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ALMOST-PRIME k-TUPLES

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Abstract. Let $k \geqslant 2$ and $\Pi(n) = \prod_{i=1}^{k}(a_i n + b_i)$ for some integers a_i, b_i $(1 \leqslant i \leqslant k)$. Suppose that $\Pi(n)$ has no fixed prime divisors. Weighted sieves have shown for infinitely many integers n that the number of prime factors $\Omega(\Pi(n))$ of $\Pi(n)$ is at most r_k, for some integer r_k depending only on k. We use a new kind of weighted sieve to improve the possible values of r_k when $k \geqslant 4$.

§1. *Introduction.*

We consider a set of distinct integer linear functions

$$L_i(x) = a_i x + b_i, \quad i \in \{1, \dots, k\}. \tag{1.1}$$

We say such a set of functions is *admissible* if their product has no fixed prime divisor. That is, for every prime p there is an integer n_p such that none of $L_i(n_p)$ is a multiple of p. We are interested in the following conjecture.

CONJECTURE (Prime k-tuples conjecture). *Given an admissible set of integer linear functions $L_i(x)$ $(i \in \{1, \dots, k\})$, there are infinitely many integers n for which all the $L_i(n)$ are prime.*

With the current technology it appears impossible to prove any case of the prime k-tuples conjecture for $k \geqslant 2$.

Although we cannot prove that the functions are simultaneously prime infinitely often, we are able to show that they are *almost prime* infinitely often, in the sense that their product has only a few prime factors. This was most notably achieved by Chen [1] who showed that there are infinitely many primes p for which $p + 2$ has at most two prime factors. His method naturally generalises to show that for a pair of admissible functions the product $L_1(n)L_2(n)$ has at most three prime factors infinitely often.

Similarly, sieve methods can prove analogous results for any k. We can show that the product of k admissible functions $\Pi(n) = L_1(n) \cdots L_k(n)$ has at most r_k prime factors infinitely often, for some explicitly given value of r_k. The prime k-tuples conjecture is equivalent to showing that we can have $r_k = k$ for all k. The current best values of r_k grow asymptotically like $k \log k$ and explicitly for small k we can take $r_2 = 3$ (Chen [1]), $r_3 = 8$ (Porter [9]), $r_4 = 12$, $r_5 = 16$, $r_6 = 20$ (Diamond and Halberstam [2]), $r_7 = 24$, $r_8 = 28$, $r_9 = 33$, $r_{10} = 38$

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Table 1: Bounds for $\Omega(\Pi(n))$.

k	3	4	5	6	7	8	9	10
r_k	8	11	15	18	22	26	30	34

Table 2: Bounds for $\Omega(\Pi(n))$.

k	3	4	5	6	7	8	9	10
Previous best bound	8	12	16	20	24	28	33	38
New bound	8	11	15	18	22	26	30	34

(Ho and Tsang [6]). Heath-Brown [5] showed that infinitely often there are k-tuples where all the functions L_i have individually at most $C \log k$ prime factors, for an explicit constant C.

A different approach was taken by Goldston *et al* [4] in their work on small gaps between primes. Under the Elliott–Halberstam conjecture, they showed that there are infinitely many n for which at least two of $n, n+4, n+6, n+10, n+12, n+16$ are prime. Thus, there must be at least one specific 2-tuple where both functions are prime infinitely often if the Elliott–Halberstam conjecture holds.

§2. *Statement of results.* Our main result is the following theorem.

THEOREM 2.1. *Given a set of k admissible linear functions, for infinitely many $n \in \mathbb{N}$ the product $\Pi(n)$ has at most r_k prime factors, where r_k is given in Table 1.*

Theorem 2.1 improves the previous best known bounds for $k \geqslant 4$, which were obtained by Diamond and Halberstam [2] for $4 \leqslant k \leqslant 6$ and by Ho and Tsang [6] for $7 \leqslant k \leqslant 10$. We fall just short of proving $r_k \leqslant 7$ for $k = 3$, and so fail to improve upon a result of Porter [9]. This comparison is shown in Table 2. We prove these results using a sieve which is a combination of a weighted sieve similar to Selberg's $\Lambda^2\Lambda^-$ sieve (see [11]), and the Graham–Goldston–Pintz–Yıldırım sieve (see [3]) used to count numbers with a specific number of prime factors. We comment that the assumption that the functions are distinct is not strictly necessary. Essentially the same argument can give the bounds which are at least as strong when the functions are not distinct.

We note that for large k our method only improves lower order terms, and so we do not improve the asymptotic bound $r_k \sim k \log k$. Indeed, it is possible to show using the methods of this paper that for large k and infinitely many n we have $\Omega(\Pi(n)) = (1 + o(1))k \log k$ with $\Omega(L_i(n)) = O_\epsilon(k^\epsilon)$ for any $\epsilon > 0$ and all $1 \leqslant i \leqslant k$. This extends a result of Ramaré [10].

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In a forthcoming paper [7], we will also improve the bound when $k = 3$, using an argument based on the Diamond–Halberstam–Richert sieve rather than Selberg's sieve.

§3. *Key ideas.* We wish to show that for any sufficiently large N we have

$$\sum_{N\leqslant n\leqslant 2N}(c-\Omega(\Pi(n)))\Big(\sum_{d|\Pi(n)}\lambda_d\Big)^2>0 \tag{3.1}$$

for some real numbers λ_d and some constant integer $c > 0$. From this it is clear that there must be some $n \in [N, 2N]$ such that $\Omega(\Pi(n)) \leqslant c$. Since this is true for all sufficiently large N, it follows that there are infinitely many integers n such that $\Omega(\Pi(n)) \leqslant c$.

The work of Heath–Brown [5] and Ho and Tsang [6] considered a similar sum, but used the divisor function $d(\Pi(n))$ instead of the number-of-prime-factors function Ω. Using the divisor function has the advantage that there are stronger level-of-distribution results available, but we find that this is outweighed by the fact that the Ω function is relatively much smaller than the divisor function on numbers with many prime factors.

The Ω function has Bombieri–Vinogradov style equidistribution results (as shown by Motohashi [8]), and so we would expect we should be able to estimate the above sum directly, in a method similar to Heath–Brown [5] or Selberg [11] when they considered the divisor function instead. We encounter some technical difficulties when attempting to translate this argument, however.

Instead we express $\Omega(n)$ as a weighted sum over small prime factors (as in the weighted sieve method of Diamond and Halberstam [2]) and a remaining positive contribution which we split up depending on the number of prime factors of each of the $L_j(n)$.

Diamond and Halberstam used a weighted sieve. The method relied on the fact that for n square-free we have the inequality

$$\Omega(n)\leqslant\sum_{\substack{p|n\\ p\leqslant y}}\Big(1-\frac{\log p}{\log y}\Big)+\frac{\log n}{\log y}. \tag{3.2}$$

We note that this inequality is strict if n has a prime factor which is larger than y. This results in a loss in the argument, which has a noticeable effect when we apply this to k-tuples when k is small. Assuming that $y \geqslant n^{1/2}$ and n is square-free, we can write instead an equality

$$\Omega(n)=\sum_{\substack{p|n\\ p\leqslant y}}\Big(1-\frac{\log p}{\log y}\Big)+\frac{\log n}{\log y}+\sum_{r=1}^{\infty}\chi_r(n), \tag{3.3}$$

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where

$$\chi_r(n) = \begin{cases} 1 - \dfrac{\log p_r}{\log y} & \text{if } n = p_1 \cdots p_r \text{ with} \\ & \quad p_1 \leqslant \cdots \leqslant p_{r-1} < y < p_r, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} -\left(\dfrac{\log n}{\log y} - 1 - \displaystyle\sum_{i=1}^{r-1} \frac{\log p_i}{\log y}\right) & \text{if } n = p_1 \cdots p_r \quad \text{with} \\ & \quad p_1 \leqslant \cdots \leqslant p_{r-1} < y < p_r \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

For fixed r we can evaluate Selberg-type weighted sums over $\chi_r(L_i(n))$ using the method of Graham *et al* in [3] as an extension of the original GPY method. We note that the contribution from $\chi_r(n)$ is always negative, so we can obtain a lower bound by simply omitting terms when $r > h$ for some constant h. The contribution of the χ_r terms decreases quickly with r, and so we in practice only need to calculate the contribution when r is small (in this paper we only consider the contributions of χ_r when $r \leqslant 4$). This is the key difference in our approach to previous methods, and allows us to obtain the improvements given by Theorem 2.1.

§4. *Initial considerations.* We adopt similar notation to that of Graham *et al* in [3].

Let $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$ be an admissible k-tuple of linear functions. We define

$$\Pi(n) = \prod_{i=1}^{k} L_i(n) = (a_1 n + b_1) \cdots (a_k n + b_k), \tag{4.1}$$

$$\nu_p(\mathcal{L}) = \#\{1 \leqslant n \leqslant p : \Pi(n) \equiv 0 \pmod p\}. \tag{4.2}$$

We note that admissibility is equivalent to the condition

$$\nu_p(\mathcal{L}) < p \quad \text{for all primes } p. \tag{4.3}$$

We also see that $\nu_p(\mathcal{L}) \leqslant k$ for all primes p, and so the above condition holds automatically for $p > k$.

For technical reasons we adopt a normalisation of our linear functions, as done originally by Heath-Brown in [5]. Since we are only interested in showing that any admissible k-tuple has at most r_k prime factors infinitely often (for some explicit r_k), by considering the functions $L_i(An + B)$ for suitably chosen constants A and B, we may assume without loss of generality that our functions satisfy the following hypothesis.

HYPOTHESIS 1. *$\mathcal{L} = \{L_1, \ldots, L_k\}$ is an admissible k-tuple of linear functions. The functions $L_i(n) = a_i n + b_i$ $(1 \leqslant i \leqslant k)$ are distinct with $a_i > 0$. Each of the coefficients a_i is composed of the same primes, none of which divides the b_j. If $i \neq j$, then any prime factor of $a_i b_j - a_j b_i$ divides each of the a_l.*

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For a set of linear functions satisfying Hypothesis 1 we define

$$A = \prod_{i=1}^{k} a_i. \tag{4.4}$$

We note that in this case

$$\nu_p(\mathcal{L}) = \begin{cases} 0 & \text{if } p|A, \\ k & \text{if } p \nmid A. \end{cases} \tag{4.5}$$

We also define the *singular series* $\mathfrak{S}(\mathcal{L})$ of $\mathcal{L}$ when $\mathcal{L}$ satisfies Hypothesis 1,

$$\mathfrak{S}(\mathcal{L}) = \prod_{p|A}\left(1-\frac{1}{p}\right)^{-k}\prod_{p\nmid A}\left(1-\frac{k}{p}\right)\left(1-\frac{1}{p}\right)^{-k}. \tag{4.6}$$

We note that $\mathfrak{S}(\mathcal{L})$ is positive.

As is common with the Selberg sieve, for some parameter R_2 we impose the condition

$$\lambda_d = 0 \quad \text{if } d \geqslant R_2 \text{ or } d \text{ is not square-free or } (d, A) \neq 1. \tag{4.7}$$

We wish to choose the λ_d to maximise the sum (3.1), but this will be difficult to do optimally. We proceed by reparameterising the form in λ_d into new variables y_r and y_r^* which will almost diagonalise it. We define

$$y_r = \mu(r) f_1(r) \sum_d{}' \frac{\lambda_{dr}}{f(dr)}, \tag{4.8}$$

$$y_r^* = \mu(r) f_1^*(r) \sum_d{}' \frac{\lambda_{dr}}{f^*(dr)}, \tag{4.9}$$

where here and from now on, the $'$ by the summation indicates that the sum is taken over all values of the indices which are square-free and coprime to A. For square-free d coprime to A, the functions f, f_1, f^* and f_1^* are defined by

$$f(d) = \prod_{p|d}\frac{p}{k}, \tag{4.10}$$

$$f_1(d) = (f*\mu)(d) = \prod_{p|d}\frac{p-k}{k}, \tag{4.11}$$

$$f^*(d) = \prod_{p|d}\frac{p-1}{k-1}, \tag{4.12}$$

$$f_1^*(d) = (f^**\mu)(d) = \prod_{p|d}\frac{p-k}{k-1}. \tag{4.13}$$

We note that by Möbius inversion we have

$$\lambda_d = \mu(d) f(d) \sum_r{}' \frac{y_{rd}}{f_1(rd)}. \tag{4.14}$$

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Thus, the λ_d (and hence also the y_r^*) are defined uniquely by a choice of the y_r. The conditions (4.7) will be satisfied if the same conditions apply to the y_r.

For some polynomial P (to be determined later), we choose

$$y_r = \begin{cases} \mu^2(r)\mathfrak{S}(\mathcal{L})P\left(\dfrac{\log R_2/r}{\log R_2}\right) & \text{if } r \leqslant R_2 \text{ and } (r, A) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

The exact polynomial P we choose is given later in Table 3. We now turn our attention to the proof of the theorem.

§5. *Proof of theorem.* We consider the sum

$$S = S(\nu; N, R_1, R_2, \mathcal{L}) = \sum_{N \leqslant n \leqslant 2N} w(n)\Lambda^2(n), \tag{5.1}$$

where

$$w(n) = \nu - \sum_{p \mid \Pi(n)} \left(1 - \frac{\log p}{\log R_1}\right), \tag{5.2}$$

$$\Lambda^2(n) = \left(\sum_{\substack{d \mid \Pi(n) \\ d \leqslant R_2}} \lambda_d\right)^2. \tag{5.3}$$

We note that if $\Pi(n)$ is square-free then

$$w(n) = \nu - \Omega(\Pi(n)) + \frac{\log \Pi(n)}{\log R_1}. \tag{5.4}$$

We see that for $n \in [N, 2N]$ and some fixed $h \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} w(n) &= \nu - \sum_{j=1}^{k} \sum_{p \mid L_j(n)} \left(1 - \frac{\log p}{\log R_1}\right) \\ &\geqslant \nu - \sum_{j=1}^{k} \sum_{\substack{p \mid L_j(n) \\ p \leqslant R_1 \text{ or } \Omega(L_j(n)) \leqslant h}} \left(1 - \frac{\log p}{\log R_1}\right) \\ &\geqslant \nu - \sum_{j=1}^{k} \sum_{\substack{p \mid L_j(n) \\ p \leqslant R_1}} \left(1 - \frac{\log p}{\log R_1}\right) + \sum_{j=1}^{k} \sum_{r=1}^{h} \chi_r(L_j(n)), \end{aligned} \tag{5.5}$$

where

$$\chi_r(n) = \begin{cases} \dfrac{\log N}{\log R_1} - 1 - \displaystyle\sum_{i=1}^{r-1} \frac{\log p_i}{\log R_1} & \text{if } n = p_1 \cdots p_r \text{ with } n^{\epsilon} < p_1 < \cdots \\ & \quad < p_{r-1} \leqslant n^{\log R_1/\log N} < p_r, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

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Thus,

$$\sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ square-free}}}\Big(\nu-\Omega(\Pi(n))+\frac{\log\Pi(n)}{\log R_1}\Big)\Lambda^2(n)=S-S'$$

$$\geqslant \nu S_0-S'-T_0+\sum_{j=1}^{k}\sum_{r=1}^{h}T_{r,j}, \tag{5.7}$$

where

$$S_0=\sum_{N\leqslant n\leqslant 2N}\Lambda^2(n), \tag{5.8}$$

$$S'=\sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ not square-free}}}w(n)\Lambda^2(n), \tag{5.9}$$

$$T_0=\sum_{N\leqslant n\leqslant 2N}\sum_{\substack{p\mid\Pi(n)\\ p\leqslant R_1}}\Big(1-\frac{\log p}{\log R_1}\Big)\Lambda^2(n), \tag{5.10}$$

$$T_{r,j}=\sum_{N\leqslant n\leqslant 2N}\chi_r(L_j(n))\Lambda^2(n). \tag{5.11}$$

We can evaluate S_0, S', T_0 and T_i using weighted forms of the Selberg sieve. We state the results here and prove them in the following sections. To ease notation we now fix as constants

$$r_1=\frac{\log R_1}{\log N},\qquad r_2=\frac{\log R_2}{\log N}. \tag{5.12}$$

We view r_1, r_2, k, A and our polynomial P as fixed, and so any constants implied by the use of O or $\ll$ notation may depend on these quantities without explicit reference.

PROPOSITION 5.1. *Let $\mathcal{L}$ satisfy Hypothesis 1. Let $W_0:[0,r_1/r_2]\to\mathbb{R}_{\geqslant 0}$ be a piecewise smooth non-negative function. Let λ_d, y_d be defined in terms of a polynomial P as given in (4.14) and (4.15). Assume that $r_1\geqslant r_2$. Then there exists a constant C such that if $R_1R_2^2\leqslant N(\log N)^{-C}$ then we have*

$$\sum_{N\leqslant n\leqslant 2N}\Big(\sum_{\substack{p\mid\Pi(n)\\ p\leqslant R_1}}W_0\Big(\frac{\log p}{\log R_2}\Big)\Big)\Big(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}}\lambda_d\Big)^2=\frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!}J_0$$
$$+O_{W_0}(N(\log N)^{k-1}(\log\log N)^2),$$

where

$$J_0=J_{01}+J_{02}+J_{03},$$
$$J_{01}=k\int_0^1\frac{W_0(y)}{y}\int_0^{1-y}(P(1-x)-P(1-x-y))^2x^{k-1}\,dx\,dy,$$

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$$J_{02} = k \int_0^1 \frac{W_0(y)}{y} \int_{1-y}^1 P(1-x)^2 x^{k-1}\, dx\, dy,$$
$$J_{03} = k \int_1^{r_1/r_2} \frac{W_0(y)}{y} \int_0^1 P(1-x)^2 x^{k-1}\, dx\, dy.$$

PROPOSITION 5.2. *Let $\mathcal{L}$ satisfy Hypothesis 1. Given $\epsilon > 0$ and $r \in \mathbb{Z}_{>0}$, let*

$$\mathcal{A}_r := \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1-r_2, 1-x_{r-1}) \right\}.$$

Let $W_r : [0,1]^{r-1} \to \mathbb{R}_{\geqslant 0}$ be a piecewise smooth function supported on $\mathcal{A}_r$ such that

$$\frac{\partial}{\partial x_j} W_r(x) \ll W_r(x) \quad \textit{uniformly in } \left\{ x \in \mathcal{A}_r : \frac{\partial}{\partial x_j} W_r(x) \textit{ exists} \right\}.$$

Let

$$\beta_r(n) = \begin{cases} W_r\left(\dfrac{\log p_1}{\log n}, \ldots, \dfrac{\log p_{r-1}}{\log n} \right) & \textit{if } n = p_1 p_2 \cdots p_r \textit{ with} \\ & \quad p_1 < \cdots < p_r, \\ 0 & \textit{otherwise.} \end{cases}$$

Let λ_d, y_d be defined in terms of a polynomial P as given in (4.14) and (4.15). Then there is a constant C such that if $R_2^2 \leqslant N^{1/2} (\log N)^{-C}$, we have

$$\sum_{N \leqslant n \leqslant 2N} \beta_r(L_j(n)) \left(\sum_{\substack{d \mid \Pi(n) \\ d \leqslant R_2}} \lambda_d \right)^2 = \frac{\mathfrak{S}(\mathcal{L}) N (\log R_2)^{k+1}}{(k-2)! (\log N)} J_r + O_{W_r}(N (\log\log N)^r (\log N)^{k-1}),$$

where

$$J_r = \int_{(x_1,\ldots,x_{r-1}) \in \mathcal{A}_r} \frac{W_r(x_1, \ldots, x_{r-1}) I_1(r_2^{-1} x_1, \ldots, r_2^{-1} x_{r-1})}{(\prod_{i=1}^{r-1} x_i)(1 - \sum_{i=1}^{r-1} x_i)} \times dx_1 \cdots dx_{r-1},$$

$$I_1 = \int_0^1 \left(\sum_{J \subset \{1,\ldots,r-1\}} (-1)^{|J|} \tilde{P}^+\left(1 - t - \sum_{i \in J} x_i\right) \right)^2 t^{k-2}\, dt,$$

$$\tilde{P}^+(x) = \begin{cases} \displaystyle\int_0^x P(t)\, dt & \textit{if } x \geqslant 0, \\ 0 & \textit{otherwise.} \end{cases}$$

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PROPOSITION 5.3. *Let $\mathcal{L}$ satisfy Hypothesis 1. Let λ_d, y_d be defined in terms of a polynomial P as given in (4.14) and (4.15). Then there exists a constant C such that if $R_2^2 \leqslant N^{1/2}(\log N)^{-C}$ then*

$$\sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ not square-free}}}\Big(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}}\lambda_d\Big)^2 \ll N(\log N)^{k-1}\log\log N.$$

We also quote a result [**3**, Theorem 7] which is based on the original result of Goldston *et al* in [**4**].

PROPOSITION 5.4. *Let $\mathcal{L}$ satisfy Hypothesis 1. Let λ_d, y_d be defined in terms of a polynomial P as given in (4.14) and (4.15). Then there is a constant C such that if $R_2^2 \leqslant N(\log N)^{-C}$, we have*

$$\sum_{N\leqslant n\leqslant 2N}\Big(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}}\lambda_d\Big)^2 = \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!}J + O(N(\log N)^{k-1}),$$

where

$$J = \int_0^1 P(1-t)^2 t^{k-1}\,dt.$$

Using Propositions 5.1, 5.2, 5.4 and 5.3 we can now bound our sum S in terms of the integers k and h and the polynomial P. For some $\epsilon > 0$ we choose

$$r_1 = \tfrac{1}{2} + \epsilon, \qquad r_2 = \tfrac{1}{4} - \epsilon, \tag{5.13}$$

so that the conditions of all the propositions are satisfied.

Proposition 5.4 gives the size of S_0 immediately.

Using Proposition 5.3 we have

$$\begin{aligned} S' &= \sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ not square-free}}} w(n)\Lambda^2(n)\\ &\leqslant \sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ not square-free}}}\Big(\nu + \frac{\log\Pi(n)}{\log R_1}\Big)\Lambda^2(n)\\ &\leqslant \sum_{\substack{N\leqslant n\leqslant 2N\\ \Pi(n)\text{ not square-free}}}\Big(\nu + \frac{k+\epsilon}{r_1}\Big)\Lambda^2(n)\\ &\ll N(\log N)^{k-1}\log\log N. \end{aligned} \tag{5.14}$$

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To estimate T_0 and the $T_{r,j}$ we choose

$$W_0(x) = 1 - \frac{r_2}{r_1}x, \tag{5.15}$$

$$W_j(x_1, \dots, x_{j-1}) = \begin{cases} \dfrac{1}{r_1} - 1 - \dfrac{1}{r_1}\displaystyle\sum_{i=1}^{j-1} x_i & \text{if } \epsilon < x_1 < \cdots < x_{j-1} \text{ and } \displaystyle\sum_{i=1}^{r-1} x_i < 1 - r_1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.16}$$

which satisfy the conditions of Propositions 5.1 and 5.2, respectively.

By Proposition 5.1 we have

$$\begin{aligned} T_0 &= \sum_{N \leqslant n \leqslant 2N} \Bigg(\sum_{\substack{p \mid \Pi(n) \\ p \leqslant R_1}} W_0\Big(\frac{\log p}{\log R_2}\Big)\Bigg)\Bigg(\sum_{\substack{d \mid \Pi(n) \\ d \leqslant R_2}} \lambda_d\Bigg)^2 \\ &= \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!} J_0 + O(N(\log N)^{k-1} \log\log N), \end{aligned} \tag{5.17}$$

where

$$J_0 = J_{01} + J_{02} + J_{03}, \tag{5.18}$$

$$J_{01} = k \int_0^1 \frac{r_1 - r_2 y}{r_1 y} \int_0^{1-y} (P(1-x) - P(1-x-y))^2 x^{k-1}\, dx\, dy, \tag{5.19}$$

$$J_{02} = k \int_0^1 \frac{r_1 - r_2 y}{r_1 y} \int_{1-y}^1 P(1-x)^2 x^{k-1}\, dx\, dy, \tag{5.20}$$

$$J_{03} = k \int_1^{r_1/r_2} \frac{r_1 - r_2 y}{r_1 y} \int_0^1 P(1-x)^2 x^{k-1}\, dx\, dy. \tag{5.21}$$

By Proposition 5.2 we have

$$\begin{aligned} T_{r,j} &= \sum_{N \leqslant n \leqslant 2N} \chi_r(L_j(n))\Lambda^2(n) \\ &= \sum_{N \leqslant n \leqslant 2N} \beta_r(L_j(n))\Lambda^2(n) \\ &= \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^{k+1}}{(k-2)!(\log N)} J_r + O_r(N(\log\log N)^{r+1}(\log N)^{k-1}), \end{aligned} \tag{5.22}$$

where

$$\beta_r(n) = \begin{cases} W_r\Big(\dfrac{\log p_1}{\log n}, \dots, \dfrac{\log p_{r-1}}{\log n}\Big) & \text{if } n = p_1 p_2 \cdots p_r \\ & \quad \text{with } p_1 < \cdots < p_r, \\ 0 & \text{otherwise,} \end{cases} \tag{5.23}$$

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$$J_r = \int_{(x_1,\dots,x_{r-1})\in\mathcal{A}_r} \frac{W_r(x_1,\dots,x_{r-1})I_1(r_2^{-1}x_1,\dots,r_2^{-1}x_{r-1})}{(\prod_{i=1}^{r-1}x_i)(1-\sum_{i=1}^{r-1}x_i)} \times dx_1\cdots dx_{r-1}. \tag{5.24}$$

Therefore, we see that

$$\nu S_0 - S' + T_0 + \sum_{j=1}^{k}\sum_{r=1}^{h} T_{r,j} = \frac{N\mathfrak{S}(\mathcal{L})(\log R_2)^k}{(k-1)!} \times \left(\nu J - J_0 + r_2 k(k-1)\left(\sum_{r=1}^{h} J_r\right)\right) + O\left(\frac{N(\log N)^k}{\log\log N}\right). \tag{5.25}$$

Therefore, we put

$$\nu = \frac{J_0 - r_2 k(k-1)(\sum_{r=1}^{h} J_r)}{J} + \epsilon. \tag{5.26}$$

We then see that for any sufficiently large N we have

$$\nu S_0 - S' - T_0 + \sum_{j=1}^{k}\sum_{r=1}^{h} T_{r,j} > 0. \tag{5.27}$$

Thus, we have

$$\Omega(\Pi(n)) \leqslant \left\lfloor \frac{J_0 - r_2 k(k-1)(\sum_{r=1}^{h} J_r)}{J} + \frac{k}{r_1} + 2\epsilon \right\rfloor \tag{5.28}$$

infinitely often.

With these fixed, given k, h and a polynomial P, we obtain a bound on $\Omega(\Pi(n))$. To make calculations feasible we choose $h = 3$ (except we take $h = 4$ when $k = 10$). Numerical experiments indicate that the bounds of Theorem 2.1 cannot be improved by increasing h except possibly when $k = 5$.

We can now explicitly write down the integrals J_1, J_2 and J_3, splitting the integral up depending on whether $\tilde{P}^+$ (defined in Proposition 5.2) is positive or not. We put

$$\tilde{P}(x) = \int_0^x P(t)\,dt. \tag{5.29}$$

Then we have that

$$J_1 = \left(\frac{1-r_1}{r_1}\right)\int_0^1 \tilde{P}(1-x)^2 x^{k-2}\,dx + O(\epsilon). \tag{5.30}$$

Similarly,

$$J_2 = J_{21} + J_{22} + J_{23} + O(\epsilon), \tag{5.31}$$

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where

$$J_{21} = \int_0^1 \frac{1 - r_1 - r_2 y}{r_1 y(1 - r_2 y)} \int_0^{1-y} (\tilde{P}(1-x) - \tilde{P}(1-x-y))^2 x^{k-2}\, dx\, dy, \tag{5.32}$$

$$J_{22} = \int_0^1 \frac{1 - r_1 - r_2 y}{r_1 y(1 - r_2 y)} \int_{1-y}^1 \tilde{P}(1-x)^2 x^{k-2}\, dx\, dy, \tag{5.33}$$

$$J_{23} = \int_1^{(1-r_1)/r_2} \frac{1 - r_1 - r_2 y}{r_1 y(1 - r_2 y)} \int_0^1 \tilde{P}(1-x)^2 x^{k-2}\, dx\, dy. \tag{5.34}$$

Finally,

$$J_3 = J_{31} + J_{32} + J_{33} + J_{34} + J_{35} + J_{36} + J_{37} + J_{38} + O(\epsilon), \tag{5.35}$$

where

$$J_{31} = \int_1^{(1-r_1)/2r_2} \int_y^{(1-r_1)/r_2 - y} \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \times \int_0^1 \tilde{P}(1-x)^2 x^{k-2}\, dx\, dz\, dy, \tag{5.36}$$

$$J_{32} = \int_0^1 \int_y^{(1-r_1)/r_2 - y} \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \int_{1-y}^1 \tilde{P}(1-x)^2 x^{k-2}\, dx\, dz\, dy, \tag{5.37}$$

$$J_{33} = \int_0^1 \int_1^{(1-r_1)/r_2 - y} \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \times \int_0^{1-y} (\tilde{P}(1-x) - \tilde{P}(1-x-y))^2 x^{k-2}\, dx\, dz\, dy, \tag{5.38}$$

$$J_{34} = \int_0^1 \int_y^1 \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \times \int_{1-z}^{1-y} (\tilde{P}(1-x) - \tilde{P}(1-x-y))^2 x^{k-2}\, dx\, dz\, dy, \tag{5.39}$$

$$J_{35} = \int_{1/2}^1 \int_y^1 \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \times \int_0^{1-z} (\tilde{P}(1-x) - \tilde{P}(1-x-y) - \tilde{P}(1-x-z))^2 x^{k-2}\, dx\, dz\, dy, \tag{5.40}$$

$$J_{36} = \int_0^{1/2} \int_{1-y}^1 \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \times \int_0^{1-z} (\tilde{P}(1-x) - \tilde{P}(1-x-y) - \tilde{P}(1-x-z))^2 x^{k-2}\, dx\, dz\, dy, \tag{5.41}$$

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Table 3: Bounds for $\Omega(\Pi(n))$.

k	Bound on $\Omega(\Pi(n))$	Polynomial $P(x)$
3	8.220...	$1+14x$
4	11.653...	$1+22x$
5	15.306...	$1+33x$
6	18.936...	$1+10x+40x^2$
7	22.834...	$1+10x+60x^2$
8	26.860...	$1+10x+80x^2$
9	30.942...	$1+30x+300x^3$
10	35.158...	$1+35x-10x^2+400x^3$

$$J_{37}=\int_0^{1/2}\int_y^{1-y}\frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))}\times\int_{1-y-z}^{1-z}(\tilde{P}(1-x)-\tilde{P}(1-x-y)-\tilde{P}(1-x-z))^2x^{k-2}\,dx\,dz\,dy, \tag{5.42}$$

$$J_{38}=\int_0^{1/2}\int_y^{1-y}\frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))}\int_0^{1-y-z}(\tilde{P}(1-x)-\tilde{P}(1-x-y)-\tilde{P}(1-x-z)+\tilde{P}(1-x-y-z))^2x^{k-2}\,dx\,dz\,dy. \tag{5.43}$$

We now have explicit representations of J, J_0, J_1, J_2 and J_3. We can calculate these by numerical integration given k and a polynomial P.

Table 3 gives close to optimal polynomials for $3\leqslant k\leqslant 10$ and the corresponding bounds obtained if we take ϵ sufficiently small. These give the results claimed in Theorem 2.1 except for $k=10$.

For $k=10$ we find an improvement if we also include the contribution when one of the $L_i(n)$ has four prime factors (we omit the explicit integrals here). In this case we choose the polynomial

$$P(x)=1+10x+150x^2. \tag{5.44}$$

This gives us the bound 34.77 . . . and so 10-tuples infinitely often have at most 34 prime factors, verifying Theorem 2.1.

§6. *The quantities T_δ and T_δ^*.* Before proving the propositions, we first establish some results about the quantities

$$T_\delta=\sum_{d,e}{}'\frac{\lambda_d\lambda_e}{f([d,e,\delta]/\delta)}, \tag{6.1}$$

$$T_\delta^*=\sum_{d,e}{}'\frac{\lambda_d\lambda_e}{f^*([d,e,\delta]/\delta)}. \tag{6.2}$$

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Here we use $[a_1, \dots, a_n]$ to denote the least common multiple of $a_1, \dots, a_n$, and we recall the definition of $\sum'$ from (4.8) as being restricted to square-free values coprime to A.

Most of these results already exist in some form in the literature. These results will underlie the proof of the propositions. We note that in [3] Graham *et al* used slightly different notation (there, our quantity T_δ^* is labelled T_δ).

We first put T_δ and T_δ^* into an almost-diagonalised form.

LEMMA 6.1. *We have*

$$T_\delta = \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|\delta} \mu(s) y_{as} \right)^2,$$

$$T_\delta^* = \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) y_{as}^* \right)^2,$$

where

$$y_a^* = \frac{\mu^2(a)a}{\phi(a)} \sum_m{}' \frac{y_{ma}}{\phi(m)}.$$

Proof. The result for T_δ is shown, for example, in [11, p. 85]. The result for T_δ^* is proven in [3, Lemma 6]. □

We now again quote a lemma from [3], which expresses the y_a^* in terms of the polynomial P which we used to define the variables y_a.

LEMMA 6.2. *Let*

$$y_a = \begin{cases} \mu^2(a)\mathfrak{S}(\mathcal{L}) P\left(\dfrac{\log R_2/a}{\log R_2} \right) & \text{if } 0 \leqslant a < R_2 \text{ and } (a, A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for $(a, A) = 1$ *and* $a < R_2$ *that*

$$y_a^* = \mu^2(a) \frac{\phi(A)}{A} \mathfrak{S}(\mathcal{L})(\log R_2) \tilde{P}\left(\frac{\log R_2/a}{\log R_2} \right) + O(\log\log R_2),$$

where

$$\tilde{P}(x) = \int_0^x P(t)\, dt.$$

If $(a, A) \neq 1$ *or* $a \geqslant R_2$ *then we have*

$$y_a^* = 0.$$

Proof. This is proven in [3, Lemma 7]. □

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We will repeatedly use the following result.

LEMMA 6.3. *For $u \geqslant 1$ we have*

$$\sum_{a<u}{}' \frac{\mu^2(a)}{f_1(a)} = \frac{(\log u)^k}{\mathfrak{S}(\mathcal{L})k!} + O((\log 2u)^{k-1}),$$
$$\sum_{a<u}{}' \frac{\mu^2(a)}{f_1^*(a)} = \frac{A}{\phi(A)} \frac{(\log u)^{k-1}}{\mathfrak{S}(\mathcal{L})(k-1)!} + O((\log 2u)^{k-2}).$$

Proof. This follows, for example, from [3, Lemma 3]. □

In order to estimate the terms T_δ^* we wish to remove the condition $(a, \delta) = 1$ in the summation over a, and remove the constraint caused by y_a and y_a^* only being supported on square-free a. We let

$$P_a = \begin{cases} \mathfrak{S}(\mathcal{L}) P\left(\dfrac{\log R_2/a}{\log R_2}\right) & \text{if } 0 \leqslant a < R_2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.3}$$

$$P_a^* = \begin{cases} \dfrac{\phi(A)}{A} \mathfrak{S}(\mathcal{L})(\log R_2) \tilde{P}\left(\dfrac{\log R_2/a}{\log R_2}\right) & \text{if } 0 \leqslant a < R_2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.4}$$

so that these are equal to y_a and $y_a^* + O(\log\log R_2)$, respectively, when a is square-free and coprime to A.

LEMMA 6.4. *Let $(\delta, A) = 1$. Then we have*

$$T_\delta = \sum_a{}' \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|\delta} \mu(s) P_{as}^*\right)^2 + O(d(\delta)^2 (\log R_2)^{k-1} \log\log R_2),$$
$$T_\delta^* = \sum_a{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) P_{as}^*\right)^2 + O(d(\delta)^2 (\log R_2)^{k} \log\log R_2).$$

Proof. We only prove the result for the T_δ^* here, the result for the T_δ follows from a completely analogous argument. We see that since $P_a^* \ll \log R_2$ we have

$$\begin{aligned} T_\delta^* &= \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) P_{as}^* + O(\log\log R_2)\right)^2 \\ &= \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) P_{as}^*\right)^2 \\ &\quad + O\left(d(\delta)^2 (\log R_2)(\log\log R_2) \sum_{a<R_2} \frac{\mu^2(a)}{f_1^*(a)}\right). \end{aligned} \tag{6.5}$$

By Lemma 6.3 the error term above is $O(d(\delta)^2 (\log R_2)^k \log\log R_2)$.

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We see that to prove the result it is sufficient to prove

$$\sum_{\substack{a\\(a,\delta)\neq 1}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 \ll (\log R_2)^k d(\delta)^2(\log\log R_2). \tag{6.6}$$

Since all terms in the sum are non-negative, we have

$$\sum_{\substack{a\\(a,\delta)\neq 1}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 \leqslant \sum_{p|\delta}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2. \tag{6.7}$$

We consider the inner sum. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 &= \sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta/p}\mu(s)(P^*_{as}-P^*_{asp})\right)^2\\ &\ll d(\delta)\sum_{s|\delta/p}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}(P^*_{as}-P^*_{asp})^2. \end{aligned}\tag{6.8}$$

We split the summation over a depending on whether the P^*_{as} and P^*_{asp} terms vanish (since $P^*_b = 0$ for $b \geqslant R_2$).

$$\begin{aligned}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 &\ll d(\delta)\sum_{s|\delta/p}\sum_{a'<R_2/sp^2}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}(P^*_{a'ps}-P^*_{a'sp^2})^2\\ &\quad + d(\delta)\sum_{s|\delta/p}\sum_{R_2/sp^2\leqslant a'<R_2/sp}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}(P^*_{a'ps})^2. \end{aligned}\tag{6.9}$$

We substitute in the value of P^*:

$$\begin{aligned}&\frac{1}{d(\delta)}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|q}\mu(s)P^*_{as}\right)^2\\ &\ll (\log R_2)^2\sum_{s|\delta/p}\sum_{a'<R_2/sp^2}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}\left(\tilde{P}\left(1-\frac{\log a'ps}{\log R_2}\right)\right.\\ &\quad\left.-\tilde{P}\left(1-\frac{\log a'sp^2}{\log R_2}\right)\right)^2\\ &\quad+(\log R_2)^2\sum_{s|\delta/p}\sum_{R_2/sp^2\leqslant a'<R_2/sp}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}\tilde{P}\left(1-\frac{\log a'ps}{\log R_2}\right)^2. \end{aligned}\tag{6.10}$$

In the first sum above both the arguments of the polynomials differ by $\log p/\log R_2$. Since they are fixed polynomials on a bounded range, the

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derivative of the polynomial is $\ll 1$ and so the difference is $\ll \log p/\log R_2$. In the second sum we just use the trivial bound $\tilde{P}(x) \ll 1$.

This gives

$$\frac{1}{d(\delta)}\sum_{\substack{a\\ p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\Big(\sum_{s|q}\mu(s)P_{as}^*\Big)^2 \ll \frac{(\log p)^2}{f_1^*(p)}\sum_{s|\delta/p}\sum_{a<R_2/sp^2}{}' \frac{\mu^2(a)}{f^*(a)} + \frac{(\log R_2)^2}{f(p)}\sum_{s|\delta/p}\sum_{R_2/sp^2\leqslant a<R_2/sp}{}' \frac{\mu^2(a)}{f^*(a)}. \tag{6.11}$$

Using Lemma 6.3 we see that the first sum is $\ll d(\delta)(\log p)^2(\log R_2)^{k-1}/f_1^*(p)$ and the second sum is $\ll d(\delta)(\log p)(\log R_2)^k/f_1^*(p)$ because of the range of summation over a. Thus,

$$\sum_{\substack{a\\ p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\Big(\sum_{s|\delta}\mu(s)P_{as}^*\Big)^2 \ll d(\delta)^2\frac{\log p}{f_1^*(p)}(\log R_2)^k \ll d(\delta)^2\frac{\log p}{p}(\log R_2)^k. \tag{6.12}$$

Summing over all $p|\delta$ gives the bound

$$d(\delta)^2(\log R_2)^k\sum_{p|\delta}\frac{\log p}{p}. \tag{6.13}$$

Splitting the sum into a sum over $p \leqslant \log R_2$ and a sum over $p > \log R_2$ we get the bound

$$d(\delta)^2(\log R_2)^k(\log\log R_2). \tag{6.14}$$

This gives (6.6), and hence the lemma. □

Essentially the same argument as above also yields a useful bound on the size of T_δ and T_δ^*.

LEMMA 6.5. *Let* $(\delta, A) = 1$. *Then we have*

$$T_\delta \ll \min_{p|\delta}(\log p)d(\delta)^2(\log R_2)^{k-1},$$
$$T_\delta^* \ll \min_{p|\delta}(\log p)d(\delta)^2(\log R_2)^k + d(\delta)^2(\log R_2)^k\log\log R_2.$$

Proof. For $p|\delta$ we have (using that all terms are non-negative)

$$T_\delta^* = \sum_{\substack{a\\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1^*(a)}\Big(\sum_{s|\delta}\mu(s)P_{as}^*\Big)^2 + O(d(\delta)^2(\log R_2)^k\log\log R_2)$$
$$\ll d(\delta)\sum_a{}' \frac{\mu^2(a)}{f_1^*(a)}\sum_{s|\delta/p}(P_{as}^* - P_{asp}^*)^2 + d(\delta)^2(\log R_2)^k\log\log R_2$$

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$$\ll d(\delta)(\log R_2)^2 \sum_{s|\delta/p} \sideset{}{'}\sum_{a<R_2/sp} \frac{\mu^2(a)}{f_1^*(a)} \left(\tilde{P}\left(1 - \frac{\log as}{\log R_2}\right) - \tilde{P}\left(1 - \frac{\log asp}{\log R_2}\right)\right)^2$$
$$+ d(\delta)(\log R_2)^2 \sum_{s|\delta/p} \sideset{}{'}\sum_{R_2/sp\leqslant a<R_2/s} \frac{\mu^2(a)}{f_1^*(a)} \tilde{P}\left(1 - \frac{\log as}{\log R_2}\right)^2$$
$$+ d(\delta)^2(\log R_2)^k \log\log R_2. \tag{6.15}$$

Noting the difference of the polynomials in the first sum is $\ll \log p/\log R_2$, and the polynomial in the second sum is $\ll 1$, we have

$$T_\delta^* \ll d(\delta)(\log p)^2 \sum_{s|\delta/p} \sideset{}{'}\sum_{a<R_2/sp} \frac{\mu^2(a)}{f_1^*(a)}$$
$$+ d(\delta)(\log R_2)^2 \sum_{s|\delta/p} \sideset{}{'}\sum_{R_2/sp\leqslant a<R_2/s} \frac{\mu^2(a)}{f_1^*(a)}$$
$$+ d(\delta)^2(\log R_2)^k \log\log R_2. \tag{6.16}$$

Appealing to Lemma 6.3 as in the previous lemma we obtain

$$T_\delta^* \ll d(\delta)^2(\log p)(\log R_2)^k + d(\delta)^2(\log R_2)^k \log\log R_2. \tag{6.17}$$

The result for T_δ follows by a completely analogous argument. In this case the first line holds without the $O(d(\delta)^2(\log R_2)^k \log\log R_2)$ term, and so the final expression also holds without this term. □

With these results we are able to get an integral expression for T_δ and T_δ^* when δ has a bounded number of prime factors.

LEMMA 6.6. *Let $p_1, \dots, p_{r-1} \nmid A$ for some primes $p_1, \dots, p_{r-1}$. Then we have*

$$T_{p_1\cdots p_{r-1}} = (\log R_2)^k \frac{\mathfrak{S}(\mathcal{L})}{(k-1)!} I_0\left(\frac{\log p_1}{\log R_2}, \dots, \frac{\log p_{r-1}}{\log R_2}\right)$$
$$+ O_r((\log R_2)^{k-1} \log\log R_2),$$
$$T^*_{p_1\cdots p_{r-1}} = (\log R_2)^{k+1} \frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!} I_1\left(\frac{\log p_1}{\log R_2}, \dots, \frac{\log p_{r-1}}{\log R_2}\right)$$
$$+ O_r((\log R_2)^k \log\log R_2).$$

Here

$$I_0(x_1, \dots, x_{r-1}) = \int_0^1 \left(\sum_{J\subset\{1,\dots,r-1\}} P^+\left(1 - t - \sum_{j\in J} x_j\right)(-1)^{|J|}\right)^2 t^{k-1}\, dt,$$

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$$I_1(x_1, \dots, x_{r-1}) = \int_0^1 \left(\sum_{J \subset \{1,\dots,r-1\}} \tilde{P}^+\left(1 - t - \sum_{j \in J} x_j\right)(-1)^{|J|} \right)^2 t^{k-2}\, dt,$$

$$P^+(x) = \begin{cases} P(x) & \text{if } x \geqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{P}^+(x) = \begin{cases} \displaystyle\int_0^x P(t)\, dt & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\delta = p_1 \cdots p_{r-1}$.

By Lemmas 6.1 and 6.4 we have that

$$T_\delta^* = \sum_a{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) P_{as}^* \right)^2 + O_r((\log R_2)^k \log\log R_2). \tag{6.18}$$

Substituting in the value P_a^* (and noting that $(\delta, A) = 1$) we obtain

$$T_\delta^* = \frac{\phi(A)^2}{A^2} (\log R_2)^2 \mathfrak{S}(\mathcal{L})^2 \sum_{(a,A)=1} \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) \tilde{P}^+\left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 + O_r((\log R_2)^k \log\log R_2). \tag{6.19}$$

We again use Lemma 6.3 which shows that

$$\sum_{\substack{a \leqslant R_2 \\ (a,A)=1}} \frac{\mu^2(a)}{f_1^*(a)} \ll (\log R_2)^{k-1}. \tag{6.20}$$

Thus,

$$T_{p_1 \cdots p_{r-1}}^* = \frac{A^2}{\phi(A)^2} (\log R_2)^2 \mathfrak{S}(\mathcal{L})^2 \sum_{(a,A)=1} \frac{\mu^2(a)}{f_1^*(a)} \times \left(\sum_{s|p_1\cdots p_{r-1}} \mu(s) \tilde{P}^+\left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 + O_r((\log R_2)^k (\log\log R_2)). \tag{6.21}$$

We also have

$$T_{p_1 \cdots p_{r-1}} = \mathfrak{S}(\mathcal{L})^2 \sum_{(a,A)=1} \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|p_1\cdots p_{r-1}} \mu(s) P^+\left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 + O_r((\log R_2)^{k-1}). \tag{6.22}$$

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We can now estimate the main term using [3, Lemma 4]. First we put

$$\gamma(p) = \begin{cases} \dfrac{p(k-1)}{p-1} & \text{if } p \nmid A, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(d) = \prod_{p \mid d} \frac{\gamma(p)}{p - \gamma(p)}, \tag{6.23}$$

$$F(t) = F_{x_1,\dots,x_{r-1}}(t) = \sum_{J \subset \{1,\dots,r-1\}} (-1)^{|J|} \tilde{P}^+\Big(t - \sum_{j \in J} x_j\Big).$$

We note that by our normalisation, $\gamma(p) < p$ for all p, so $g(d)$ is always positive and well-defined.

If we put $x_i = \log p_i / \log R_2$ for each $i \in \{1, \dots, r-1\}$ then we see that

$$\sum_{(a,A)=1} \frac{\mu^2(a)}{f_1^*(a)} \Big(\sum_{s \mid p_1 \cdots p_{r-1}} \mu(s) \tilde{P}^+\Big(\frac{\log R_2/as}{\log R_2}\Big)\Big)^2 = \sum_{d \leqslant R_2} \mu^2(d) g(d) F\Big(\frac{\log R_2/d}{\log R_2}\Big). \tag{6.24}$$

Since F is a continuous piecewise differentiable function we can apply [3, Lemma 4] which gives

$$\sum_{d \leqslant R_2} \mu^2(d) g(d) F\Big(\frac{\log R_2/d}{\log R_2}\Big) = \frac{A}{\phi(A)} \frac{(\log R_2)^{k-1}}{\mathfrak{S}(\mathcal{L})(k-2)!} \int_0^1 F(1-t) t^{k-2}\, dt + O((\log R_2)^{k-2}). \tag{6.25}$$

Similarly, we follow the same procedure instead with

$$\gamma(p) = \begin{cases} k & \text{if } p \nmid A, \\ 0 & \text{otherwise,} \end{cases}$$

$$G(t) = \sum_{J \subset \{1,\dots,r-1\}} (-1)^{|J|} P^+\Big(t - \sum_{j \in J} x_j\Big). \tag{6.26}$$

This yields

$$\sum_{(a,A)=1} \frac{\mu^2(a)}{f_1(a)} \Big(\sum_{s \mid p_1 \cdots p_{r-1}} \mu(s) P^+\Big(\frac{\log R_2/as}{\log R_2}\Big)\Big)^2 = \frac{(\log R_2)^k}{\mathfrak{S}(\mathcal{L})(k-1)!} \int_0^1 G(1-t)^2 t^{k-1}\, dt + O((\log R_2)^{k-1}). \qquad \square$$

We also require a bound on the size of the sieve coefficients λ_d.

LEMMA 6.7. *We have that*

$$\lambda_d \ll (\log R_2)^k.$$

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Proof. This is proved in [3, Proof of Theorem 7]. □

We finish this section with a partial summation lemma, which will be useful later on.

LEMMA 6.8. *Let $0 \leqslant a < b$ be fixed constants. Let $V : [a, b] \to \mathbb{R}_{\geqslant 0}$ be a continuous piecewise smooth function. If V satisfies $V(x) \ll x$ uniformly for $x \in [a, b]$ then we have*

$$\sum_{R^a \leqslant p \leqslant R^b} \frac{1}{p} V\left(\frac{\log p}{\log R}\right) = \int_a^b \frac{V(u)}{u}\, du + O\left(\frac{M(V)\log\log R}{\log R}\right),$$

where

$$M(V) = \sup_{t\in[a,b]} (1 + |V'(t)|).$$

Proof. The result follows straightforwardly by partial summation and the prime number theorem.

If $a = 0$ then we replace a with $2/\log R$. This leaves the left-hand side of the result unchanged, and introduces an error

$$\int_a^{2/\log R} \frac{V(u)}{u}\, du \ll \frac{1}{\log R} \tag{6.27}$$

to the right-hand side, which can be absorbed into the error term.

By the prime number theorem,

$$\pi(y) = \frac{y}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right). \tag{6.28}$$

Therefore, by partial summation we have

$$\begin{aligned}
&\sum_{R^a \leqslant p_j \leqslant R^b} \frac{1}{p} V\left(\frac{\log p}{\log R}\right)\\
&= O\left(\frac{1}{\log R}\right) + \int_{R^a}^{R^b} \frac{t}{t^2 \log t} V\left(\frac{\log t}{\log R}\right)\left(1 + O\left(\frac{1}{\log t}\right)\right) dt\\
&\quad - \int_{R^a}^{R^b} \frac{t}{t^2(\log t)(\log R)} V'\left(\frac{\log t}{\log R}\right)\left(1 + O\left(\frac{1}{\log t}\right)\right) dt\\
&= \int_a^b \frac{V(u)}{u}\, du + O\left(\int_a^b \frac{1 + |V'(u)|}{u \log R}\, du\right) + O\left(\frac{1}{\log R}\right)\\
&= \int_a^b \frac{V(u)}{u}\, du + O\left(\frac{M(V)\log\log R}{\log R}\right).
\end{aligned}$$

□

§7. *Proof of Proposition 5.1.* We consider the weighted sum of Proposition 5.1 in a similar way to previous work on Selberg's $\Lambda^2\Lambda^-$ sieve which in

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its basic form considers the weight $W_0(x) = -1$:

$$\begin{aligned}
&\sum_{N\leqslant n\leqslant 2N}\Bigg(\sum_{\substack{p\mid\Pi(n)\\ p\leqslant R_1}} W_0\bigg(\frac{\log p}{\log R_2}\bigg)\Bigg)\Bigg(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}}\lambda_d\Bigg)^2\\
&\quad=\sum_{p\leqslant R_1} W_0\bigg(\frac{\log p}{\log R_2}\bigg)\sum_{d,e\leqslant R_2}\lambda_d\lambda_e\sum_{\substack{N\leqslant n\leqslant 2N\\ [p,d,e]\mid\Pi(n)}}1\\
&\quad=N\sum_{p\leqslant R_1}{}' W_0\bigg(\frac{\log p}{\log R_2}\bigg)\sum_{d,e\leqslant R_2}{}'\frac{\lambda_d\lambda_e}{f([d,e,p])}+O_{W_0}(E_1)\\
&\quad=N\sum_{p\leqslant R_1}{}' W_0\bigg(\frac{\log p}{\log R_2}\bigg)\frac{T_p}{f(p)}+O_{W_0}(E_1),
\end{aligned}\tag{7.1}$$

where

$$E_1=\sum_{p\leqslant R_1}\sum_{d,e\leqslant R_2}|\lambda_d\lambda_e r_{[d,e,p]}|,\qquad r_d=\sum_{\substack{N<n\leqslant 2N\\ d\mid\Pi(n)}}1-\frac{N}{f(d)}.\tag{7.2}$$

By Lemma 6.7 we have $\lambda_d\ll(\log N)^k$, and we note that $r_d\leqslant k^{\omega(d)}$. Therefore, we have

$$\begin{aligned}
E_1&\ll(\log N)^{2k}\sum_{\substack{p\leqslant R_1\\ d,e\leqslant R_2}}\mu^2([d,e,p])k^{\omega([d,e,p])}\\
&\ll(\log N)^{2k}\sum_{r\leqslant R_2^2R_1}\mu^2(r)(7k)^{\omega(r)}\\
&\ll(\log N)^{2k}R_2^2R_1\sum_{r\leqslant R_2^2R_1}\frac{\mu^2(r)(7k)^{\omega(r)}}{r}\\
&\ll(\log N)^{2k}R_2^2R_1\prod_{p\leqslant R_2^2R_1}\bigg(1+\frac{7k}{p}\bigg)\\
&\ll(\log N)^{9k}R_2^2R_1.
\end{aligned}\tag{7.3}$$

Thus, for $R_2^2R_1\leqslant N(\log N)^{-9k}$ we have $E_1\ll N$.

By Lemma 6.6 we have

$$T_p=(\log R_2)^k\frac{\mathfrak{S}(\mathcal{L})}{(k-1)!}I_0\bigg(\frac{\log p}{\log R_2}\bigg)+O((\log N)^{k-1}\log\log N),\tag{7.4}$$

where

$$I_0(x)=\int_0^1(P_1^+(1-t)-P_1^+(1-t-x))^2t^{k-1}\,dt.\tag{7.5}$$

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Recalling that $f(p) = p/k$ for $p \nmid A$, we see that the error terms from T_p contribute

$$\ll_{W_0} (\log N)^{k-1} \log\log N \sum_{p \leqslant R_1} \frac{1}{p} \ll (\log N)^{k-1} (\log\log N)^2. \tag{7.6}$$

Therefore, we are left to estimate the sum

$$\sum_{p \leqslant R_1}{}' \frac{1}{p} W_0\left(\frac{\log p}{\log R_2}\right) I_0\left(\frac{\log p}{\log R_2}\right). \tag{7.7}$$

We note that if $t \leqslant 1 - x$ then $P^+(1-t) - P^+(1-t-x) \ll x$, and so

$$I_0(x) \ll x. \tag{7.8}$$

If $1 - x \leqslant t \leqslant 1$ then since the interval has length x we also have

$$I_0(x) \ll x. \tag{7.9}$$

By the piecewise smoothness of $I_0(x)$ and $W_0(x)$ we have uniformly for $x \in [0, r_1/r_2]$

$$I_0'(x) \ll 1, \qquad W_0'(x) \ll_{W_0} 1. \tag{7.10}$$

Therefore, by Lemma 6.8, we have

$$\sum_{p \leqslant R_1} \frac{1}{p} W_0\left(\frac{\log p}{\log R_2}\right) I_0\left(\frac{\log p}{\log R_2}\right) = \int_0^{r_1/r_2} \frac{W_0(u)}{u} I_0(u)\, du + O_{W_0}\left(\frac{\log\log N}{\log N}\right). \tag{7.11}$$

By (7.8) we see that the contribution to the above sum for primes which divide A is

$$\ll \frac{1}{\log N}. \tag{7.12}$$

This gives the result.

§8. *Proof of Proposition 5.2.* We will follow a similar argument to that of Graham *et al* [**3**], where the result was obtained with $r = 2$ and $W_2(x_1, x_2) = 1$. Thorne [**12**] extended this in the natural way to consider $r > 2$, again without the weighting W_r. In order to introduce the weighting by W_r, it is necessary to establish a Bombieri–Vinogradov style result for numbers with r prime factors weighted by W_r.

LEMMA 8.1. *Let*

$$\beta_r(n) = \begin{cases} W_r\left(\dfrac{\log p_1}{\log n}, \dots, \dfrac{\log p_{r-1}}{\log n}\right) & \text{if } n = p_1 p_2 \cdots p_r \text{ with } p_1 \leqslant \cdots \leqslant p_r, \\ 0 & \text{otherwise,} \end{cases}$$

for some piecewise smooth function $W_r : [0,1]^{r-1} \to \mathbb{R}$.

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Let

$$\Delta_{\beta,r}(x;\,q)=\max_{y\leqslant x}\max_{\substack{a\\(a,q)=1}}\Bigg|\sum_{\substack{y<n\leqslant 2y\\ n\equiv a\ (\mathrm{mod}\ q)}}\beta_r(n)-\frac{1}{\phi(q)}\sum_{\substack{y<n\leqslant 2y\\(n,q)=1}}\beta_r(n)\Bigg|. \tag{8.1}$$

For every fixed integer $h>0$ and for every $C>0$, there exists a constant $C'=C'(C,h)$ such that if $Q\leqslant x^{1/2}(\log x)^{-C'}$ then we have

$$\sum_{q\leqslant Q}\mu^2(q)h^{\omega(q)}\Delta_{\beta,r}(x;\,q)\ll_{C,h,W_r} x(\log x)^{-C}. \tag{8.2}$$

Proof. This result follows from the Bombieri–Vinogradov theorem for numbers with exactly r prime factors, as proved by Motohashi [8], and the continuity of W_r.

We assume that W_r is smooth. The result can be extended to piecewise smooth functions by taking smooth approximations.

We fix a constant $C>0$, an integer h, and a function W_r.

We let

$$\chi_{\delta,\eta}(n)=\begin{cases}1 & \text{if } n=p_1p_2\cdots p_r \text{ with } n^{\eta_i}\leqslant p_i\leqslant n^{\delta_i}\ \forall i\\ & \quad\text{and } p_1<p_2<\cdots<p_r,\\ 0 & \text{otherwise.}\end{cases} \tag{8.3}$$

By Motohashi's result [8, Theorem 2] we have that uniformly for any choice of constants δ_i and η_i $(i=1,\dots,r)$ there is a constant $C'=C'(C,h)$ such that if $Q\leqslant x^{1/2}(\log x)^{-C'}$ then we have

$$\sum_{q\leqslant Q}\mu^2(q)h^{\omega(q)}\max_{\substack{y,a\\ y\leqslant x\\(a,q)=1}}\Bigg|\sum_{\substack{y\leqslant n\leqslant 2y\\ n\equiv a\ (\mathrm{mod}\ q)}}\chi_{\delta,\eta}(n)-\frac{1}{\phi(q)}\sum_{\substack{y\leqslant n\leqslant 2y\\(n,q)=1}}\chi_{\delta,\eta}(n)\Bigg|$$
$$\ll_{C,h} x(\log x)^{-(C+h)(r+1)}. \tag{8.4}$$

We choose $\delta_i\in\{(\log x)^{-C-h},2(\log x)^{-C-h},\dots,\lceil(\log x)^{C+h}\rceil(\log x)^{-C-h}\}$ separately for each $i\in\{1,\dots,r\}$, subject to the constraint $\delta_i\leqslant\delta_{i+1}$ $(1\leqslant i\leqslant r-1)$. For each choice of the δ_i we take $\eta_i=\delta_i-(\log x)^{-C-h}$ for $1\leqslant i\leqslant r$. We put

$$W_r(\delta)=W_r(\delta_1,\delta_2,\dots,\delta_{r-1}). \tag{8.5}$$

We notice that by the smoothness of W_r we have that

$$\begin{aligned}\beta_r(n)&=\sum_{\delta}\chi_{\delta,\eta}(n)(W_r(\delta)+O((\log x)^{-C-h}))\\&=\sum_{\delta}\chi_{\delta,\eta}(n)W_r(\delta)+O((\log x)^{-C-h}).\end{aligned} \tag{8.6}$$

Here $\sum_\delta$ indicates a sum over all the $O((\log x)^{r(C+h)})$ possible choices of the δ_i.

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Therefore, we have that

$$\sum_{\substack{y\leqslant n\leqslant 2y\\ n\equiv a \pmod q}}\beta_r(n)-\frac{1}{\phi(q)}\sum_{\substack{y\leqslant n\leqslant 2y\\ (n,q)=1}}\beta_r(n)$$
$$=\sum_{\delta}W_r(\delta)\Bigg(\sum_{\substack{y\leqslant n\leqslant 2y\\ n\equiv a \pmod q}}\chi_{\delta,\eta}(n)-\frac{1}{\phi(q)}\sum_{\substack{y\leqslant n\leqslant 2y\\ (n,q)=1}}\chi_{\delta,\eta}(n)\Bigg)$$
$$+O\Bigg((\log y)^{-C-h}\frac{y}{\phi(q)}\Bigg). \tag{8.7}$$

Thus, for $Q\leqslant x(\log x)^{-C'}$ we have

$$\sum_{q\leqslant Q}\mu^2(r)h^{\omega(q)}\Delta_{\beta,r}(x;q)$$
$$\leqslant\sum_{\delta}W_r(\delta)\sum_{q\leqslant Q}\mu^2(r)h^{\omega(q)}$$
$$\times\max_{\substack{a,y\\ y\leqslant x\\ (a,q)=1}}\Bigg|\sum_{\substack{y\leqslant n\leqslant 2y\\ n\equiv a \pmod q}}\chi_{\delta,\eta}(n)-\frac{1}{\phi(q)}\sum_{\substack{y\leqslant n\leqslant 2y\\ (n,q)=1}}\chi_{\delta,\eta}(n)\Bigg|$$
$$+O\Bigg((\log x)^{-(C+h)}\sum_{q\leqslant Q}\mu^2(q)h^{\omega(q)}\frac{x}{\phi(q)}\Bigg)$$
$$\ll\sum_{\delta}W_r(\delta)x(\log x)^{-(C+h)(r+1)}+x(\log x)^{-(C+h)}\prod_{p\leqslant Q}\Bigg(1+\frac{h}{p-1}\Bigg)$$
$$\ll x(\log x)^{-C}. \qquad \square$$

With this, we can adapt the argument of Thorne [12] slightly to rewrite the main term in terms of the quantities T_q^*.

LEMMA 8.2. *We have*

$$\sum_{N\leqslant n\leqslant 2N}\beta_r(L_j(n))\Bigg(\sum_{d\mid\Pi(n)}\lambda_d\Bigg)^2$$
$$=\frac{AN}{\phi(A)(\log N)}\sum_{\substack{p_1,\dots,p_{r-1}\\ N^{\epsilon}<p_1<p_2<\cdots<p_{r-1}\\ q<\min(N/R_2,N/p_{r-1})}}\frac{T_q^*}{q}\alpha\Bigg(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\Bigg)$$
$$+O_{W_r}(N(\log N)^{k-1}(\log\log N)^{r-1}),$$

where

$$q=\prod_{i=1}^{r-1}p_i,$$

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$$T_\delta^* = \sum_{\substack{d,e \\ (d,A)=(e,A)=1}} \frac{\lambda_d \lambda_e}{f^*([d,e,\delta]/\delta)},$$

$$\alpha(x_1, \dots, x_{r-1}) = \frac{W_r(r_2 x_1, \dots, r_2 x_{r-1})}{1 - r_2 \sum_{i-1}^{r-1} x_i}.$$

Proof. Thorne [12] considers essentially the same sum but without the weighting by W_r. In his argument up until equation (4.14) on p. 23, this difference only affects the argument when he appeals to the Bombieri–Vinogradov theorem for E_h numbers (where $h \leqslant r$). Lemma 8.1 gives the equivalent Bombieri–Vinogradov style result when weighting by W_r, and so exactly the same argument follows through. The only additional assumption of Thorne is that he restricts the consideration to numbers $n = p_1 \cdots p_r$ satisfying

$$\exp(\sqrt{\log N}) < p_1 < \cdots < p_r \quad \text{and} \quad R_2 < p_r. \tag{8.8}$$

This is satisfied if for a fixed $\epsilon > 0$ we require W_r to be supported on

$$\mathcal{A}_r = \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1 - r_2, 1 - x_{r-1}) \right\}. \tag{8.9}$$

This gives us in our case (the equivalent of Thorne's equation (4.14) but with the explicit error term he calculates)

$$\begin{aligned} &\sum_{N \leqslant n \leqslant 2N} \beta_r(L_j(n)) \Bigg(\sum_{d \mid \Pi(n)} \lambda_d \Bigg)^2 \\ &\quad = \sum_{d,e}{}' \lambda_d \lambda_e \sum_{p_1, \dots, p_{r-1}}{}^* \frac{d_{k-1}([d,e,q]/q)}{\phi(a_j[d,e,q]/q)} \\ &\qquad \times \sum_{\substack{a_j N/q \leqslant m \leqslant 2a_j N/q \\ m \text{ prime}}} W_r\left(\frac{\log p_1}{\log mq}, \dots, \frac{\log p_{r-1}}{\log mq} \right) \\ &\qquad + O(N). \end{aligned} \tag{8.10}$$

Here and from now on we use the symbol $\sum^*$ to indicate that we are summing over primes $p_1, \dots, p_{r-1}$ with

$$\left(\frac{\log p_1}{\log N}, \dots, \frac{\log p_{r-1}}{\log N} \right) \in \mathcal{A}_r, \tag{8.11}$$

and to ease notation we have put

$$q = \prod_{i=1}^{r-1} p_i. \tag{8.12}$$

Again we assume for simplicity that W_r is smooth. By taking smooth approximations one can establish the result for piecewise-smooth W_r.

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Estimating the inner sum gives

$$\sum_{\substack{a_jN/q\leqslant m\leqslant 2a_jN/q\\ m \text{ prime}}} W_r\left(\frac{\log p_1}{\log mq},\dots,\frac{\log p_{r-1}}{\log mq}\right)$$
$$=\left(W_r\left(\frac{\log p_1}{\log N},\dots,\frac{\log p_{r-1}}{\log N}\right)+O\left(\frac{1}{\log N}\right)\right)$$
$$\times\left(\pi\left(\frac{2a_jN}{q}\right)-\pi\left(\frac{a_jN}{q}\right)\right)$$
$$=W_r\left(\frac{\log p_1}{\log N},\dots,\frac{\log p_{r-1}}{\log N}\right)\frac{a_jN}{\log N}\left(\frac{\log N}{\log N-\log q}\right)$$
$$\times\left(1+O\left(\frac{1}{\log N}\right)\right). \tag{8.13}$$

We note that by Hypothesis 1 if $d|\Pi(n)$ then $(d,A)=1$. Therefore, $(a_j,[d,e,q]/q)=1$, so $\phi(a_j[d,e,q]/q)=\phi(a_j)\phi([d,e,q]/q)$. Together these give

$$\sum_{N\leqslant n\leqslant 2N}\beta_r(L_j(n))\left(\sum_{d|\Pi(n)}\lambda_d\right)^2$$
$$=\frac{a_jN}{\phi(a_j)\log N}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{T_q^*W_r(\log p_1/\log N,\dots,\log p_{r-1}/\log N)\log N}{q(\log N-\log q)}$$
$$\times(1+O((\log N)^{-1}))+O(N)$$
$$=\frac{a_jN}{\phi(a_j)\log N}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{T_q^*}{q}\alpha\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)\left(1+O\left(\frac{1}{\log N}\right)\right)$$
$$+O(N), \tag{8.14}$$

where

$$\alpha(x_1,\dots,x_{r-1})=\frac{W_r(r_2x_1,\dots,r_2x_{r-1})}{1-r_2\sum_{i-1}^{r-1}x_i}. \tag{8.15}$$

We note that a_j and A are composed of the same prime factors, so $a_j/\phi(a_j)=A/\phi(A)$. Therefore, the main term is that of the lemma.

By Lemma 6.5 we have

$$T_q^*\ll_r(\log N)^k\log p_1+(\log N)^k\log\log N. \tag{8.16}$$

We also have

$$\alpha(x_1,\dots,x_{r-1})\ll_{W_r}1. \tag{8.17}$$

Thus, the $O(1/\log N)$ term contributes

$$\ll_{W_r,r}N(\log N)^{k-2}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{\log p_1+\log\log N}{p_1\cdots p_{r-1}}$$
$$\ll_{W_r,r}N(\log N)^{k-1}(\log\log N)^{r-2}. \tag{8.18}$$

This gives the result. □

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LEMMA 8.3. *We have*

$$\sum_{p_1,\dots,p_{r-1}}^{*} \frac{T_q^*}{q}\alpha\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)$$
$$= (\log R_2)^{k+1}\frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!}\int\cdots\int \frac{I_1(u_1,\dots,u_{r-1})\alpha(u_1,\dots,u_{r-1})}{u_1u_2\cdots u_{r-1}}$$
$$\times du_1\cdots du_{r-1} + O((\log\log N)^r(\log N)^k), \tag{8.19}$$

where the integration is subject to the constraints

$$\epsilon < u_1 < \cdots < u_{r-1} \quad \textit{and} \quad \sum_{i=1}^{r-1} u_i \leqslant \min(r_2^{-1}-1, r_2^{-1}-u_{r-1}). \tag{8.20}$$

Proof. By Lemma 6.6 for $q = p_1p_2\cdots p_{r-1}$ we have

$$T_q^* = (\log R_2)^{k+1}\frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!}I_1\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)$$
$$+ O_r((\log N)^k\log\log N). \tag{8.21}$$

Thus, summing the error term over $p_1,\dots,p_{r-1}$ gives a contribution

$$\sum_{p_1,\dots,p_{r-1}}^{*}\frac{1}{q}\alpha\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)(\log N)^k\log\log N$$
$$\ll_{W_r} (\log N)^k\log\log N\left(\sum_{p\leqslant N}\frac{1}{p}\right)^{r-1}$$
$$\ll_{W_r} (\log N)^k(\log\log N)^r. \tag{8.22}$$

We are therefore left to evaluate the main term

$$\sum_{p_1,\dots,p_{r-1}}^{*}\frac{1}{q}\alpha\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)I_1\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right). \tag{8.23}$$

We will now apply Lemma 6.8 to $p_{r-1},\dots,p_1$ in turn to estimate the sum (8.23).

For $u_1,\dots,u_j\in[0,r_2^{-1}]$ we put

$$V_j(u_1,\dots,u_j) = \int\cdots\int \frac{1}{\prod_{i=j+1}^{r-1}u_i}\alpha(u_1,\dots,u_{r-1})I_1(u_1,\dots,u_{r-1})$$
$$\times du_{j+1}\cdots du_{r-1}, \tag{8.24}$$

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where the integration is subject to

$$u_j < u_{j+1} < \cdots < u_{r-1} \quad \text{and} \quad \sum_{i=1}^{r-1} u_i \leqslant \min(r_2^{-1} - 1, r_2^{-1} - u_{r-1}).$$

As in the proof of Lemma 6.5, since $\tilde{P}$ is continuous and its derivative is uniformly bounded on [0, 1], we have that

$$\begin{aligned} I_1(u_1, \dots, u_{r-1}) &= \int_0^1 \Bigg(\sum_{J \subset \{1,\dots,r-1\}} \tilde{P}^+\Bigg(1 - t - \sum_{i \in J} u_i\Bigg)(-1)^{|J|}\Bigg)^2 t^{k-2}\, dt \\ &\ll_r \int_0^1 \sum_{J \subset \{1,\dots,r-1\} \setminus \{j\}} \Bigg(\tilde{P}^+\Bigg(1 - t - \sum_{i \in J} u_i\Bigg) \\ &\quad - \tilde{P}^+\Bigg(1 - t - u_j - \sum_{i \in J} u_i\Bigg)\Bigg)^2 t^{k-2}\, dt \\ &\ll_r u_j^2. \end{aligned} \tag{8.25}$$

Thus, since $\alpha(u_1, \dots, u_{r-1}) \ll 1$, we have uniformly for $u_1, \dots, u_j \in [0, r_2^{-1}]$

$$\begin{aligned} V_j(u_1, \dots, u_j) &\ll u_j^2 \int \cdots \int \frac{1}{\prod_{i=j+1}^{r-1} u_i} du_{j+1} \cdots du_{r-1} \\ &\ll u_j^2 (1 + |\log 1/u_j|^r) \\ &\ll u_j. \end{aligned} \tag{8.26}$$

Moreover, essentially the same argument shows that uniformly for $u_1, \dots, u_j \in [0, r_2^{-1}]$ we have

$$\frac{\partial}{\partial u_j} I_1(u_1, \dots, u_{r-1}) \ll_r u_j. \tag{8.27}$$

Thus, since

$$\frac{\partial}{\partial u_j} \alpha(u_1, \dots, u_{r-1}) \ll_r 1 \tag{8.28}$$

we have that

$$\begin{aligned} \frac{\partial}{\partial u_j} V_j(u_1, \dots, u_j) &\ll u_j \int \cdots \int \frac{1}{\prod_{i=j+1}^{r-1} u_i} du_{j+1} \cdots du_{r-1} \\ &\ll 1. \end{aligned} \tag{8.29}$$

Thus, the condition of Lemma 6.8 applies for the function V_j. Applying Lemma 6.8 in turn to $V_{r-1}, V_{r-2}, \dots, V_1$ gives the result. We note that the error terms contribute a total which is $\ll (\log N)^k (\log\log N)^{r-1}$. □

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§9. *Proof of Proposition 5.3.* By Lemma 6.7 we have $\lambda_d \ll (\log N)^k$. Therefore, we have

$$\begin{aligned}
&\sum_{p\leqslant AN^{1/2}} \sum_{\substack{N\leqslant n\leqslant 2N\\ p^2\mid\Pi(n)}} \Bigg(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}} \lambda_d\Bigg)^2 \\
&\quad = N \sum_{p\leqslant AN^{1/2}} \sum_{d,e\leqslant R_2}{}' \frac{\lambda_d\lambda_e}{f([d,\,e,\,p^2])} + O\Bigg(\sum_{p\leqslant AN^{1/2}} \sum_{d,e\leqslant R_2} |\lambda_d\lambda_e r_{[d,e,p^2]}|\Bigg) \\
&\quad \ll N \sum_{p\leqslant N} \frac{T_p}{p^2} + O\Bigg((\log N)^{2k} \sum_{r\leqslant R_2^2AN^{1/2}} \mu^2(r)(7k)^{\omega(r)}\Bigg). \qquad (9.1)
\end{aligned}$$

We first bound the error term

$$\begin{aligned}
\sum_{r\leqslant R_2^2AN^{1/2}} \mu^2(r)(7k)^{\omega(r)} &\ll R_2^2N^{1/2} \sum_{r\leqslant AR_2^2N^{1/2}} \frac{\mu^2(r)(7k)^{\omega(r)}}{r} \\
&\ll R_2^2N^{1/2} \prod_{p\leqslant AR_2^2N^{1/2}} \left(1+\frac{7k}{p}\right) \\
&\ll R_2^2N^{1/2}(\log N)^{7k}. \qquad (9.2)
\end{aligned}$$

Thus, for $R_2 \leqslant N^{1/4}(\log N)^{-5k}$ the error term is $O(N)$.

By Lemma 6.5 we have that

$$T_p \ll (\log N)^{k-1}\log p + (\log N)^{k-1}\log\log N. \qquad (9.3)$$

Thus,

$$\begin{aligned}
\sum_{p\leqslant AN^{1/2}} \sum_{\substack{N\leqslant n\leqslant 2N\\ p^2\mid\Pi(n)}} \Bigg(\sum_{\substack{d\mid\Pi(n)\\ d\leqslant R_2}} \lambda_d\Bigg)^2 &\ll N(\log N)^{k-1} \sum_{p\leqslant N} \frac{\log p + \log\log N}{p^2} + O(N) \\
&\ll N(\log N)^{k-1}\log\log N. \qquad (9.4)
\end{aligned}$$

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