_m(\Psi_{m,n-1}(\tau_n))| - |X_m(\Psi_{m,n-1}(\varsigma_n))|| \\ &\quad \geq K - C_4 \sigma^{n-m-1} \left[1 + C_1 \left(\frac{1 + C_0 \bar{\varepsilon}^{p^n}/K}{1 - C_0 \bar{\varepsilon}^{p^n}/K} \right) \right]. \end{aligned} \tag{6.49}$$

But, by our continuity assumption, this must be less than κ. For $K > 0$ sufficiently large this cannot happen.

So now let us assume that $X(\tau) = t_{0,*}$. Then the induced line fields must satisfy $X_m(\tau_m) = t_{m,*}$, for all $m > 0$. The idea is, as before, to look at the first returns under F_m of B_m^0. We will apply $\widetilde{\mathrm{D}}_{\tau_m} F_m^{\circ p}$ to the line $X_m(\tau_m) = t_{m,n}$ and take the limit as n tends to infinity.

Proposition 6.2 implies, as

$$t_m(\tau_{m+1}) = \pm \partial_y \phi_m^{p-1}(\tau_m) / \partial_x \phi_m^{p-1}(\tau_m) = \pm \eta_m,$$

that there exists a constant $C_5 > 0$ for which

$$|t_{m,n-1}(\tau_n) + \theta_m(\tau_m)| \leq |t_m(\tau_{m+1})||K_{m,n-1}(\tau_{n+1})| \leq C_5 \bar{\varepsilon}^{p^{m+1}}. \tag{6.50}$$

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On the other hand, we know that $|\eta_m(\tau_m)| < C_0\bar{\varepsilon}^{p^{m+1}}$ and $|t_{m,n-1}(\tau_n)| < C_2\bar{\varepsilon}^{p^m}$ and hence

$$|t_{m,n-1}(\tau_n) + \eta_m(\tau_m)| \geq ||t_{m,n-1}(\tau_n)| - |\eta_m(\tau_m)|| \geq C_2\bar{\varepsilon}^{p^m} - C_0\bar{\varepsilon}^{p^{m+1}}. \tag{6.51}$$

We also know that $|\zeta_m(\tau_m)| > C_1^{-1}$. Therefore there exists a constant $C_6 > 0$ such that

$$\begin{aligned} |\widetilde{\mathrm{D}}_{\tau_m} F_m^{\circ p}(t_{m,n-1}(\tau_n))| &= |\zeta_m(\tau_m)| \left| \frac{t_{m,n-1}(\tau_n) + \eta_m(\tau_m)}{t_{m,n-1}(\tau_n) + \theta_m(\tau_m)} \right| \\ &\geq C_1^{-1} C_5^{-1} \bar{\varepsilon}^{-p^{m+1}} (C_2\bar{\varepsilon}^{p^m} - C_0\bar{\varepsilon}^{p^{m+1}}) \\ &\geq C_6\bar{\varepsilon}^{-p^{m+1}}. \end{aligned} \tag{6.52}$$

Now recall that $|t_{m,n-1}(\tau_n)| < C_2\bar{\varepsilon}^{p^m}$. Also observe that both of these estimates are independent of n. Therefore they still hold when passing to the limit, as n tends to infinity, giving

$$|X_m(\varsigma_m)| > C_6\bar{\varepsilon}^{-p^{m+1}}, \quad |X_m(\tau_m)| < C_2\bar{\varepsilon}^{p^m}. \tag{6.53}$$

Finally, applying $\Psi_{0,m-1}$ and setting $\varsigma = \Psi_{0,m-1}(\varsigma_m)$, we find that

$$\begin{aligned} |X(\varsigma) - X(\tau)| &= |[s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) + t_{0,m-1}(\varsigma_m)] \\ &\quad - [s_{0,m-1}(\tau_m)X_m(\tau_m) + t_{0,m-1}(\tau_m)]| \\ &\geq ||s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) - s_{0,m-1}(\tau_m)X_m(\tau_m)| \\ &\quad - |t_{0,m-1}(\varsigma_m) - t_{0,m-1}(\tau_m)||, \end{aligned} \tag{6.54}$$

but by the constraints (6.40)–(6.43) on the constant C_4

$$\begin{aligned} &|s_{0,m-1}(\varsigma_m)X_m(\varsigma_m) - s_{0,m-1}(\tau_m)X_m(\tau_m)| \\ &\quad \geq ||s_{0,m-1}(\varsigma_m)||X_m(\varsigma_m) - X_m(\tau_m)| - |s_{0,m-1}(\varsigma_m) - s_{0,m-1}(\tau_m)||X_m(\tau_m)|| \\ &\quad \geq C_4^{-1}|s_{0,m-1}(\tau_m)||X_m(\varsigma_m) - X_m(\tau_m)| - C_4|\varsigma_m - \tau_m||X_m(\tau_m)| \end{aligned} \tag{6.55}$$

and

$$|t_{0,m-1}(\varsigma_m) - t_{0,m-1}(\tau_m)| \leq C_4|\varsigma_m - \tau_m|. \tag{6.56}$$

Also, by definition of C_3 given above, $|s_{0,m-1}(\tau_m)| > C_3\sigma^m$. Hence, by our bounds on $|X_m(\varsigma_m)|$ and $|X_m(\tau_m)|$ and the above we find that

$$\begin{aligned} |X(\varsigma) - X(\tau)| &\geq C_4^{-1} C_3\sigma^m |X_m(\varsigma_m) - X_m(\tau_m)| \\ &\quad - C_4|\varsigma_m - \tau_m||X_m(\tau_m)| - C_4|\varsigma_m - \tau_m| \\ &\geq C_4^{-1} C_3\sigma^m (C_2\bar{\varepsilon}^{-p^{m+1}} - C_6\bar{\varepsilon}^{p^m}) \\ &\quad - C_4|\varsigma_m - \tau_m|(1 + C_6\bar{\varepsilon}^{p^m}). \end{aligned} \tag{6.57}$$

However, since $|\varsigma_m - \tau_m|$ is bounded from above there is a constant $C_7 > 0$ such that

$$|X(\varsigma) - X(\tau)| \geq C_7\sigma^m\bar{\varepsilon}^{-p^{m+1}}. \tag{6.58}$$

Therefore, as we increase $m > 0$ the points τ and ς get exponentially closer but the distance between $X(\tau)$ and $X(\varsigma)$ diverges super-exponentially. In particular, X cannot be continuous at τ as required. □

We now need to define the following type of convergence, which is stronger than Hausdorff convergence.

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Definition 6.6. Let $\mathcal{O}_* \subset M$ be a Cantor set, embedded in the metric space M, with presentation $\underline{B}_* = \{B_*^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$. Let $\mathcal{O}_*^{\mathbf{w}}$ denote the cylinder set for $\mathcal{O}_*$ associated to the word $\mathbf{w} \in \overline{W}$. Let $\mathcal{O}_n \subset M$ denote a sequence of Cantor sets, also embedded in M, with presentations $\underline{B}_n = \{B_n^{\mathbf{w}}\}_{\mathbf{w}\in W^*}$ combinatorially equivalent to $\underline{B}_*$. Then we say that $\mathcal{O}_n$ *strongly converges* to $\mathcal{O}_*$ if, for each $\mathbf{w} \in \overline{W}$, $\mathcal{O}_n^{\mathbf{w}} \to \mathcal{O}_*^{\mathbf{w}}$.

Definition 6.7. Let X_n be a line field on $\mathcal{O}_n$. Then we say that X_n *strongly converges* to a line field X_* on $\mathcal{O}_*$ if, for each $\mathbf{w} \in \overline{W}$, $X_n(\mathcal{O}_n^{\mathbf{w}})$ converges to $X_*(\mathcal{O}_*^{\mathbf{w}})$ in the projected coordinates.

PROPOSITION 6.8. *Let $F \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ and let $\mathcal{O}$ denote its renormalization Cantor set. Given any invariant line field X on $\mathcal{O}$ the induced line fields X_n on $\mathcal{O}_n$ do not strongly converge to the tangent line field X_* on $\mathcal{O}_*$.*

Proof. Let us denote the correspondence between elements of $\mathcal{O}_n$ and $\mathcal{O}_*$ by π_n. Then a sequence of line fields X_n strongly converges to X_* if $X_n \circ \pi_n$ converges to X_*, where we have identified the line fields with their projectivized coordinates.

Assume that convergence holds, and let $\epsilon > 0$ and choose $N > 0$ such that $|X_n \circ \pi_n - X_*|_{\mathcal{O}_*} < \epsilon$ for all $n > N$. Take any $m > N$ and let $n > m$ be chosen so that $\sigma^{n-m+1} \leq b^{p^m} \leq \sigma^{n-m}$. Then

$$|X_m(\tau_m) - X_*(\tau_*)|,\ |X_m(F_m(\tau_m)) - X_*(F_*(\tau_*))| < \epsilon, \tag{6.59}$$

and the same holds if we replace m by n. Let us denote the points $F_i(\tau_i)$ by ς_i.

Observe that $X_*(\varsigma_*) = \partial_x \phi_*(\tau_*)$. Therefore, as convergence of renormalization implies that $|\partial_x \phi_m(\tau_m) - \partial_x \phi_*(\tau_*)| < C\rho^m$, this tells us that

$$|X_m(\varsigma_m) - \partial_x \phi_m(\tau_m)| < \epsilon + C\rho^m. \tag{6.60}$$

We will now show that they must differ by a definite constant and achieve the required contradiction. We will show this by evaluating X_m at a point near to ς_m. Consider the points $\varsigma = \Psi_{m,n}(\varsigma_n)$ and $\varsigma' = F_m \Psi_{m,n}(\varsigma_n)$. First let us evaluate X_m at ς'. By invariance this must be

$$\widetilde{\mathrm{D}}_{\varsigma_n} F_m \Psi_{m,n}(X_n(\varsigma_n)) = \partial_x \phi_m(\varsigma) + \frac{\partial_y \phi_m(\varsigma)}{s_{m,n}(\varsigma_n) + t_{m,n}(\varsigma_n) X_n(\varsigma_n)}. \tag{6.61}$$

The second term must be bounded away from zero as $X_n(\varsigma_n)$ is bounded from above if n is sufficiently large and the hypothesis on m, n tells us that $s_{m,n}$ and $t_{m,n}$ are both comparable to b^{p^m}, as is the numerator $\partial_y \phi_m(\varsigma)$. It is clear this bound can be made uniform in m.

Second, observe that $|\varsigma' - \varsigma_m|$ can be made arbitrarily small by choosing m and $n - m$ sufficiently large, by the assumption that $\mathcal{O}_n$ converges strongly to $\mathcal{O}_*$. Combining these gives us the required contradiction, as our hypothesis implies that increasing m leads to an exponential increase in n. □

6.3. *Failure of rigidity at the tip.* Using the same method as for the period doubling case, we show that, given two Cantor attractors $\mathcal{O}$ and $\tilde{\mathcal{O}}$ for some $F, \tilde{F} \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon}_0)$ with average Jacobian b, $\tilde{b}$ respectively, there is a bound on the Hölder exponent of any conjugacy that preserves 'tips'.

THEOREM 6.9. *Let $F, \tilde{F} \in \mathcal{I}_{\Omega,\upsilon}(\bar{\varepsilon})$ be two infinitely renormalizable Hénon-like maps with respective renormalization Cantor sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$, and tips τ and $\tilde{\tau}$. If there is a*

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conjugacy $\pi : \tilde{\mathcal{O}} \to \mathcal{O}$ mapping $\tilde{\tau}$ to τ then the Hölder exponent α of π satisfies

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log \tilde{b}}{\log b}\right). \tag{6.62}$$

Proof. We will denote all objects associated with F without tildes and all objects associated with $\tilde{F}$ with them. For example, Ψ and $\tilde{\Psi}$ will denote the scope function for F and $\tilde{F}$ respectively.

Let $K > 0$ be a positive constant which we will think of as being large. Let us choose an integer $m > 0$ which ensures that $\tilde{b}^{p^m} > K b^{p^m}$ and take an integer $n > m$ which satisfies $\sigma^{n-m+1} \leq b^{p^m} < \sigma^{n-m}$. This will be the depth of the Cantor sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$ that we will consider. So let us consider F and $\mathcal{O}$. Let us denote the tip of F_{n+1} by τ and let ς be its image under F_{n+1}. Let $\dot{\tau}$ and $\dot{\varsigma}$ be the respective images of these points under $\Psi_{m,n}$. Let $\ddot{\tau}$ and $\ddot{\varsigma}$ be the respective images of $\dot{\tau}$ and $\dot{\varsigma}$ under F_m. Let $\dddot{\tau}$ and $\dddot{\varsigma}$ be the respective images of $\ddot{\tau}$, $\ddot{\varsigma}$ under $\Psi_{0,m-1}$. The equivalent points for $\tilde{F}$ will be denoted with tildes. Finally, τ_* denotes the tip of F_* and ς_* denotes its image under F_*.

Observe that τ_* and ς_* will not lie on the same vertical or horizontal line. Therefore we know that the constant

$$C_0 = \tfrac{1}{2}\min(|\pi_x(\varsigma_*) - \pi_x(\tau_*)|, |\pi_y(\varsigma_*) - \pi_y(\tau_*)|) \tag{6.63}$$

is positive. By Theorem 3.11 there exists an integer $N > 0$ such that

$$|\pi_x(\varsigma) - \pi_x(\tau)|, |\pi_y(\varsigma) - \pi_y(\tau)|, |\pi_x(\tilde{\varsigma}) - \pi_x(\tilde{\tau})|, |\pi_y(\tilde{\varsigma}) - \pi_y(\tilde{\tau})| > C_0 > 0, \tag{6.64}$$

for all integers $m > N$. Let $\delta = (\delta_x, \delta_y) = \varsigma - \tau$ and $\tilde{\delta} = (\tilde{\delta}_x, \tilde{\delta}_y) = \tilde{\varsigma} - \tilde{\tau}$. Clearly we also have an upper bound for each of these quantities, namely $C_1 = \operatorname{diam}(B)$.

First we will derive an upper bound for the distance between $\dddot{\varsigma}$ and $\dddot{\tau}$, then we will derive a lower bound for the distance between $\dddot{\tilde{\varsigma}}$ and $\dddot{\tilde{\tau}}$.

Applying $\Psi_{m,n}$ to ς and τ gives $\dot{\varsigma} - \dot{\tau} = D_{m,n}(\mathrm{id} + R_{m,n})(\varsigma - \tau)$. Let $\dot{\delta} = (\dot{\delta}_x, \dot{\delta}_y) = \dot{\varsigma} - \dot{\tau}$. Hence by Proposition 5.6 and the above paragraph there exists a constant $C_2 > 0$ such that

$$|\dot{\delta}_x| = |\sigma_{m,n} s_{m,n}[\delta_x + r_{m,n}(\delta_x, \delta_y)] + \sigma_{m,n} t_{m,n} \delta_y| \leq C_2 \sigma^{n-m}(\sigma^{n-m} + b^{p^m}), \tag{6.65}$$

$$|\dot{\delta}_y| = |\sigma_{m,n} \delta_y| \leq C_2 \sigma^{n-m}. \tag{6.66}$$

Next we apply $F_m = (\phi_m, \pi_x)$ which gives $\ddot{\varsigma} - \ddot{\tau} = F_m(\dot{\varsigma}) - F_m(\dot{\tau})$. Let $\ddot{\delta} = (\ddot{\delta}_x, \ddot{\delta}_y) = \ddot{\varsigma} - \ddot{\tau}$. First observe that by convergence of renormalization (Theorem 3.11), there is a constant $C_2 > 0$ such that $|\partial_x \phi_m| < C_2$. Second observe that, by Theorem 3.10, there exists a constant $C_3 > 0$ such that $|\partial_y \phi_m| < C_3 b^{p^m}$. Then by the mean value theorem, if $\xi = (\pi_x(\dot{\tau}), \pi_y(\dot{\varsigma}))$, there exist points† $\xi_y \in [\dot{\varsigma}, \xi]$, $\xi_x \in [\xi, \dot{\tau}]$ such that

$$\begin{aligned} |\ddot{\delta}_x| &= |\partial_x \phi_m(\xi_y)\dot{\delta}_x + \partial_y \phi_m(\xi_x)\dot{\delta}_y| \\ &\leq C_3|\dot{\delta}_x| + C_4 b^{p^m}|\dot{\delta}_y| \\ &\leq C_2 \sigma^{n-m}(C_2(\sigma^{n-m} + b^{p^m}) + C_3 b^{p^m}) \\ &\leq C_5 \sigma^{n-m}(\sigma^{n-m} + b^{p^m}) \end{aligned} \tag{6.67}$$

$$|\ddot{\delta}_y| = |\dot{\delta}_x| \leq C_2 \sigma^{n-m}(\sigma^{n-m} + b^{p^m}). \tag{6.68}$$

† Recall, for $a, b \in \mathbb{R}^2$, that $[a, b]$ denote the straight line segment between a and b.

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Now we apply $\Psi_{0,m}$ which gives

$$\ddot{\zeta} - \ddot{\tau} = D_{0,m}(\mathrm{id} + R_{0,m})(\ddot{\xi} - \ddot{\tau}).$$

Let $\dddot{\delta} = (\dddot{\delta}_x, \dddot{\delta}_y) = \dddot{\zeta} - \dddot{\tau}$. Hence, by Proposition 5.6 and the above paragraph, there is a constant $C_6 > 0$ such that

$$\begin{aligned}
|\dddot{\delta}_x| &= |\sigma_{0,m} s_{0,m}[\ddot{\delta}_x + r_{0,m}(\ddot{\delta}_x, \ddot{\delta}_y)] + \sigma_{0,m} t_{0,m} \ddot{\delta}_y| \\
&\le C_2 \sigma^m |\sigma^m[|\ddot{\delta}_x| + |\partial_x r_{0,m}||\ddot{\delta}|] + b^{p^m} |\ddot{\delta}_y|| \\
&\le C_6 \sigma^{2m} \sigma^{n-m}(\sigma^{n-m} + b^{p^m}) + C_2^2 \sigma^n b^{p^m}(\sigma^{n-m} + b^{p^m}) \\
&\le (C_6 \sigma^{n+m} + C_2^2 \sigma^n b^{p^m})(\sigma^{n-m} + b^{p^m}), \qquad (6.69) \\
|\dddot{\delta}_y| &= |\sigma_{0,m} \ddot{\delta}_y| \le C_2^2 \sigma^n (\sigma^{n-m} + b^{p^m}). \qquad (6.70)
\end{aligned}$$

For $m > 0$ sufficiently large we find $|\dddot{\delta}_x| \le |\dddot{\delta}_y|$. Since $n > m$ was chosen so that $b^{p^m} < \sigma^{n-m}$ it follows that there exists a constant $C_7 > 0$ such that $\mathrm{dist}(\dddot{\zeta}, \dddot{\tau}) \le C_7 \sigma^{2n-m}$.

Now we wish to a find a lower bound for $\mathrm{dist}(\dddot{\zeta}, \dddot{\tau})$. Applying $\tilde{\Psi}_{m,n}$ to these points gives $\dot{\zeta} - \dot{\tau} = \tilde{D}_{m,n}(\mathrm{id} + \tilde{R}_{m,n})(\zeta - \tau)$. Let $\dot{\delta} = (\dot{\delta}_x, \dot{\delta}_y) = \dot{\zeta} - \dot{\tau}$. The same argument that gave us inequality (6.66) implies $|\dot{\delta}_y| = |\tilde{\sigma}_{m,n} \delta_y| \le C_2 \sigma^{n-m}$. Let $C_8 > 1$ be a constant satisfying

$$|\tilde{\sigma}_{m,n}| > C_8^{-1} \sigma^{n-m}, \quad |\tilde{t}_{m,n}| > C_8^{-1} \tilde{b}^{p^m}, \quad |\tilde{s}_{m,n}| < C_8 \sigma^{n-m}, \quad |\tilde{r}_{m,n}| < C_8. \qquad (6.71)$$

Since $\tilde{b}^{p^m} > K \sigma^{n-m+1}$, Proposition 5.6 tells us that

$$\begin{aligned}
|\dddot{\delta}_x| &= |\tilde{\sigma}_{m,n} \tilde{s}_{m,n}[\tilde{\delta}_x + \tilde{r}_{m,n}(\tilde{\delta}_x, \tilde{\delta}_y)] + \tilde{\sigma}_{m,n} \tilde{t}_{m,n} \tilde{\delta}_y| \\
&\ge |\tilde{\sigma}_{m,n}| \big| |\tilde{s}_{m,n}||\tilde{\delta}_x + \tilde{r}_{m,n}(\tilde{\delta}_x, \tilde{\delta}_y)| - |\tilde{t}_{m,n} \tilde{\delta}_y| \big| \\
&\ge C_8^{-1} \sigma^{n-m}(C_8^{-1} C_0 b^{p^m} - C_8(C_0 + C_8)\sigma^{n-m}) \\
&\ge C_8^{-1} \sigma^{n-m} b^{p^m}(C_8^{-1} C_0 - K^{-1} \sigma^{-1} C_8(C_0 + C_8)). \qquad (6.72)
\end{aligned}$$

As $K > 0$ was assumed to be large (and the constant C_8 had no dependence upon m and n) we find there exists a constant $C_9 > 0$ such that $|\dddot{\delta}_x| > C_9 b^{p^m} \sigma^{n-m}$.

Applying $\tilde{F}_m$ to $\dot{\zeta}$ and $\dot{\tau}$ gives $\ddot{\zeta} - \ddot{\tau} = F_m(\dot{\zeta}) - F_m(\dot{\tau})$. Let $\ddot{\delta} = (\ddot{\delta}_x, \ddot{\delta}_y) = \ddot{\zeta} - \ddot{\tau}$. Then, ignoring the difference in the x-direction, we find that $|\ddot{\delta}_y| = |\dddot{\delta}_x| \ge C_9 b^{p^m} \sigma^{n-m}$.

Now we apply $\tilde{\Psi}_{0,m}$ which gives

$$\dddot{\zeta} - \dddot{\tau} = \tilde{D}_{0,m}(\mathrm{id} + \tilde{R}_{0,m})(\ddot{\zeta} - \ddot{\tau}).$$

Let $\dddot{\delta} = (\dddot{\delta}_x, \dddot{\delta}_y) = \dddot{\zeta} - \dddot{\tau}$. Then from Lemma 5.5 we find that $|\dddot{\delta}_y| = |\tilde{\sigma}_{0,m} \ddot{\delta}_y|$. But Proposition 5.6 implies that there exists a constant $C_{10} > 0$ such that $|\tilde{\sigma}_{0,m}| \ge C_{10} \sigma^m$, so combining this with the estimate from the previous paragraph gives $|\dddot{\delta}_y| \ge C_9 C_{10} \sigma^n b^{p^m}$.

Now let us combine these upper and lower bounds. Let $C_{11}, C_{12} > 0$ be constants satisfying $\mathrm{dist}(\dddot{\zeta}, \dddot{\tau}) > C_{11} \sigma^n b^{p^m}$ and $\mathrm{dist}(\dddot{\zeta}, \dddot{\tau}) < C_{12} \sigma^{2n-m}$. Then, assuming that the Hölder condition holds for some $C_{13}, \alpha > 0$, we have

$$C_{11} \sigma^n \tilde{b}^{p^m} \le \mathrm{dist}(\dddot{\tau}, \dddot{\zeta}) \le C\, \mathrm{dist}(\dddot{\tau}, \dddot{\zeta})^\alpha \le C_{13} C_{12}^\alpha (\sigma^{2n-m})^\alpha \qquad (6.73)$$

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which implies, after collecting all constant factors, that there is a $C > 0$ such that

$$\sigma^m b^{p^m} \tilde{b}^{p^m} \leq C(\sigma^m b^{p^m} b^{p^m})^\alpha \tag{6.74}$$

and hence, after taking the logarithm of both sides and passing to the limit,

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log \tilde{b}}{\log b}\right). \tag{6.75}$$

The proof of the theorem is complete. □

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A. *Appendix. Elementary results*

PROPOSITION A.1. *Let $C > 0$ and $0 \leq \rho \leq \delta < 1$. Then the product $\prod_{i=0}^{\infty}(1 + C\rho^i)$ converges and, moreover, there exists a $C_0 > 0$ such that*

$$\prod_{i=m}^{\infty}(1 + C\rho^i) < 1 + C_0\rho^m. \tag{A.1}$$

Proof. First let us show convergence. Observe that concavity of log implies that $\log(1 + C\rho^i) < \log(1) + \log'(1)C\rho^i = C\rho^i$. Therefore taking logarithms gives

$$\log\left[\prod_{0}^{n}(1 + C\rho^i)\right] \leq \sum_{i=0}^{n}\log(1 + C\rho^i) \leq C\sum_{i=0}^{n}\rho^i \leq \frac{C}{1-\rho}. \tag{A.2}$$

Therefore, since the sequence given by the left-most term of the above inequality is increasing in n, the Bolzano–Weierstrass theorem implies that $\log[\prod_{i=0}^{\infty}(1 + C\rho^i)]$ exists. Hence, applying exp gives us convergence.

Now let $F_{m,n}(\rho) = \prod_{i=m}^{n}(1 + C\rho^i)$. Observe that, by the product rule,

$$\frac{d}{d\rho}F_{m,n}(\rho) = \prod_{i=m}^{n}(1 + C\rho^i)\sum_{i=m}^{n}\frac{Ci\rho^{i-1}}{1 + C\rho^i} \tag{A.3}$$

$$= CF_{m,n}(\rho)\rho^{m-1}\sum_{i=0}^{n-m}\frac{(m+i)\rho^i}{1 + C\rho^i} \tag{A.4}$$

but since C, $\rho > 0$,

$$\begin{aligned}\sum_{i=0}^{n-m}\frac{(m+i)\rho^i}{1 + C\rho^i} &\leq m\sum_{i=0}^{n-m}\rho^i + \rho\sum_{i=0}^{n-m}i\rho^{i-1}\\ &\leq m\sum_{i=0}^{\infty}\rho^i + \rho\frac{d}{d\rho}\left(\sum_{i=0}^{\infty}\rho^i\right)\\ &\leq \frac{m}{1-\rho} + \frac{\rho}{(1-\rho)^2}.\end{aligned} \tag{A.5}$$

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So, setting

$$M = C F_{m,n}(\delta)\left(\frac{m}{1-\delta} + \frac{\delta}{(1-\delta)^2}\right) \quad \text{and} \quad G_m(\rho) = \left(1 + \frac{M}{m}\rho^m\right),$$

we find that

$$\frac{d}{d\rho}F_{m,n}(\rho) \le M\rho^{m-1} \le \frac{d}{d\rho}G_m(\rho). \tag{A.6}$$

Hence, as $F_{m,n}(0) = 0 = G_m(0)$ the result follows by setting $C_0 = M/m$. □

LEMMA A.2. *Let $C > 0$ and $0 < \rho < 1$. Then there exists a constant $C_0 > 0$ such that*

$$\frac{1 + C\rho}{1 - C\rho} < 1 + C_0\rho^2. \tag{A.7}$$

The following lemma and proposition are straightforward and are left to the reader.

LEMMA A.3. *Given constants $0 \le \bar{\varepsilon}$, ρ, $\sigma < 1$ and C_0, $C_1 > 0$ and a fixed integer $p > 1$, there exists a constant $C > 0$ such that for all integers $0 < m < M$:*

(i) $C_0\bar{\varepsilon}^{p^m} + C_1\bar{\varepsilon}^{p^{m+1}} \le C\bar{\varepsilon}^{p^m}$;

(ii) $C_0\bar{\varepsilon}^{p^m} + C_1\rho^m \le C\rho^m$;

(iii) $\sum_{m<n<M} \sigma^{i-m-1}\bar{\varepsilon}^{p^n - p^m}(1 + C_0\rho^n) < C$;

(iv) $\sum_{n>M} \bar{\varepsilon}^{p^n} \le C\bar{\varepsilon}^{p^M}$.

PROPOSITION A.4. *Given any $\rho > 0$, there exists a $\overline{\varepsilon} > 0$ such that $\sum_{i>0} \rho^i \varepsilon^{p^i}$ converges for all $\varepsilon < \overline{\varepsilon}$. Moreover, for $0 < \overline{\overline{\varepsilon}} < \overline{\varepsilon}$ there exists a constant $C = C(\overline{\overline{\varepsilon}}) > 0$ such that $\sum_{i>0} \rho^i \varepsilon^{p^i} < C\varepsilon$ for all $0 < \varepsilon < \overline{\overline{\varepsilon}}$.*

LEMMA A.5. *Let P, Q, P', $Q' \in \mathbb{R}$ with Q, Q' non-zero. Then*

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| \le C \max(|P - P'|, |Q - Q'|) \tag{A.8}$$

where $C = 2|Q|^{-1} \max(1, |P'/Q'|)$.

Proof. This is immediate given the inequality

$$\begin{aligned}\left|\frac{P}{Q} - \frac{P'}{Q'}\right| &= \left|\frac{1}{Q}(P - P') + P'\left(\frac{1}{Q} - \frac{1}{Q'}\right)\right| \\ &\le \frac{1}{|Q|}\left[|P - P'| + \left|\frac{P'}{Q'}\right||Q' - Q|\right].\end{aligned} \tag{A.9}$$

□

B. *Appendix. Variational properties of composition operators*

In this section we derive properties of the composition operator. We show how the remainder term from Taylor's theorem behaves under composition and we derive the first variation of the n-fold composition operator. Although we only state these for maps on $\mathbb{R}^2$ or $\mathbb{C}^2$, they work in full generality.

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PROPOSITION B.1. *Let* $F, G \in \mathrm{Emb}^2(\mathbb{R}^2, \mathbb{R}^2)$. *For any* $z_0, z_1 \in \mathbb{R}^2$, *consider the decompositions*

$$F(z_0 + z_1) = F(z_0) + \mathrm{D}_{z_0} F(\mathrm{id} + \mathrm{R}_{z_0} F)(z_1) \tag{B.1}$$

and

$$G(z_0 + z_1) = G(z_0) + \mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1). \tag{B.2}$$

Then

$$\mathrm{R}_{z_0} FG(z_1) = \mathrm{R}_{z_0} G(z_1) + \mathrm{D}_{z_0} G^{-1} \mathrm{R}_{G(z_0)} F(\mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1)). \tag{B.3}$$

Proof. Observe that

$$FG(z_0 + z_1) = FG(z_0) + \mathrm{D}_{z_0} FG(z_1) + \mathrm{D}_{z_0} FG(\mathrm{R}_{z_0} FG)(z_1) \tag{B.4}$$

must be equal to

$$\begin{aligned} F(G(z_0 + z_1)) &= F(G(z_0) + \mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1)) \\ &= F(G(z_0)) + \mathrm{D}_{G(z_0)} F(\mathrm{id} + \mathrm{R}_{G(z_0)} F)(\mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1)) \\ &= F(G(z_0)) + \mathrm{D}_{G(z_0)} F \mathrm{D}_{z_0} G(z_1) + \mathrm{D}_{G(z_0)} F \mathrm{D}_{z_0} G(\mathrm{R}_{z_0} G(z_1)) \\ &\quad + \mathrm{D}_{G(z_0)} F \mathrm{R}_{G(z_0)} F(\mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1)). \end{aligned} \tag{B.5}$$

This implies that

$$\mathrm{R}_{z_0} FG(z_1) = \mathrm{R}_{z_0} G(z_1) + \mathrm{D}_{z_0} G^{-1} \mathrm{R}_{G(z_0)} F(\mathrm{D}_{z_0} G(\mathrm{id} + \mathrm{R}_{z_0} G)(z_1)) \tag{B.6}$$

and hence the proposition is shown. □

PROPOSITION B.2. *For each integer* $n > 0$ *let* $C_n : C^\omega(B, B)^n \to C^\omega(B, B)$ *denote the n-fold composition operator*

$$C_n(G_1, \ldots, G_n) = G_1 \circ \cdots \circ G_n. \tag{B.7}$$

For $i = 1, \ldots, n$, *assume that we are given* $F_i, G_i \in C^\omega(B, B)$ *and let* E_i *be defined by* $G_i = F_i + E_i$. *Then*

$$C(G_1, \ldots, G_n) = C(F_1, \ldots, F_n) + \delta C_n(F_1, \ldots, F_n; E_1, \ldots, E_n) + \mathrm{O}(|E_i||E_j|) \tag{B.8}$$

where

$$\delta C_n(F_1, \ldots, F_n; E_1, \ldots, E_n) = \sum_{i=1}^{n-1} \mathrm{D}_{F_{i+1,\ldots,n}(z)} F_{1,\ldots,i}(E_{i+1}(F_{i+2,\ldots,n}(z))) \tag{B.9}$$

where we have set $F_\emptyset$, $E_{n+1} = \mathrm{id}$.

Proof. For notational simplicity let $F_{1,\ldots,n} = F_1 \circ \cdots \circ F_n$ and $G_{1,\ldots,n} = G_1 \circ \cdots \circ G_n$. Let $E_{1,\ldots,n}$ satisfy $G_{1,\ldots,n} = F_{1,\ldots,n} + E_{1,\ldots,n}$. Then equating $G_{1,2,\ldots,n}$ with $G_1 \circ G_{2,\ldots,n}$ and using the power series expansion of G_1 gives

$$\begin{aligned} G_{1,\ldots,n}(z) &= G_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z)) \\ &= F_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z)) + E_1(F_{2,\ldots,n}(z) + E_{2,\ldots,n}(z)) \\ &= F_1(F_{2,\ldots,n}(z)) + \mathrm{D}_{F_{2,\ldots,n}(z)} F_1(E_{2,\ldots,n}(z)) + \mathrm{O}(|E_{2,\ldots,n}|^2) \\ &\quad + E_1(F_{2,\ldots,n}(z)) + \mathrm{O}(|\mathrm{D}E_1||E_{2,\ldots,n}|) \end{aligned} \tag{B.10}$$

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while equating $G_{1,2,\dots,n}$ with $G_{1,\dots,n-1} \circ G_n$ and using the power series expansion of $G_{1,\dots,n-1}$ gives

$$\begin{aligned} G_{1,\dots,n}(z) &= G_{1,\dots,n-1}(F_n(z) + E_n(z)) \\ &= F_{1,\dots,n-1}(F_n(z) + E_n(z)) + E_{1,\dots,n-1}(F_n(z) + E_n(z)) \\ &= F_{1,\dots,n}(z) + \mathrm{D}_{F_n(z)} F_{1,\dots,n-1}(E_n(z)) + \mathrm{O}(|E_n|^2) \\ &\quad + E_{1,\dots,n-1}(F_n(z)) + \mathrm{O}(|\mathrm{D}E_{1,\dots,n}||E_n|). \end{aligned} \tag{B.11}$$

From the second of these expressions, we find inductively, setting $F_\emptyset$, $E_{n+1} = \mathrm{id}$, that

$$E_{1,\dots,n}(z) = \sum_{i=1}^{n-1} \mathrm{D}_{F_{i+1,\dots,n}(z)} F_{1,\dots,i}(E_{i+1}(F_{i+2,\dots,n}(z))) + \mathrm{O}(|E_i||E_j|). \tag{B.12}$$

□

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