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Holomorphic foliations with Liouvillian first integrals

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Abstract. Intuitively, a Liouvillian function on $\mathbb{C}P(n)$ is one which is obtained from rational functions by a finite process of integrations, exponentiations and algebraic operations. This paper is devoted to the study of foliations determined by polynomial 1-forms which have a Liouvillian first integral. Our main result states that, under some mild restrictions on the singularities of the foliation, such a foliation must be either a linear foliation or an exponent two Bernoulli foliation after some rational pull-back. This proves that the highest level of transcendence for the ordinary differential equations which can be integrated by the use of elementary functions is reached at the Riccati equations.

0. *Introduction*

Let us first recall some basic notions from differential algebra. Let R be a ring (commutative with unit $1 \in R$). A *derivation* of R is a map $\delta : R \to R$ satisfying: (i) $\delta(a+b) = \delta(a) + \delta(b)$ (δ is an additive homomorphism), (ii) $\delta(a \cdot b) = a \cdot \delta b + b \cdot \delta a$ (δ satisfies the product rule). A *differential field* is a couple (k, Δ) where k is a field and $\Delta = \{\delta_i\}_{i \in I}$ is a set of derivations of k. We shall consider only *commutative* differential fields, that is the derivations $\delta_i \in \Delta$ commute, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, $\forall\, i, j \in I$. The *constants* of (k, Δ) are the elements $c \in k$ such that $\delta_i c = 0$, $\forall\, i \in I$; they form a subfield $c(k, \Delta)$ of k. A *differential extension* of (k, Δ) is a differential field $(\tilde{k}, \tilde{\Delta})$ where $\tilde{k}$ is an extension of k and each derivation $\tilde{\delta} \in \tilde{\Delta}$ induces by restriction an element $\delta \in \Delta$ and conversely each element $\delta \in \Delta$ extends to an element $\tilde{\delta} \in \tilde{\Delta}$. Thus it is natural to think of $\tilde{\Delta}$ as Δ extended to $\tilde{k}$ and write $(\tilde{k}, \Delta)$ in the place of $(\tilde{k}, \tilde{\Delta})$.

A *Liouvillian extension* of (k, Δ) is a differential extension $(K, \hat{\Delta})$ of (k, Δ) for which there exists a chain of differential extensions: $k = k_0 \subset k_1 \subset \cdots \subset k_m = K$, such that $k_{i+1}/k_i = k_i(t_i)/k_i$ is either an algebraic extension or it is of the type adjuntion of an integral or the exponential of an integral. In other words, we have: (i) the field of constants

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of (K_i, Δ_i) is the same and $\Delta_i|_{K_{i-1}} = \Delta_{i-1}$, $\forall\, i = 1, \ldots, n$; (ii) $K_i = K_{i-1}(t_i)$ for some $t_i \in K_i$ such that either: (a) t_i is algebraic over K_{i-1}, or (b) $\delta t_i \in K_{i-1}$, $\forall\, \delta \in \Delta_i$, or (c) $\delta t_i / t_i \in K_{i-1}$, $\forall\, \delta \in \Delta_i$, $\forall\, i = 1, \ldots, n$. A *Liouvillian function* is by definition an element of a Liouvillian extension. All this terminology is found in [**16**].

Now let $\mathcal{F}$ be a codimension-one holomorphic foliation on a projective non-singular manifold M, given by an integrable rational 1-form Ω on M (see Remark 1). A non-constant Liouvillian function F, belonging to a Liouvillian extension of the differential field of rational functions over M, is said to be a *first integral* of $\mathcal{F}$ if we have $dF \wedge \Omega = 0$.

A well-known theorem due to M. Singer assures that a polynomial foliation on $\mathbb{C}^2$, admitting a Liouvillian first integral, admits a suitable integrating factor as follows.

THEOREM A. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ given in some affine space $\mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)$ by the polynomial 1-form $w = P\,dy - Q\,dx$. Then $\mathcal{F}$ admits a Liouvillian first integral if, and only if, w has an integrating factor of the form $h = \exp \int \eta$ where η is a closed rational 1-form.*

Now let η be a closed rational 1-form on $\mathbb{C}P(2)$. According to the Integration Lemma [**3, 14**] we can write

$$\eta|_{\mathbb{C}^2} = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d\left(\frac{g}{\prod_{j=1}^{r} f_j^{n_j - 1}}\right),$$

for some $\lambda_j \in \mathbb{C}$, $n_j \in \mathbb{N}$ and for some irreducible polynomials f_j and g on $\mathbb{C}^2$ with $(\eta)_\infty \cap \mathbb{C}^2 = \cup(f_j = 0)$—without loss of generality we may assume that $(\eta)_\infty \cap (\mathbb{C}P(2) \setminus \mathbb{C}^2)$ is finite. Therefore, we have that

$$h = \exp \int \eta = \prod_{j=1}^{r} f_j^{\lambda_j} \cdot \exp\left(\frac{g}{\prod_{j=1}^{r} f_j^{n_j - 1}}\right)$$

on $\mathbb{C}^2$. Theorem A can thus be stated in the following equivalent way [**3**].

THEOREM B. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$. The following three conditions are equivalent:*

(1) *$\mathcal{F}$ admits a Liouvillian first integral;*

(2) *$\mathcal{F}$ is given in some affine space by a polynomial 1-form $\Omega = P\,dy - Q\,dx$ which has an integrating factor of the form*

$$h = \prod_{j=1}^{r} f_j^{\lambda j} \cdot \exp\left(\frac{g}{\prod_{j=1}^{r} f_j^{n_j - 1}}\right);$$

(3) *there exists a pair (Ω, η) of 1-forms with Ω polynomial and η rational in some affine chart $\mathbb{C}^2 \subset \mathbb{C}P(2)$ satisfying:* (i) *$\mathcal{F}|_{\mathbb{C}^2}$ is given by $\Omega = 0$ and* (ii) *$d\Omega = \eta \wedge \Omega$, $d\eta = 0$.*

In the situation of Theorem B above we define $\eta = dh/h$, then $h = \exp \int \eta$ so that h is an integrating factor for w if and only if $dw = \eta \wedge w$.

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Again we write

$$h = \exp \int \eta = \prod_{j=1}^{r} f_j^{\lambda_j} \exp\left(\frac{g}{\prod_{j=1}^{r} f_j^{n_j}}\right).$$

We will see that some of the $(f_j = 0)$ are $\mathcal{F}$-invariant. Therefore a foliation admitting a Liouvillian first integral on $\mathbb{C}P(n)$ has some algebraic invariant hypersurface (see Lemma 1, §2). In this paper we regard the problem of classifying such foliations. Let us first give some remarkable examples of this situation:

Example 1. (Darboux foliations) Let M be a complex manifold and let $f_j : M \to \overline{\mathbb{C}}$ be meromorphic functions and $\lambda_j \in \mathbb{C}^*$ complex numbers, $j = 1, \ldots, r$. The meromorphic integrable 1-form

$$\Omega = \left(\prod_{j=1}^{r} f_j\right) \sum_{j=1}^{r} \lambda_i \frac{df_i}{f_i}$$

defines a *Darboux foliation* $\mathcal{F} = \mathcal{F}(w)$ on M. $\mathcal{F}$ has $f = \prod_{j=1}^{r} f_j^{\lambda_j}$ as a *logarithmic* first integral. We remark that $f = \exp\left(\sum_{j=1}^{r} \lambda_j \log f_j\right)$ so that its singularities are of logarithmic type.

Example 2. (Bernoulli foliations) Let $\mathcal{F}$ be a *Bernoulli foliation* of exponent k on $\mathbb{C}^2$ given by $w = 0$, where $w = p(x)\,dy - (y^k a(x) - y b(x))\,dx$, where $a(x), b(x)$ and $p(x)$ are polynomials and $(x, y) \in \mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)$ in some affine chart. We will construct a Liouvillian first integral for $\mathcal{F}$. First we observe that $\mathcal{F}$ can be given by

$$(k-1)\frac{w}{py^k} = (k-1)\frac{dy}{y^k} - (k-1)\left[\frac{a(x)}{p(x)} - \frac{b(x)}{p(x)y^{k-1}}\right] dx = 0.$$

Now let $f(x)$ be such that $f'(x)/f(x) = (k-1)b(x)/p(x)$ and let $g(x)$ be such that $g'(x) = (k-1)a(x)/p(x)f(x)$. Then $\mathcal{F}$ can be given by

$$(k-1)\frac{dy}{y^k} - (k-1)\frac{a(x)}{p(x)}\,dx + \frac{f'(x)}{y^{k-1}f(x)}\,dx = 0,$$

or equivalently by

$$d\left(\frac{1}{f(x)y^{k-1}}\right) + \frac{(k-1)a(x)}{p(x)f(x)}\,dx = 0.$$

Therefore

$$F(x, y) = g(x) + \frac{1}{f(x)y^{k-1}}$$

defines a first integral for $\mathcal{F}$, which is of Liouvillian type.

Our main result is the following.

THEOREM 1. (Main result) *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{C}P(n)$ admitting a Liouvillian first integral. Assume that one of the algebraic leaves of $\mathcal{F}$ has only non-dicritical infinitesimally hyperbolic singularities. Then either $\mathcal{F}$ is a Darboux foliation or there is a rational map F from $\mathbb{C}P(n)$ to $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ and an exponent two Bernoulli foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$:*

$$\mathcal{R} : p(x)\,dy - (a(x)y^2 + b(x)y)\,dx = 0$$

where $a(x)$, $b(x)$, $p(x)$ are polynomials, such that $\mathcal{F} = F^*\mathcal{R}$.

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A singularity p of a foliation in dimension two is *non-dicritical* if at most a finite number of analytic invariant curves pass through p. The notion of infinitesimally hyperbolic singularity is found in §1 for dimension two. We refer to §3 for the notions of dicritical singularity and infinitesimally hyperbolic singularity in dimension ≥ 2. Theorem 1 can also be stated in the case $n = 2$ if we replace $\mathbb{C}P(2)$ by a simply-connected projective surface M and require that $M \backslash S$ is affine (that is, a Stein manifold), where S is an irreducible invariant analytic curve; for example, if $M = \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ and S is the diagonal $\Delta \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

Idea of the proof of Theorem 1. Our approach in the proof of Theorem 1 is motivated by the existence of an affine transverse structure for $\mathcal{F}$ outside some invariant algebraic set (see §2). The proof can be outlined as follows. Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{C}P(2)$ and denote by $\operatorname{sing}\mathcal{F}$ its (discrete) singular set. Assume that $\mathcal{F}$ admits a Liouvillian first integral and let Ω be a rational 1-form which defines $\mathcal{F}$ on $\mathbb{C}P(2)$ (see Remark 1 below). Take a rational logarithmic derivative η for Ω as in Theorem B above. This 1-form η is unique unless $\mathcal{F}$ is given by a closed rational 1-form (and in this case due to the hypothesis on the singularities $\mathcal{F}$ will be of Darboux type). Some of the poles of η are *algebraic leaves* of $\mathcal{F}$, that is, algebraic invariant hypersurfaces (see Lemma 1, §2). Let us fix S as one of these leaves and assume that $\operatorname{sing}\mathcal{F} \cap S$ satisfies the hypothesis of Theorem 1. According to [**14**] the existence of η implies that the foliation $\mathcal{F}$ is transversely affine *outside the invariant part of the polar set of* η. With our hypothesis on $\operatorname{sing}\mathcal{F} \cap S$ it follows from [**14**] that each holonomy group of the components of the invariant divisor arising in the resolution of $\operatorname{sing}\mathcal{F} \cap S$, is *solvable* analytically normalizable. Moreover, there exists a *global compatibility* between these holonomy groups. In order to define this compatibility relation we introduce the *singular holonomy groups* associated to the resolution of $\operatorname{sing}\mathcal{F} \cap S$ (this concept first appeared in [**6**]). These are groups of germs of diffeomorphisms containing the holonomy groups of the components and which are obtained by 'passing' the holonomy group of one component of the resolution divisor to another adjacent component. This passage is given by the Dulac correspondence associated to the singular point defined by the intersection of these two components. The existence of a global 1-form η and its uniqueness implies that these singular holonomy groups are also solvable†. Using this information we can find suitable local expressions for η so that we can prove the extension of the affine transverse structure to a *ramified covering* of a projective transverse structure for $\mathcal{F}$ defined in a neighborhood of the fixed component S of $(\eta)_\infty$ (see Examples 3 and 4). This is done with the introduction of a 1-form η_1 which satisfies the relation $d\Omega = \eta_1 \wedge \Omega$, but which has polar set *transverse* to S. The form η_1 is called a logarithmic derivative *transverse* to S. Once this extension is obtained, some techniques from [**14**] apply to show that $\mathcal{F}$ is a rational pull-back of a Bernoulli differential equation as in Theorem 1.

This paper is organized as follows. In §1 we recall the notion of holonomy group of a leaf for a complex foliation. In §2 we introduce the affine and projective transverse structures for a foliation. In this section we prove that a projective foliation admitting a

† The singular holonomy groups are sorts of differential Galois groups of the foliation (see Theorem 3, the Structure theorem, in §3).

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Liouvillian first integral has some algebraic leaf and prove Theorem 2 which describes the holonomy groups in the resolution divisor of the singularities in such leaves. In §3 we introduce the concept of singular holonomy for an invariant divisor and state the Structure theorem which characterizes the foliations that admit a Liouvillian first integral, in terms of the solvability of the singular holonomy of the elements of the divisor in the resolution diagram. This result is similar to the Galois fundamental theorem for algebraic field extensions. §4 is devoted to the construction of suitable logarithmic derivatives for a given foliation having a compact leaf whose projective holonomy groups satisfy the necessary conditions stated in the Structure theorem. The results in this section complete the equivalence between the existence of Liouvillian first integrals, the existence of suitable logarithmic derivatives and the solvability of certain generalized holonomy groups. In this section we also give explicit local forms and constructions for these logarithmic derivatives. §5 is devoted to the proof of an equivalence between the existence of logarithmic derivatives and the solvability of the singular holonomy groups associated to a suitable compact leaf of a foliation. §6 contains the proof of Theorem 1 in dimension two. The last section finishes the proof of Theorem 1 in dimension n.

Germs of holomorphic foliations admitting Liouvillian first integrals are studied in [**13**] where one can find an interesting notion of solvability for the projective holonomy expressed in terms of the solvability of certain Lie algebras of vector fields. A local version of Singer's result is proved in [**12**] using geometric techniques. Theorem 1 contains as a particular case the main theorem of [**5**].

1. *Holonomy group of a leaf*

Let $\mathcal{F}$ be a codimension one foliation on a complex manifold M. Let L be a *leaf* of $\mathcal{F}$ (that is, a leaf of the corresponding non-singular foliation). Fix $q_0 \in L$. The *holonomy group* of the leaf L at q_0 is defined as follows: choose a small open disk Σ_0, transverse to $\mathcal{F}$, centered at $q_0 \in L$. Given any closed path γ in L, based at $q_0 \in L$, we denote by $H_\gamma : (\Sigma_0, q_0) \hookleftarrow$ the return map to Σ_0 of the leaves of $\mathcal{F}$ over the path γ. The map H_γ is a local holomorphic diffeomorphism fixing $q_0 \in \Sigma_0$ and its germ at $q_0 \in \Sigma_0$ only depends on the homotopy class of γ. Thus the map $\gamma \overset{H}{\longmapsto}$ germ of H_γ at $q_0 \in \Sigma$ induces a homomorphism $H : \pi_1(L, \mathcal{F}; q_0) \to \mathrm{Diff}(\Sigma_0, q_0) \cong \mathrm{Diff}(\mathbb{C}, 0)$ from the fundamental group of L at q_0 to the group of germs of local holomorphic diffeomorphisms of $\mathbb{C}$, fixing $0 \in \mathbb{C}$. The image of H is called *the holonomy group of L at q_0*, denoted $\mathrm{Hol}(\mathcal{F}, L, \Sigma_0)$ (see [**1**] for more details).

Now let $q_1 \in L$ be another *base point* and let Σ_1 be another local transverse section to $\mathcal{F}$ with $\Sigma_1 \cap L = \{q_1\}$. Let $\delta : [0, 1] \to L$ be a path joining $\delta(0) = q_1$ to $\delta(1) = q_0$. Then the isomorphism $\delta_\# : \pi_1(L, q_0) \to \pi_1(L, q_1)$ induces a natural conjugacy between the holonomy groups $\mathrm{Hol}(\mathcal{F}, L, \Sigma_0)$ and $\mathrm{Hol}(\mathcal{F}, L, \Sigma_1)$. In fact, let $f_\delta : (\Sigma_1, q_1) \to (\Sigma_0, q_0)$ be the local diffeomorphism obtained by lifting the path δ to the leaves of $\mathcal{F}$ as in the definition of the holonomy group above. Then f_δ conjugates these holonomy groups in a natural way. Thus, up to conjugacies as above the holonomy group of the leaf L depends neither on the base point nor on the local transverse section. This remark will be used in order to simplify the notation in §3, where we introduce the concept of singular holonomy.

If $\dim M = 2$ and $S \subset M$ is a compact invariant divisor, then the resolution theorem

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[**15**] asserts the existence of a proper holomorphic map $\pi : \tilde{M} \to M$ which is a finite composition of quadratic blow-ups at the singular points of $\mathcal{F}$ in S such that the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$ by π exhibits in $\pi^{-1}S$ only *irreducible* singularities, which can be of the following two types:

(i) $x\,dy - \lambda y\,dx + \text{h.o.t.} = 0$ and $\lambda \notin \mathbb{Q}_+$;

(ii) $y^{p+1}\,dx - [x(1 + \lambda y^p) + \text{h.o.t.}]\,dy = 0$, $p \geq 1$. This last case is called a *saddle node*.

An irreducible singular point of type (i) is *resonant* if $\lambda \in \mathbb{Q}_-$. The *divisor* $D = \pi^{-1}(S) \subset \tilde{M}$ of the resolution π can be written as $D = \bigcup_{j=1}^{m} D_j$, where D_1 is diffeomorphic to S by π and $\{D_j\}_{j=2}^{m}$ is the set of projective lines introduced as divisors of the successive blow-ups. We denote by $\tilde{S} := D_1$ the *strict transform of* S. The D_j's are called *components* of the divisor D. A singularity $q \in \operatorname{sing}\mathcal{F} \cap S$ is *non-dicritical* if $\pi^{-1}(q)$ is invariant by $\tilde{\mathcal{F}}$. Any two D_i, D_j, $i \neq j$, intersect in at most one point. These points of intersection are called *corners*.

The notion of generic singularity we introduce below is for technical reasons and motivated by the geometric approach we use here.

Definition 1. A singularity $q \in S$ of the foliation $\mathcal{F}$ is called *infinitesimally hyperbolic* (relative to S) if:

(i) each saddle-node singularity in $\operatorname{sing}\tilde{\mathcal{F}} \cap D$ in the resolution of $q \in S$ is *in good position with respect to* D, i.e. their strong invariant manifolds ($(y = 0)$ in (ii) above) are contained in D;

(ii) each component of the divisor in the resolution of q exhibits at least one non-resonant linearizable singularity.

Remark 1. Let $\mathcal{F}$ be a foliation on a projective manifold M. We denote by $\operatorname{sing}\mathcal{F}$ the *singular set* of $\mathcal{F}$. This is a codimension ≥ 2 analytic subset of M, with the property that $M \setminus \operatorname{sing}\mathcal{F}$ is the maximal open subset of M, for which $\mathcal{F}|_{M\setminus\operatorname{sing}\mathcal{F}}$ is a regular holomorphic foliation. We may assume that $\mathcal{F}$ *is given* in M by a rational integrable 1-form Ω on M. This means the following. Ω is a rational 1-form on M. Denote by $(\Omega)_0$, respectively $(\Omega)_\infty$, the union of codimension-one irreducible components of the zero divisor, respectively of the polar divisor of Ω, and let $X = (\Omega)_0 \cup (\Omega)_\infty$. Then Ω is integrable, that is, $\Omega \wedge d\Omega = 0$ outside X. Let $\operatorname{sing}\mathcal{F} \cap (M \setminus X) := \{p \in M \setminus X, \Omega(p) = 0\}$. By the Frobenius theorem Ω defines a codimension one holomorphic foliation (non-singular) $\mathcal{F}$ in $M \setminus (X \cup \operatorname{sing}\mathcal{F})$. Now, given a point $p \in X \setminus [(\Omega)_0 \cap (\Omega)_\infty]$, we choose a local holomorphic function $h : U \subset M \to \mathbb{C}$, $p \in U$, $h(p) = 0$, such that $\Omega|_U = h^n \Omega_U$, for some $n \in \mathbb{Z}$, and Ω_U a holomorphic 1-form in U, with zero set $\operatorname{sing}\Omega_U$, of codimension ≥ 2. Then Ω_U is also integrable and we may define $\operatorname{sing}\mathcal{F} \cap U := \operatorname{sing}\Omega_U$. Moreover, since $\Omega_U \wedge \Omega = 0$ in U, we may extend $\mathcal{F}$ to $U \setminus \operatorname{sing}\mathcal{F}$, as a regular holomorphic foliation. Thus we may define $\mathcal{F}$ as a regular holomorphic foliation in $M \setminus \operatorname{sing}\mathcal{F}$ as announced. The pair $(\mathcal{F}, \operatorname{sing}\mathcal{F})$ is called a *singular holomorphic foliation* of codimension one on M and $\operatorname{sing}\mathcal{F}$ is the *singular set* of $\mathcal{F}$.

Now we assume that $\dim M = 2$. Thus $\operatorname{sing}\mathcal{F}$ is a discrete subset of M. We consider a resolution morphism $\pi : \tilde{M} \to M$ as above and denote by $\tilde{\Omega}$ the pull-back 1-form $\pi^*\Omega$.

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The 1-form $\tilde{\Omega}$ is meromorphic in $\tilde{M}$, and may have non-isolated singularities along the projective lines D_j introduced by π. On the other hand, for the cases we are interested in, we may assume that $(\Omega)_\infty$ (and therefore $(\tilde{\Omega})_\infty$), has non-invariant codimension-one irreducible components, and meets the divisor transversely at regular points of $\tilde{\mathcal{F}}$. In this case, the singular set sing $\tilde{\mathcal{F}}$ is contained in the zero divisor $(\tilde{\Omega})_0$.

Finally, we recall some basic facts about groups of germs of conformal diffeomorphisms. Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a subgroup of the group of germs of conformal local diffeomorphisms of $\mathbb{C}$ fixing $0 \in \mathbb{C}$. We denote by $[G, G] \subset G$ the subgroup generated by all the elements of the form $[\varphi_1, \varphi_2] := \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1}$ for $\varphi_1, \varphi_2 \in G$. Clearly G is abelian if $[G, G] = 1$. We say that G is *solvable* if there is $n_0 \in \mathbb{N}$ such that the sequence $G = G_0 \supset G_1 \supset G_2 \supset \cdots$, where $G_{n+1} = [G_n, G_n]$ is constant and equal to 1 for all $n \geq n_0$. If $n_0 = 2$ G is called *metabelian*. Denote by $\mathrm{Diff}_1(\mathbb{C}, 0) \subset \mathrm{Diff}(\mathbb{C}, 0)$ the subgroup of elements tangent to the identity, i.e. of the form $z \mapsto z + a_2 z^2 + \cdots$. It is well known that the following conditions are equivalent: (i) G is solvable, (ii) G is metabelian, (iii) the subgroup $G \cap \mathrm{Diff}_1(\mathbb{C}, 0)$ is abelian (see, for instance, [**4**]). It follows from [**4**] that a non-commutative solvable group containing an element φ of the form $\varphi(z) = \mu z + \cdots$, $\mu^n \neq 1$, $\forall\, n \in \mathbb{N}$, is isomorphic to a subgroup of

$$\mathbb{H}_k = \left\{ \varphi \in \mathrm{Diff}(\mathbb{C}, 0);\, \varphi(z)^k = \frac{\mu_\varphi z^k}{1 + a_\varphi z^k},\, \mu_\varphi \in \mathbb{C}^*,\, a_\varphi \in \mathbb{C} \right\}$$

for some $k \in \mathbb{N}$. The number k is called the *order of ramification of the group*.

Remark 2. Let, therefore, $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a solvable subgroup containing a linearizable non-resonant element $f(z) = \lambda z + \text{h.o.t.}$, $\lambda^n \neq 1$, $\forall\, n \in \mathbb{N}$. If G is abelian then it is linearizable say, $g(w) = \lambda_g w$, $\forall\, g \in G$, for a suitable coordinate $w \in (\mathbb{C}, 0)$. Otherwise, we have $g(w) = \lambda_g w/(1 + a_g w^k)^{1/k}$, $\forall\, g \in G$ for a suitable $w \in (\mathbb{C}, 0)$. Such a coordinate w will be called a *normalizing coordinate* for G.

2. *Logarithmic derivatives and transverse structures*

In this section $\tilde{\mathcal{F}}$ is a foliation on a complex n-manifold $\tilde{M}$ defined by a meromorphic 1-form $\tilde{\Omega}$ on $\tilde{M}$. The 1-form $\tilde{\Omega}$ may have a singular set of codimension one. We denote by $(\tilde{\Omega})_0$ the codimension one component of the zero divisor of $\tilde{\Omega}$. Let $U \subset \tilde{M}$ be an open subset of $\tilde{M}$. We say that $\mathcal{F}|_U$ is *transversely affine* if there exists an open covering $(U_\alpha)_{\alpha \in A}$ of $U \setminus \mathrm{sing}\, \mathcal{F}$ such that; (i) for every $\alpha \in A$ there is a holomorphic submersion $y_\alpha : U_\alpha \to \mathbb{C}$ such that $\mathcal{F}|_{U_\alpha} = (dy_\alpha = 0)$, (ii) for any non-empty intersection $U_\alpha \cap U_\beta \neq \varnothing$ we have $y_\alpha = a_{\alpha\beta} y_\beta + b_{\alpha\beta}$, $a_{\alpha\beta} \in \mathbb{C}^*$, $b_{\alpha\beta} \in \mathbb{C}$. If instead of (ii) we have (ii)$'$ $y_\alpha = a_{\alpha\beta} y_\beta + b_{\alpha\beta}/c_{\alpha\beta} y_\beta + d_{\alpha\beta}$, $a_{\alpha\beta} d_{\alpha\beta} - b_{\alpha\beta} c_{\alpha\beta} = 1$, $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta} \in \mathbb{C}$, then we say that $\mathcal{F}|_U$ is *transversely projective*.

Definition 2. Let $D \subset \tilde{M}$ be an invariant divisor (not necessarily irreducible). A meromorphic 1-form $\tilde{\eta}$ defined on $\tilde{M}$ is called a *logarithmic derivative of* $\tilde{\Omega}$ *adapted to* D if: (1) $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$, $d\tilde{\eta} = 0$; (2) the polar divisor $(\tilde{\eta})_\infty$ of $\tilde{\eta}$ has order one along (any component of) D, and consists of the union of $(\tilde{\Omega})_\infty \cup (\tilde{\Omega})_0$ and an invariant divisor of $\mathcal{F}$; and (3) the residue of $\tilde{\eta}$ along any *non-invariant* irreducible component L of $(\tilde{\Omega})_\infty \cup (\tilde{\Omega})_0$

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is equal to either $-$(the order of the poles of $\tilde{\Omega}$ along L), or (the order of $(\tilde{\Omega})_0$ along L), respectively. We remark that, *a priori*, $\tilde{\eta}$ may have non-simple poles on $\tilde{M}$.

Remark 3. The notion of logarithmic derivative we have introduced above explicates the one used in [**14**], which has been settled for the case when $\tilde{\Omega}$ has isolated singularities. This is just a technical feature, and the ideas remain the same as in [**14**].

The importance of the definition above relies on the following theorem.

THEOREM. [14] *If $S \subset M$ is an irreducible invariant divisor of Ω and if the singularities of $\mathcal{F}$ in S are non-dicritical and infinitesimally hyperbolic, then the following conditions are equivalent:*

(a) *Ω admits a logarithmic derivative η adapted to S, with* $\mathrm{Res}_S\, \eta = a$;
(b) *$\mathcal{F}$ is transversely affine in a neighborhood of S minus S and its local separatrices;*
(c) *the holonomy of S is solvable. Moreover, the holonomy of* the leaf $S \setminus \mathrm{sing}\, \mathcal{F}$ *is linearizable if $a - 1 \notin \mathbb{N}$ and embedded in $\mathbb{H}_k$ if $a - 1 = k \in \mathbb{N}$.*

Now we introduce another useful notion.

Definition 3. Let $D \subset \tilde{M}$ be an invariant divisor, and let $\tilde{\eta}$ be a logarithmic derivative of a 1-form $\tilde{\Omega}$ adapted to D. We say that a 1-form $\tilde{\eta}_1 = \tilde{\eta} + \tilde{h}\tilde{\Omega}$, where $\tilde{h}$ is a meromorphic function, is a *logarithmic derivative of $\tilde{\Omega}$ transverse to D* if $(\tilde{\eta}_1)_\infty$ is transverse to D.

In what follows we will see, on the other hand, that a logarithmic derivative of $\tilde{\Omega}$ *transverse* to D puts in evidence the existence of a projective structure for $\tilde{\mathcal{F}}$ near (and through) D.

Example 3. A *Riccati foliation* on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is given in some affine chart $(x, y) \in \mathbb{C} \times \mathbb{C}$ by a polynomial 1-form $\Omega = p(x)\,dy - (y^2 c(x) - y b(x) - a(x))\,dx$. Such a foliation is transverse to the fibration $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $(x, y) \mapsto x$, except for a finite number of invariant fibers given in the affine part by $p(x) = 0$. This transversality allows us to define a *global holonomy* of the horizontal projective line $S = \overline{(y = 0)}$ which gives us a group of Moebius transformations $G \subset SL(2; \mathbb{C})$ of a non-invariant vertical fiber (see [**8**] for a precise definition). If $a(x) \equiv 0$, then we have an exponent two Bernoulli foliation, S is an invariant divisor and G is the usual holonomy of the leaf $S \setminus \mathrm{sing}\, \mathcal{F}$ as defined above. In this case the elements of G are of the form $f(z) = \alpha z/(1 + \beta z)$. Thus the holonomy of the leaf $S \setminus \mathrm{sing}\, \mathcal{F}$ is solvable. Moreover, since the elements of G are affine maps after the change of coordinates $Z = 1/z$ on $\overline{\mathbb{C}}$ it is easy to see that the foliation is transversely affine outside the union of S and the invariant vertical fibers given by the zeros of $p(x)$. This can also be seen if we define

$$\eta = 2\frac{dy}{y} + \frac{p' + b}{p}\,dx$$

and verify that $d\Omega = \eta \wedge \Omega$, $d\eta = 0$, $(\eta)_\infty = S \cup \overline{(p(x) = 0)}$, and indeed η is a logarithmic derivative of Ω adapted to S (see [**14**]). Now we proceed to apply the strategy of our proof in this example. Let us continue with the assumption $a(x) \equiv 0$. We can define

$$\eta_1 = \eta - \frac{2}{py}\Omega = \frac{p' - b + 2yc}{p}\,dx$$

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and $\xi = (2c/p^2)\,dx$ in order to obtain a triple satisfying $d\Omega = \eta_1 \wedge \Omega$, $d\eta_1 = \Omega \wedge \xi$, and $d\xi = \xi \wedge \eta_1$, where η_1 is *not* closed and ξ is *not* identically zero. This triple (Ω, η_1, ξ) puts in evidence the transverse projective structure of the foliation in the manifold $(\overline{\mathbb{C}} \times \overline{\mathbb{C}}) \backslash \overline{\{p(x) = 0\}}$ (see [**14**]), while if we choose

$$\eta = 2\frac{dy}{y} + \frac{p' + b}{p}\,dx,$$

then we have the same projective relations, η closed, and $\xi \equiv 0$.

Example 4. Let $\mathcal{R} : \Omega = p(x)\,dy - (y^2c(x) - yb(x))\,dx = 0$ be an exponent two Bernoulli foliation as above. Let $\pi : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ be the map $(x, y) \mapsto (x, y^k)$, $k \in \mathbb{N}$. Then π preserves the vertical fibration and induces a k to 1 ramified covering for each fiber $x \times \overline{\mathbb{C}}$, $y \mapsto y^k$. The pull-back foliation is an exponent $(k + 1)$ *Bernoulli* foliation $\mathcal{B} = \pi^*\mathcal{R}$. Since π ramifies along the horizontal $S := \overline{(y = 0)}$, it follows from the example above that $\mathcal{B}$ is transversely projective in $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \setminus S \cup \overline{(p(x) = 0)}$. Moreover, the holonomy groups of the resolution divisor of $\mathcal{B}|_S$ are solvable and the holonomy of the associated leaf $S \backslash \operatorname{sing} \mathcal{F}$ is naturally contained in $\mathbb{H}_k$. This shows that, in general, for $k > 1$, the projective transverse structure of $\mathcal{B}$ does not extend to S, however it extends as a ramified covering of a projective structure, given by the subjacent Riccati foliation. This can also be seen as follows. Consider the pull-back 1-forms

$$\begin{aligned}
\Omega^* &= \pi^*(\Omega) = y^{k-1}.[p(x)\,dy - (y^{k+1}c(x) - yb(x))\,dx],\\
\eta^* &= \pi^*(\eta) = 2k\frac{dy}{y} + \frac{p'(x) + b(x)}{p(x)}\,dx,\\
\eta_1^* &= \pi^*(\eta_1) = \frac{p' - b + 2y^k c}{p}\,dx,\\
\xi^* &= \pi^*(\xi) = \xi = \frac{2c}{p^2}\,dx.
\end{aligned}$$

Then we have that

$$\overline{\Omega} := \frac{1}{y^{k-1}}\Omega^* = kp(x)\,dy - (y^{k+1}c(x) - yb(x))\,dx$$

and

$$\overline{\eta} := \eta^* - \frac{d(y^{k-1})}{d(y^{k-1})} = (k+1)\frac{dy}{y} + \frac{p'(x) + kb(x)}{p(x)}\,dx$$

define an affine transverse structure for $\mathcal{B}$ in $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \setminus (S \cup \overline{(p(x) = 0)})$ [**14**]. Indeed, $\overline{\eta}$ is a logarithmic derivative of $\overline{\Omega}$ adapted to S. On the other hand, if we put

$$\overline{\eta}_1 := \eta_1^* - (k-1)\frac{dy}{y} = \frac{p' - b + 2y^k c}{p}\,dx - (k-1)\frac{dy}{y},$$

and

$$\overline{\xi} := y^{k-1}\xi^* = y^{k-1}\frac{2c}{p^2}\,dx$$

then, by [**14**], the triple $(\overline{\Omega}, \overline{\eta}_1, \overline{\xi})$ is a *projective triple* in $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \setminus (S \cup \overline{(p(x) = 0)})$, in the sense that: $\overline{\Omega}$ is holomorphic on $\mathbb{C}^2$, $\operatorname{codim} \operatorname{sing} \Omega \geq 2$, $\overline{\eta}_1$ is a logarithmic derivative of $\overline{\Omega}$

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in $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \setminus (S \cup \overline{(p(x) = 0)})$, and $\overline{\xi}$ is holomorphic on $\mathbb{C}^2 \setminus \overline{(p(x) = 0)}$ (see also §6 for the notion of projective triple). Notice that, as in Example 3, the 1-form $\overline{\xi}$ satisfies $\overline{\xi} \wedge dx \equiv 0$.

The following proposition motivates the geometric approach we use in this paper.

PROPOSITION 1. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{C}P(n)$ admitting a Liouvillian first integral. Then $\mathcal{F}$ is transversely affine outside some algebraic invariant subset of codimension one.*

Proof. We fix a polynomial 1-form Ω which has codimension (at least) two singular set and defines $\mathcal{F}$ in $\mathbb{C}^n \subset \mathbb{C}P(n)$, where $\mathbb{C}P(n) \setminus \mathbb{C}^n$ is generically transverse to $\mathcal{F}$. Since $\mathcal{F}$ admits a Liouvillian first integral, it follows from [**16**] that there exists a closed rational 1-form η which is a logarithmic derivative of Ω. According to [**14**] this implies that $\mathcal{F}$ is transversely affine outside $(\eta)_\infty$. It remains to show that $(\eta)_\infty$ is invariant. This is proved in the lemma below. □

LEMMA 1. *Let Ω be a* meromorphic *1-form, in an open subset $U \subset \mathbb{C}^n, n \geq 2$. Denote by X the union of the irreducible components of the zero set and polar set of Ω. Assume that there exists a meromorphic 1-form η in U, which satisfies $d\Omega = \eta \wedge \Omega$, and which is closed. Then Ω is integrable and the poles of η outside X are invariant by the foliation defined by Ω (see Remark 1).*

Proof. We have $\Omega \wedge d\Omega = \Omega \wedge \eta \wedge \Omega = 0$, so that Ω is integrable. First we note the following (see Definition 2).

CLAIM. *Let X be as above. Then, the polar divisor $(\eta)_\infty$ of η has order one along (any irreducible component of) X, and consists of the union of X and an invariant divisor of $\mathcal{F}$. Moreover, the residue of η along any non-invariant irreducible component Λ of X is equal to either $-$(the order of the poles of Ω along Λ), or (the order of the zero set of Ω along Λ).*

Proof. Just for simplicity of the notation we assume that $n = 2$. Given a generic point $p \in X$ we may choose a small neighborhood $U \supset V \ni p$ and local coordinates $(x, y) \in V$ so that $X \cap V = \Lambda \cap V = \{y = 0\}$ and $\Omega|_V = y^k g\, dx$, for some holomorphic non-vanishing function $g \in \mathcal{O}(V)^*$, $k \in \mathbb{Z}$ (recall that Λ is non-invariant). We write $\eta|_V = \lambda\, dy/y + dg/g + dR$ for some meromorphic function R in V and some $\lambda \in \mathbb{C}$. Let also $\eta_1 := k\, dy/y + dg/g$. Then the difference $\eta - \eta_1$ satisfies $(\eta - \eta_1) \wedge \Omega = d\Omega - d\Omega = 0$, so that $\eta - \eta_1 = h\, dx$ for some meromorphic function h in V. Since $d\eta = d\eta_1 = 0$, we have $h = h(x)$ and therefore $(\lambda - k)\, dy/y + dR = h(x)\, dx$. This implies $\lambda = k$ and $dR = h(x)\, dx$. Therefore, $\eta|_V = k\, dy/y + dg/g + dR(x)$. This proves the claim. □

Assume by contradiction that there exists some component $Y \subset (\eta)_\infty \setminus X$ which is not invariant. Again we assume $n = 2$. If $q \in Y$ is a generic point, we have that $\Omega(q) \neq 0$, and Y is smooth at q. Therefore, there are local coordinates $(x, y) \in V \subset U$, centered at q and such that $\Omega|_V = g\, dx$ and $Y \cap V = \{y = 0\}$, where $g \in \mathcal{O}^*(V)$. We can also assume that $Y \cap V = (\eta)_\infty \cap V$. We have that $d\Omega = dg/g \wedge \Omega$, so that $\eta - (dg/g) = h\, dx$ for some meromorphic function h in V. Since η is closed we have $dh \wedge dx = 0$, so that $h = h(x)$. Therefore, it follows that $\eta|_V = dg/g + h(x)\, dx$, and it has polar set transverse to Y, a contradiction. □

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COROLLARY 1. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{C}P(n)$. If $\mathcal{F}$ admits a Liouvillian first integral then it has some algebraic leaf.*

Proof. Let us fix an affine system $\mathbb{C}^n \subset \mathbb{C}P(n)$ such that $\mathbb{C}P(n)\backslash\mathbb{C}^n$ is generically transverse to $\mathcal{F}$ and contains no singularity of $\mathcal{F}$. Given Ω a polynomial 1-form with singular set of codimension ≥ 2, defining $\mathcal{F}$ in $\mathbb{C}^n \subset \mathbb{C}P(n)$, we take η given by Singer's theorem [**16**]. If η is polynomial then we have $\eta = dP$ for some polynomial and $\Omega/\exp(P)$ is closed and entire. In particular, there exists an entire first integral for $\mathcal{F}$ in $\mathbb{C}^n$. This implies that the leaves of $\mathcal{F}$ are closed in $\mathbb{C}^n$. Now, since $\mathbb{C}P(n)\backslash\mathbb{C}^n$ is generically transverse to $\mathcal{F}$ and contains no singularity of $\mathcal{F}$, we conclude that the leaves of $\mathcal{F}$ are closed in $\mathbb{C}P(n)$. Therefore it follows that the closure of each leaf of $\mathcal{F}$ is an algebraic curve. This implies, by a theorem of Darboux [**7**], that $\mathcal{F}$ has a rational first integral. Now we assume that η has poles in $\mathbb{C}^n$. In this case we only have to apply Lemma 1 above. □

The following result motivates the notion of *singular holonomy* to be introduced in §3 (see also [**6**]), and gives important information on the desingularization of a foliation admitting a Liouvillian first integral.

THEOREM 2. *Let $\mathcal{F}$ be a foliation given by a meromorphic 1-form Ω on the surface M. Assume that Ω admits a logarithmic derivative η on M and let $S \subset (\eta)_\infty$ be an irreducible analytic invariant curve such that* $\operatorname{sing}\mathcal{F} \cap S$ *is non-dicritical and infinitesimally hyperbolic. Denote by $D = \bigcup_{j=1}^m D_j$ the resolution divisor of* $\operatorname{sing}\mathcal{F} \cap S$. *Denote by G_j the holonomy group of D_j. Then:*

(i) *each holonomy group G_j is either abelian linearizable, or solvable analytically conjugated to some subgroup of some $\mathbb{H}_{k_j}$;*

(ii) *if $q = D_i \cap D_j$ is a saddle-node corner whose strong manifold is contained in D_i, then G_i is solvable non-abelian, and G_j is abelian linearizable;*

(iii) *if $q = D_i \cap D_j$ is a corner such that $G_i \subset \mathbb{H}_{k_i}$ and $G_j \subset \mathbb{H}_{k_j}$ are non-abelian, then q is a resonant singularity. Moreover, we can find local coordinates $(x, y) \in U$ centered at q such that $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$ and $\mathcal{F}|_U : k_j x\,dy + k_i y\,dx + \text{h.o.t.} = 0$.*

Proof. As usual we denote by $\tilde{\mathcal{F}}$ the foliation obtained from $\mathcal{F}$ by the resolution process of $\mathcal{F}|_S$. Denote also by $\tilde{\Omega}$ and $\tilde{\eta}$ the pull-back of the 1-forms Ω and η, respectively, by the proper resolution morphism. Fix a component D_j of D. By the hypothesis, $\operatorname{sing}\tilde{\mathcal{F}} \cap D_j$ contains a non-resonant linearizable singularity. According to [**14**] this implies that the 1-form $\tilde{\eta}$ must have simple poles along D_j: in fact, let $q \in \operatorname{sing}\tilde{\mathcal{F}} \cap D_j$ be such a singularity. Choose local coordinates $(z, w) \in V$ centered at q, such that $D_j \cap V = (w = 0)$ and

$$\tilde{\Omega}|_V(z, w) = g(z\,dw - \lambda w\,dz)$$

for some meromorphic function g, where $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. Then by [**14**] we know that

$$\tilde{\eta}|_V(z, w) = a\frac{dz}{z} + b\frac{dw}{w} + \frac{dg}{g},$$

for some constants $a, b \in \mathbb{C}^*$. Now, we have $b = \operatorname{Res}_{D_j}\tilde{\eta}$. According to [**14**] we know that if $b = k + 1$ for $k \in \mathbb{N}$ then the holonomy group $G_j \subset \mathbb{H}_k$ is therefore solvable. On

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the other hand, if $b-1 \notin \mathbb{N}$ then the holonomy group G_j is abelian linearizable. This proves (i).

Let $q = D_i \cap D_j$ be a saddle node whose strong manifold lies over D_i. If G_i is abelian then, since by hypothesis it contains a non-resonant linearizable diffeomorphism, it follows that G_i is linearizable. In particular, if $h \in G_i$ is tangent to the identity then $h = \mathrm{Id}$. This implies that the local holonomy of the strong manifold of q is trivial, which is absurd. Thus G_i is solvable non-abelian. Therefore, $G_i \subset \mathbb{H}_k$ for some $k \in \mathbb{N}$. In particular, the local holonomy of the saddle node q can be written in the form $h(z) = z/(1+az^k)^{1/k}$, $a \in \mathbb{C}^*$, for some holomorphic coordinate $z \in (\mathbb{C}, 0)$. Therefore, it is analytically conjugated to the holonomy of the strong manifold $(v = 0)$, of the normal form saddle node $u^{k+1}\,dv - v\,du = 0$. It follows from [**10**] that we can find local holomorphic coordinates $(x, y) \in U$ centered at q such that $\tilde{\Omega}|_U(x, y) = g(x^{k+1}\,dy - y\,dx)$ for some meromorphic function g in U. Now, we define the 1-form

$$\eta_q = \frac{dy}{y} + (k+1)\frac{dx}{x} + \frac{dg}{g}.$$

Clearly this 1-form is a logarithmic derivative for $\tilde{\Omega}$ in U. Therefore, we have $\tilde{\eta} - \eta_q = h\tilde{\Omega}$ for some meromorphic function h in U. Moreover, since both logarithmic derivatives are closed 1-forms we must have $d(h\tilde{\Omega}) = 0$. This last equation we can write as

$$d(hgx^{k+1}y) \wedge \left(\frac{dy}{y} - \frac{dx}{x^{k+1}}\right) = 0.$$

But it is true that a saddle-node 1-form does not admit any meromorphic first integral, so that $hgx^{k+1}y = \mu \in \mathbb{C}$ is constant. Thus we must have

$$\tilde{\eta} - \eta_q = \mu\left(\frac{dy}{y} - \frac{dx}{x^{k+1}}\right).$$

Now, the fact that both η_q and $\tilde{\eta}$ have simple poles along D_i implies that $\mu = 0$ and therefore $\tilde{\eta} = \eta_q$ in U. It follows that $\tilde{\eta}$ has residue 1 along D_j. As we have seen above, this implies that the holonomy group G_j is abelian linearizable. This proves (ii).

Now let $q = D_i \cap D_j$ be as in (iii) above. Choose local coordinates $(x, y) \in U$ centered at q such that we have the local expression $\tilde{\Omega}|_U(x, y) = g(x\,dy - \lambda y\,dx)$ where $\lambda \in \mathbb{Q}_-$ and g is a meromorphic function. Moreover, we may assume that $D_i \cap U = (x = 0)$ and $D_j \cap U = (y = 0)$. Write

$$\tilde{\eta}|_U = a\frac{dx}{x} + b\frac{dy}{y} + \frac{dg}{g} + \theta$$

for some closed *holomorphic* 1-form θ (recall that $\tilde{\eta}$ has simple poles). Using the relation $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$ it follows that $1 + \lambda = a + b\lambda$. On the other hand, as we have seen above, if $G_j \subset \mathbb{H}_{k_j}$ and $G_i \subset \mathbb{H}_{k_i}$ are non-abelian, then we must have $a = k_i + 1$ and $b = k_j + 1$. Therefore, it follows that $\lambda = -k_i/k_j$. This proves (iii). □

Finally, the hypothesis made on the saddle nodes in D in Theorem 1 is motivated by the following.

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Remark 4. Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ given by a polynomial 1-form Ω with isolated singularities in some affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$, and admitting a Liouvillian first integral given by a rational logarithmic derivative η of Ω. Denote $S = (\eta)_\infty \cap \mathbb{C}^2$. The saddle nodes appearing in the resolution of $\operatorname{sing}\mathcal{F} \cap S$ are in good position with respect to the resolution divisor†.

Proof. Indeed, denote by D the resolution divisor of $\operatorname{sing}\mathcal{F} \cap S$. If $q_0 \in D_j \cap \operatorname{sing}\tilde{\mathcal{F}}$ is a saddle node which is not in good position with respect to D, then the strong invariant manifold S through q has a linearizable holonomy which is absurd. This follows from the fact that $\tilde{\mathcal{F}}$ has an affine transverse structure (if the form $\tilde{\eta}$ is holomorphic) in a neighborhood of $S\backslash\{q_0\}$ [**14**]. □

3. *Singular holonomy group of an invariant divisor*

Let us consider the following situation motivated by Theorem 2: $\mathcal{F}$ is a foliation on a compact complex surface M, $D \subset M$ is a compact (codimension one) invariant divisor with normal crossings. We write $D = \bigcup_j D_j$ where the D_j are irreducible smooth components, fix local transverse sections $\mathcal{S}_j$, $\mathcal{S}_j \cap D_j = p_j \notin \operatorname{sing}\mathcal{F}$, $(\mathcal{S}_j, p_j) \cong (\mathbb{C}, 0)$. Let us write G_j for the holonomy group $\operatorname{Hol}(\mathcal{F}, D_j, \mathcal{S}_j)$ of D_j.

Now we fix a *corner* $q_0 = D_i \cap D_j$. We assume that the holonomy group G_ν is either solvable analytically normalizable, or abelian linearizable, for $\nu \in \{i, j\}$. Thus, in the non-abelian case, if we denote by k_ν the *ramification order* of G_ν, we have an analytic embedding $G_\nu \subset \mathbb{H}_{k_\nu}$. We will proceed to define the *adjunction holonomy group of G_i to G_j* which is, roughly speaking, the result of passing one holonomy group to the other adjacent transverse section, by the use of the so-called Dulac correspondence. This Dulac correspondence will be defined when q_0 is either a linearizable singularity, or a saddle node with strong manifold along D_i, as follows. There is a neighborhood U of q_0 where $\mathcal{F}$ can be written in normal form. In U we take small transverse sections to $\mathcal{F}$, $\Sigma_i, \Sigma_j, \subset U$ such that $\Sigma_i \cap D_i = q_i, \Sigma_j \cap D_j = q_j$. The Dulac correspondence is a multivalued correspondence $\mathcal{D} : \Sigma_i \to \Sigma_j$, which is obtained by following the local leaves of $\mathcal{F}|_U$. Given an element $h \in \operatorname{Hol}(\mathcal{F}, D_i, \Sigma_i)$, we will associate to h a collection of elements $h^{\mathcal{D}} \in \operatorname{Diff}(\Sigma_j, q_j)$, which will verify the following relation,

$$h^{\mathcal{D}} \circ \mathcal{D} = \mathcal{D} \circ h$$

called the *adjunction equation*. Up to a conjugacy induced by a simple path in $D_i \setminus \operatorname{sing}\mathcal{F}$ joining p_i to q_i, we can assume that $G_i = \operatorname{Hol}(\mathcal{F}, D_i, \Sigma_i)$ and similarly that $G_j = \operatorname{Hol}(\mathcal{F}, \Sigma_j, q_j)$ (see §1). Therefore, the adjunction equation induces a subgroup

$$G_j * (\mathcal{D}_* G_i) \subset \operatorname{Diff}(\mathcal{S}_j, p_j)$$

generated by G_j and all the elements $h^{\mathcal{D}}$, $h \in G_i$. This subgroup is unique up to this kind of conjugacy. Notice that the subgroup $G_j * (\mathcal{D}_* G_i)$ does not need to be finitely generated.

3.1. *The case of a resonant linearizable corner.* Let us assume that q_0 is a resonant linearizable corner. We may consider the *Dulac correspondence* given in a neighborhood

† Notice that in Theorem 1 the hypothesis is made only on part of $(\eta)_\infty$.

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of the singularity q_0. This correspondence is defined as follows: there are local holomorphic coordinates $(x, y) \in U$ such that $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$, and such that $\mathcal{F}|_U$ is given by $px\,dy + qy\,dx = 0$, $q_0 : x = y = 0$, where $p/q \in \mathbb{Q}_+$, $\langle p, q\rangle = 1$. We fix the local transverse sections as $\Sigma_j = (x = 1)$ and $\Sigma_i = (y = 1)$, such that $\Sigma_i \cap D_i = q_i \neq q_0$ and $\Sigma_j \cap D_j = q_j \neq q_0$. Let us denote by $h_0 \in \mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$ the element corresponding to the corner q_0. Then we have $h_0(x) = \exp(-2\sqrt{-1}\,\pi p/q) \cdot x$. The Dulac correspondence is therefore given by

$$\mathcal{D} : (\Sigma_i, q_i) \to (\Sigma_j, q_j), \quad \mathcal{D}(x_0) = x_0^{q/p}.$$

Case 1. G_i is abelian. Take any element $h \in \mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$. Since $\mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$ is abelian and contains h_0, we have $h(x) = \mu x \tilde{h}(x^q)$ for some $\tilde{h} \in \mathcal{O}_1$, $\tilde{h}(0) = 1$. We take $\mu_1 = \mu^{q/p}$ and $h_1 = \tilde{h}^{q/p}$ to be one of the p-roots of μ^q and $\tilde{h}^q$, respectively. Then we define $h^{\mathcal{D}} : (\Sigma_j, q_j) \to (\Sigma_j, q_j)$ by

$$h^{\mathcal{D}}(y) = \mu_1 y h_1(y^p).$$

Case 2. G_i is solvable non-abelian and $pk_i/q \in \mathbb{N}$ where $G_i \subset \mathbb{H}_{k_i}$. Take an element $h(x) = \lambda x/(1 + ax^{k_i})^{1/k_i} \in \mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$. Then the maps $h^{\mathcal{D}}$ are all the determinations of

$$y \mapsto \frac{\lambda^{q/p} y}{(1 + ay^{pk_i/q})^{q/pk_i}}.$$

3.2. *The case of a linearizable non-resonant corner.* Let us assume that q_0 is a linearizable non-resonant corner, say, $\mathcal{F} : x\,dy - \lambda y\,dx = 0$ in a neighborhood U of $q_0 : x = y = 0$, where $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. We will consider only the following case: G_i *is abelian*. This is a consequence of item (iii) in Theorem 2.

Assume that we have $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$, $\Sigma_i = (y = 1)$, and $\Sigma_j = (x = 1)$. By the hypothesis G_i is linearizable. Moreover, we can assume that the holonomy group $\mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$ is linear in the local coordinate $x|_{\Sigma_i}$. Therefore any element $h \in G_i$ corresponds to an element $h(x) = \mu_h x$ in $\mathrm{Diff}(\Sigma_i, q_i)$. In the present case the Dulac correspondence is given by $\mathcal{D}(x_0) = x_0^{-\lambda}$. We are looking for elements $h^{\mathcal{D}} \in \mathrm{Diff}(\Sigma_j, q_j)$, which satisfy $h^{\mathcal{D}} \circ \mathcal{D} = \mathcal{D} \circ h$. Therefore we must have

$$h^{\mathcal{D}}(y) = \mu_h^{-\lambda} y$$

where $\mu_h^{-\lambda}$ runs over the solutions of $z^{-1/\lambda} = \mu_h$.

3.3. *The case of an analytically normalizable saddle-node corner.* Assume now that q_0 is a saddle-node singularity which will be assumed to be analytically conjugate to its normal form. Thus there exists a system of local coordinates $(x, y) \in U$ centered at q_0 such that $\mathcal{F}|_U$ is given by $x^{k+1}\,dy - y(1 + \lambda x^k)\,dx = 0$. We assume that $D_i \cap U = (x = 0)$ and $D_j \cap U = (y = 0)$. In order to simplify the notation we consider the transverse sections $\Sigma_i = (y = e^{1/k})$ and $\Sigma_j = (x = 1)$ (eventually performing an homothety on x). The leaves of $\mathcal{F}|_U$ are the level curves of the multivalued first integral $f(x, y) = yx^{-\lambda} \exp(1/kx^k)$. Therefore, the *Dulac correspondence* is defined by

$$\mathcal{D} : \Sigma_i \to \Sigma_j, x_0 \mapsto y_0 = \mathcal{D}(x_0) = x_0^{-\lambda} \exp\left(\frac{1}{kx_0^k}\right).$$

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Now we show how the Dulac correspondence can still be used to define an adjunction for the holonomy. This adjunction will be from the holonomy group of the strong manifold to the holonomy group of the central manifold, that is, from G_i to G_j above.

Case 1. G_i is solvable non-abelian. In this case we have $\lambda = 0$ and $h_0(x) = x/(1+ax^k)^{1/k}$ and therefore $k_i = k$. This also implies that we can assume that x is a normalizing coordinate for G_i. The Dulac correspondence $\mathcal{D}$ is given by $\mathcal{D}(x_0) = \exp(1/kx_0^k)$. Now we take any element $h \in G_i$ and write $h(x) = \mu x/(1+ax^k)^{1/k}$. Since h is of the form given above we have the adjunction equation as

$$h^{\mathcal{D}}\left(\exp\left(\frac{1}{kx^k}\right)\right) = \exp\left(\frac{a}{k\mu^k}\right)\left(\exp\left(\frac{1}{kx^k}\right)\right)^{\mu^{-k}}.$$

Thus $h^{\mathcal{D}}(y) = \exp(a/k\mu^k)\cdot y^{\mu^{-k}}$. However, these are not uniform analytic functions. *Assume that* h^k *is tangent to* 1, that is, $\mu^k = 1$. In this case we can take solutions of the form $h^{\mathcal{D}}(y) = \exp(a/k)\cdot y$.

Remark 5. The adjunction holonomy group, as soon as it is defined, never reduces to the holonomy group. In fact, this is clear perhaps except in the case above of a saddle-node corner: take an element tangent to the identity $h \in G_i$, say $h(x) = x/(1+ax^p)^{1/p}$. Then the corresponding element $h^{\mathcal{D}} \in G_j * \mathcal{D}_*(G_i)$ is given by $h^{\mathcal{D}}(y) = e^{a/p}y$. The holonomy group G_i contains a non-resonant linearizable element say $h_0(x) = \mu x + \text{h.o.t.}$, $\mu^n \neq 1$, $\forall\, n \in \mathbb{N}$, and the flat element h. Therefore, it is non-abelian and we may choose x in such a way that $h_0(x) = \mu x$. Thus G_i contains elements of the form

$$h_0^{-m} \circ h \circ h_0^m = \frac{x}{(1+\mu^{pm}ax^p)^{1/p}} = \frac{x}{(1+bx^p)^{1/p}},$$

where $b/p \notin 2\pi\sqrt{-1}\,\mathbb{Q}$. This gives elements in $G_j * (\mathcal{D}_* G_i)$ which are not in the holonomy group G_j.

Case 2. G_i is abelian. This case does not occur because G_i would be linearizable, on the other hand the local holonomy h_0 is tangent to the identity and non-trivial.

Remark 6. The case of a nonlinearizable non-degenerate corner. Now we consider the case where $q_0 = D_i \cap D_j$ is a nonlinearizable non-degenerate corner. We will see that no adjunction may be defined in this case. The first claim is the following.

CLAIM. *Both holonomy groups* G_i *and* G_j *are solvable non-abelian.*

Proof. In fact, if one of the holonomy groups is abelian, then since it contains a non-resonant linearizable diffeomorphism, this group is linearizable [**2**] and hence the local holonomy around q_0 is linearizable. Therefore, according to [**11**] (recall that q_0 is not a saddle node) it follows that q_0 is itself linearizable, a contradiction. □

Thus we have analytic embeddings $G_j \subset \mathbb{H}_{k_j}$ and $G_i \subset \mathbb{H}_{k_i}$, for some $k_i, k_j \in \mathbb{N}$. Now, we claim the following.

CLAIM. q_0 *is a resonant singularity.*

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Proof. In fact, we write $\tilde{\mathcal{F}}$ around q_0 as $x\,dy - \lambda y\,dx + \text{h.o.t.} = 0$, for $\lambda \in \mathbb{C}^*$. Assume by contradiction that $\lambda \notin \mathbb{Q}$. In this case the local holonomy associated to q_0 along the separatrix contained in D_j is of the form $h(y) = e^{2\pi i\lambda}y + \text{h.o.t.}$ On the other hand, using the embedding of G_j above, we can find an analytic coordinate $z \in (x = 1)$, such that $h(z) = e^{2\pi i\lambda}z/(1 + \mu z^{k_j})^{1/k_j}$. Since $\lambda \notin \mathbb{Q}$, the subjacent homography is not tangent to the identity, and it is therefore linearizable. This implies that the local holonomy h is itself linearizable, and therefore q_0 is linearizable as a singularity [**11**], a contradiction. □

Summarizing, we have concluded that q_0 is a resonant singularity, and therefore its analytic normal form is given by [**9**]. On the other hand, using this normal form, and the fact that both holonomy groups are solvable non-abelian, it is not difficult to see that no adjunction can be defined.

Remark 7. Later on, we will see that such a resonant nonlinearizable singularity has an analytic normal form, like

$$\Omega_{k,\ell} = kx\,dy + \ell y\left(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k\right)dx = 0,$$

which admits the integrating factor $h = x^{\ell+1}y^{k+1}$, and gives us the closed meromorphic 1-form $(1/x^{\ell+1}y^{k+1})\Omega_{k,\ell}$, which has non-simple poles along $(xy = 0)$. Moreover, this closed form is unique up to multiplication by constants (this is a consequence of the fact that the singularity admits no meromorphic first integral, except the constants). This will imply that *if q_0 is a resonant nonlinearizable singularity, given for instance by $\Omega = 0$, then there exists a unique local logarithmic derivative η associated to Ω, around q_0*. This last remark shows that in fact it is not necessary for our purposes, to define any adjunction associated to such a corner.

In order to proceed to the definition of the singular holonomy groups of the components D_j we introduce a partial order in the resolution divisor. This partial order is defined as follows. Let $\mathcal{Z}$ be the (finite) family of connected components of $D^* = D\backslash\{\text{saddle nodes} \cup \text{resonant nonlinearizable singularities}\}$. Given two elements $A, B \in \mathcal{Z}$ we will say that $A > B$ if and only if there exist components D_i, D_j of D such that $D_i \cap D_j = \{q\}$ is a saddle node, $D_i^* = D_i - \{q\} \subset A$, $D_j^* = D_j - \{q\} \subset B$, and the strong manifold of q lies over D_i. Clearly $>$ defines a partial order on $\mathcal{Z}$. Since $\mathcal{Z}$ is finite each totally ordered subfamily $\mathcal{Z}_1 \subset \mathcal{Z}$ has a maximal element.

The family $\mathcal{Z}$ has a maximal element say, $A_0 \in \mathcal{Z}$. It is clear that given a component such that D_i such $D_i\backslash\{\text{saddle-nodes and resonant nonlinearizable singularities}\}$ belongs to A_0, then given any other component D_j not belonging to A_0 but such that $D_i \cap D_j \neq \varnothing$, then $D_i \cap D_j$ is either a resonant nonlinearizable singularity, or a saddle node whose strong manifold lies in D_i. Therefore no adjunction can be defined from any other component B of $\mathcal{Z}$ to A_0. Thus we consider the *singular holonomy* of any fixed component D_i in A_0 as the maximal subgroup $\text{Hol}^{\text{sing}}(\mathcal{F}, D_i, \mathcal{S}_i) = G_i^{\text{sing}} \subset \text{Diff}(\mathcal{S}_i, p_i)$ obtained by iterating the adjunction process at all the other corners of D_i, $D_i \cap D_k, k \neq i$ where D_k is in A_0. Now we consider the new family $\mathcal{Z}_1 = \mathcal{Z} - \{A_0\}$. Take a maximal element $A_1 \in \mathcal{Z}_1$ with respect to the (induced partial order) $>$. Clearly either $D_i \cap D_j = \varnothing$ for all $D_j \in A_1$ and

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all $D_i \in A_0$, or there exists a saddle node or resonant nonlinearizable corner $q = D_i \cap D_j$ with $D_j \in \mathcal{A}_1$, $D_i \in A_0$, and whose strong manifold lies in D_i. Thus we may adjunct the singular holonomy of the component D_i (defined above) to the holonomy of the component D_j. Now we consider the same iteration of the adjunction process for the corners in A_1 and add to this the adjuncted singular holonomy of D_j. This defines the singular holonomy group of any component $D_k \in A_1$. Proceeding in this way we will exhaust $\mathcal{Z}$ and therefore define the adjunction of the holonomy for all the components of D.

Definition 4. The subgroup $\mathrm{Hol}^{\mathrm{sing}}(\mathcal{F}, D_j, \mathcal{S}_j) = G_j^{\mathrm{sing}} \subset \mathrm{Diff}(\mathcal{S}_j, p_j)$ obtained by the algorithmic process above will be called the *singular holonomy group* associated to the component D_j of D.

Choose any holomorphic coordinate $w \in (\mathcal{S}_j, p_j)$, such that the elements in $g \in G_j$ write either as $g(w) = \lambda_g w/(1 + a_g w^{k_j})^{1/k_j}$, $\forall\, g \in G_j$ or as $g(w) = \lambda_g w$, $\forall\, g \in G_j$.

Given a singularity $q_0 = D_i \cap D_j$ as in §§3.1, 3.2 or 3.3 above, we choose normalizing local coordinates $(x, y) \in U$ as in these cases, i.e. coordinates that give the foliation $\mathcal{F}|_U$ in its local normal form. We choose $D_i \cap U = (x = 0)$ and $D_j \cap U = (y = 0)$, and such that if q_0 is a saddle node then the strong manifold is (as above) contained in $(x = 0)$. According to the fact that the local holonomy associated to q_0 and the separatrix contained in $(x = 0)$ classifies analytically the singularity [**9**–**11**], we may choose x in such a way that the restriction $x|_{\Sigma_i}$ is a normalizing coordinate for the holonomy group G_i. However, it is not always true that $y|_{\Sigma_j}$ also normalizes G_j. This is the subject of the following proposition.

PROPOSITION 2. (Compatibility lemma) *The adjunction holonomy group $G_j * (\mathcal{D}_* G_i)$ is a solvable group if, and only if, we have $y|_{\Sigma_j} = \varphi(w)$, for some φ such that either $\varphi \in \mathbb{H}_{k_j}$ or $\varphi(w) = aw$, $a \in \mathbb{C}^*$, that is, if and only if $y|_{\Sigma_j}$ normalizes G_j.*

Proof. First we prove the only if part. Assume that $G_j * (\mathcal{D}_* G_i)$ is solvable non-abelian. According to [**4**] we have $G_j * (\mathcal{D}_* G_i) \subset \mathbb{H}_k$ for some $k \in \mathbb{N}$ (notice that $G_j \subset G_j * (\mathcal{D}_* G_i)$ contains a non-resonant linearizable element). Let $u \in (\mathbb{C}, 0)$ be a normalizing coordinate for this adjunction holonomy group. Now, by Remark 5 there is $f = h^{\mathcal{D}} \in G_j * (\mathcal{D}_* G_i)$, such that $\lambda = f'(0)$ is not a root of unity. If we put $y|_{\Sigma_j} = \varphi(u)$ for $\varphi \in \mathrm{Diff}(\mathbb{C}, 0)$, then we have $(\varphi \circ f \circ \varphi^{-1})(y) = \lambda y/(1 + a y^k)^{1/k}$. Therefore we may use Lemma 2 below in order to conclude that $\varphi(u) \in \mathbb{H}_k$. This shows that the change of coordinates from $y|_{\Sigma_j}$ to u belongs to $\mathbb{H}_k$ and therefore $y|_{\Sigma_j}$ also normalizes the holonomy group G_j. The converse part is left to the reader. The abelian case is done in a similar way. □

LEMMA 2. [**14**] *Let $f \in \mathrm{Diff}(\mathbb{C}, 0)$ with $f(z) = \lambda z + \cdots$ where λ is not a root of unity. Suppose that we have $(\varphi \circ f \circ \varphi^{-1})(z)^k = \lambda^k z^k/(1 + bz^k)$ for some $k \in \mathbb{N}$, $a \in \mathbb{C}^*$ and some $\varphi \in \mathrm{Diff}(\mathbb{C}, 0)$. Then $f(z)^k = \lambda^k z^k/(1 + \mu z^k)$ if and only if $\varphi(z)^k = cz^k/(1 + dz^k)$.*

In other words, the Compatibility lemma (Proposition 2) says that if $q_0 = D_i \cap D_j$ with G_i, G_j solvable as above, then we can normalize simultaneously G_i, G_j and the singularity q_0 if, and only if, the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ is a solvable group.

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Comment. The iteration of the adjunction process in the definition of singular holonomy above is used in the construction of logarithmic derivatives that we will perform in the next section, §4 (Proposition 3). This is necessary in order to assure that the construction of the logarithmic derivatives associated to the components D_j of the resolution divisor (Lemma 6 below) exhibits no ambiguities (see the proof of Proposition 3), as follows from the Compatibility lemma above.

Theorem 2 motivates the following definition: let $\mathcal{F}$ be a foliation on a compact complex surface M and let $S \subset M$ be an irreducible invariant divisor.

Definition 5. The foliation $\mathcal{F}$ has a *Liouvillian resolution* (*relative to* S) if the divisor $D = \bigcup_{j=1}^{m} D_j$ of the resolution of sing $\mathcal{F} \cap S$ satisfies:

(i) each singular holonomy group G_j of D_j is either abelian linearizable or solvable analytically conjugated to a non-abelian subgroup of $\mathbb{H}_{k_j}$;

(ii) if $q = D_i \cap D_j$ is a saddle-node corner whose strong manifold is contained in D_i, then G_i is solvable non-abelian, and G_j is abelian;

(iii) if $q = D_i \cap D_j$ is a corner such that $G_i \subset \mathbb{H}_{k_i}$ and $G_j \subset \mathbb{H}_{k_j}$ are non-abelian, then q is a resonant singularity, and we can find local coordinates $(x, y) \in U$ centered at q such that $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$ and $\mathcal{F}|_U : k_j x\, dy + k_i y\, dx + \text{h.o.t.} = 0$.

Now we extend the definitions of Liouvillian resolution, dicritical singularity and infinitesimally hyperbolic singularity to dimension $n \geq 3$. Let $\mathcal{F}$ be a foliation of codimension one on $\mathbb{C}P(n)$ with $n \geq 3$. It follows from [**2**] that the set of linearly embedded r-planes $N \subset \mathbb{C}P(n), r \geq 2$, which are in *general position* with respect to $\mathcal{F}$ is a Zariski's open dense subset of the Grassmanian of r-planes. Therefore, we will say that singularities of $\mathcal{F}$ on S are *non-dicritical*, respectively *infinitesimally hyperbolic*, if for some linearly embedded 2-plane $N \subset \mathbb{C}P(n)$, in general position with respect to $\mathcal{F}$, the induced foliation $\mathcal{F}|_N$ on N has only non-dicritical, respectively infinitesimally hyperbolic, singularities on $S \cap N$. We will also say that $\mathcal{F}$ has a *Liouvillian resolution (relative to S)* if there exists such an N for which the induced foliation $\mathcal{F}|_N$ has a Liouvillian resolution relative to $S \cap N$.

Theorem 1 follows from the following theorems.

THEOREM 3. (Structure theorem) *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{C}P(n)$ and suppose that one of its algebraic leaves, say S, has only non-dicritical infinitesimally hyperbolic singularities. Then the following two assertions are equivalent:*

(a) *$\mathcal{F}$ has a Liouvillian first integral on $\mathbb{C}P(n)$;*

(b) *$\mathcal{F}$ has a Liouvillian resolution relative to S.*

THEOREM 4. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(n)$ with an irreducible invariant divisor S such that* sing $\mathcal{F} \cap S$ *consists of non-dicritical infinitesimally hyperbolic singularities. Suppose that $\mathcal{F}$ has a Liouvillian resolution relative to S. Then either $\mathcal{F}$ is a Darboux foliation or it is a rational pull-back of an exponent two Bernoulli foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.*

A logarithmic rational 1-form Ω on $\mathbb{C}P(n)$ writes in homogeneous coordinates $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$, as $\Omega = \sum_{j=1}^{r} \lambda_j\, dP_j/P_j$, with polar divisor $(\Omega)_\infty = \bigcup_{j=1}^{r}$

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$\{P_j = 0\}$, where λ_j is the residue of Ω along $\{P_j = 0\}$, and the P_j are homogeneous polynomials satisfying $\sum_{j=1}^{r} \lambda_j \deg(P_j) = 0$. A *Darboux foliation* on $\mathbb{C}P(n)$ is one defined by a logarithmic 1-form.

4. *Construction of logarithmic derivatives*

The following proposition assures the existence of a logarithmic derivative in the presence of a Liouvillian resolution for a certain invariant analytic curve, for a given foliation. Let $\mathcal{F}$ be a foliation on a complex surface M, and let $S \subset M$ be an analytic compact invariant (irreducible) curve, and assume that $\mathcal{F}$ is given by a meromorphic 1-form Ω on M. As before, we denote by $\pi : \tilde{M} \to M$ the morphism given by the resolution of $\mathcal{F}|_S$, by $D = \pi^{-1}(S)$ the resolution divisor, and by $\tilde{\Omega}$ the pull-back 1-form $\pi^*\Omega$ (see Remark 1).

PROPOSITION 3. *Suppose that* $\dim M = 2$ *and that* $S \subset M$ *is a compact irreducible divisor invariant by* $\mathcal{F}$ *such that the set* $\operatorname{sing}\mathcal{F} \cap S$ *consists of non-dicritical infinitesimally hyperbolic singularities. If* $\mathcal{F}$ *has a Liouvillian resolution relative to* S*, then the 1-form* $\tilde{\Omega}$ *has a logarithmic derivative* $\tilde{\eta}$ *adapted to* D *in a neighborhood of* D*. In particular, if* $M = \mathbb{C}P(2)$ *and* S *is an algebraic curve, then* $\mathcal{F}$ *admits a Liouvillian first integral.*

With the hypothesis of Proposition 3 we fix D_j to be one of the irreducible components of D.

LEMMA 3. *Assume that* $G_j^{\text{sing}} \subset \mathbb{H}_{k_j}$ *is non-abelian. Given a singularity* $q \in D_j$*, then it has one of the following normal forms in some local chart* $(x, y) \in U$ *such that* $x(q) = y(q) = 0$, $D_j \cap U = (y = 0)$*:*

(a) $\Omega_\lambda = x\,dy - \lambda y\,dx = 0$ *(linearizable case);*

(b) $\Omega_{k_j,\ell} = k_j x\,dy + \ell y(1 + (\sqrt{-1}/2\pi)x^\ell y^{k_j})\,dx = 0$ *(resonant nonlinearizable case);*

(c) $\Omega_{k_j} = x\,dy - y^{k_j+1}\,dx = 0$ *(saddle node in good position with respect to* D_j*);*

(d) $\Omega^{k_i} = y\,dx - x^{k_i+1}\,dy = 0$ *(saddle node in good position with respect to* D_i *where* $q = D_i \cap D_j$ *and* $G_i \subset \mathbb{H}_{k_i}$*).*

In particular, there are local logarithmic derivatives 1-forms $\eta_{j,q}$ *in* U *such that* $d\tilde{\Omega} = \eta_{j,q} \wedge \tilde{\Omega}$, $(\eta_{j,q})_\infty$ *consists of local separatrices*† *of* $\tilde{\mathcal{F}}|_U$*, has order one along* D_j*, and if* $\ell_j = \operatorname{ord}(\tilde{\Omega} = 0, D_j)$*, then we have* $\operatorname{Res}_{D_j}(\eta_{j,q}) = \ell_j + k_j + 1$ *in cases (a), (b), (c). In case (d) we have* $\operatorname{Res}_{D_j}(\eta_{j,q}) = \ell_j + 1$*. Moreover, these local normal forms induce local Liouvillian first integrals* $\widehat{y}$ *of* $\tilde{\Omega}|_U$ *such that* $\tilde{\Omega}|_U = \widehat{g}\,d\widehat{y}$*.*

Proof. In order to simplify the notation we put $k_j = k$. By hypothesis, given any local transverse section Σ_j to $\tilde{\mathcal{F}}$ at some point $q_j \in D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$, there is a holomorphic coordinate $z \in (\mathbb{C}, 0) = (\Sigma_j, q_j)$ such that for any element φ of the holonomy of D_j we have $\varphi(z)^k = \lambda_\varphi z^k/(1 + a_\varphi z^k)$. Now, if $q \in \operatorname{sing}\tilde{\mathcal{F}} \cap D_j$ is a linearizable singularity, then $\tilde{\Omega}|_U = g(x\,dy - \lambda y\,dx) = \widehat{g}\,d\widehat{y}$, where g is a meromorphic function, $\widehat{g} = gx^{1+\lambda}$ and $\widehat{y} = yx^{-\lambda}$. We define

$$\eta_{j,q} := (k+1)\frac{d\widehat{y}}{\widehat{y}} + \frac{d\widehat{g}}{\widehat{g}} = (k+1)\frac{dy}{y} + (1 - \lambda k)\frac{dx}{x} + \frac{dg}{g}.$$

† We are assuming that the polar set of $\tilde{\Omega}$ does not meet D at singular points of $\tilde{\mathcal{F}}$.

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Now we remark that whenever $\tilde{\Omega}$ has a local expression in U as

$$\tilde{\Omega}|_U = g(x\,dy - \lambda y\,dx + \cdots)$$

with $\lambda \notin \mathbb{Q}$ and $(y = 0) \subset D_j$, then $\tilde{\Omega}|_U$ is linearizable. In fact, its local holonomy is of the form $\varphi(z)^k = \mu z^k/(1 + az^k)$ with $\mu \neq 1$, therefore it is linearizable and by [**11**] this singularity is linearizable. So we have two remaining cases.

(i) $q \in U$ is a singularity of the form $\tilde{\Omega}|_U = g(x\,dy - \lambda y\,dx + \text{h.o.t.})$, $\lambda = -m/n \in \mathbb{Q}_-$, $(m, n) = 1$, and is nonlinearizable with $(y = 0) \subset D_j$. In this case the local holonomy φ of q in D_j satisfies $\varphi(z)^k = z^k/(1 + az^k)$. This implies that $mk = ln$ for some $l \in \mathbb{N}$. Furthermore, we can assume that $a = -\ell$ and so this holonomy is conjugated to the holonomy of $(y = 0)$ of the germ of foliation

$$\Omega_{k,\ell} = kx\,dy + \ell y\left(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k\right)dx = 0.$$

Thus by [**9**] the foliation $\tilde{\mathcal{F}}$ is conjugated near q to the foliation $\Omega_{k,\ell}$. Therefore, there are local coordinates (x, y) centered at $q \in D_j$ such that for some meromorphic function $\tilde{g}$, $\tilde{\Omega}|_U = \tilde{g}\Omega_{k,\ell}$. Moreover, $\Omega_{k,\ell} = g_{k,\ell}\,dy_{k,\ell}$ where

$$y_{k,\ell} = \frac{x^\ell y^k}{(\sqrt{-1}\,k/2\pi)\ell x^\ell y^k \log x - 1}$$

and

$$g_{k,\ell} = -\frac{((\sqrt{-1}\,\ell/2\pi)x^\ell y^k \log x - 1)^2}{x^{\ell-1}y^{k-1}}.$$

We define $\widehat{g} = \tilde{g}g_{k,\ell}$, $\widehat{y} = y_{k,\ell}$ and also

$$\eta_{j,q} := (k+1)\frac{dy}{y} + (\ell+1)\frac{dx}{x} + \frac{d\tilde{g}}{\tilde{g}}.$$

(ii) q is a saddle-node and in local coordinates (x, y) with $(y = 0) \subset D_j$, $\tilde{\mathcal{F}}$ is defined by $x(1 + \lambda y^{k+1})\,dy - y^{k+1}\,dx + (\text{h.o.t.})\,dy = 0$. In this case since the local holonomy of $(y = 0)$ at q is of the form $\varphi(z)^k = \mu z^k/(1 + az^k)$ and $(y = 0)$ is the strong manifold of the saddle node, we have $\mu = 1$. Thus this is conjugated to the holonomy of $(y = 0)$ of the germ of saddle node $\Omega_k := x\,dy - y^{k+1}\,dx = 0$. Therefore by [**10**] there is a system of coordinates $(x, y) \in U$ centered a q with $(y = 0) \subset D_j$ such that $\tilde{\mathcal{F}}$ is given by $\tilde{\Omega}|_U = g(x\,dy - y^{k+1}\,dx)$, for some meromorphic function g, so that $\tilde{\Omega}|_U = g\widehat{g}_k\,d\widehat{y}$, where

$$\widehat{g}_k = \frac{x(ky^k \log x + 1)^2}{k^2 y^{k-1}}$$

and

$$\widehat{y} = \frac{ky^k}{ky^k \log x + 1}.$$

We define

$$\widehat{g} = g\widehat{g}_k, \quad \eta_{j,q} := (k+1)\frac{dy}{y} + \frac{dx}{x} + \frac{dg}{g}.$$

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If the singularity is a saddle node which is not in good position with respect to D_j then we have the normal form (d) because since it is in good position with respect to D we can use the case we have just finished (which gives us $\lambda = 0$). Thus, if we set $l = k_i$, we have local coordinates (x, y) such that $\tilde{\Omega}|_U(x, y) = \tilde{g}\Omega^l$. We notice that the 1-form

$$\eta^l = (l+1)\frac{dx}{x} + \frac{dy}{y} + \frac{dg}{g},$$

is closed meromorphic and satisfies the relation $d\tilde{\Omega} = \eta^l \wedge \tilde{\Omega}$. Moreover, $(\eta^l)_\infty$ is invariant and contains $(y = 0)$ as a simple pole of residue one. We define $\eta_{j,q} = \eta^l$. □

For the abelian case we have more restrictions on the analytical type of the singularities.

LEMMA 4. *Assume that G_j^{sing} is abelian. Then given a singularity $q \in D_j \cap \operatorname{sing}\tilde{\mathcal{F}}$, there are analytical coordinates $(x, y) \in U \ni q$, centered at q, such that $D_j \cap U = (y = 0)$, and $\tilde{\mathcal{F}}|_U$ has one of the following types: (a) $\Omega_\lambda = 0$, (b) $\Omega^{k_i} = 0$.*

Proof. Indeed, G_j is abelian linearizable. Therefore, the non-degenerate singularities in $\operatorname{sing}\tilde{\mathcal{F}} \cap D_j$ are linearizable. Now let $q \in \operatorname{sing}\tilde{\mathcal{F}} \cap D_j$ be a saddle node. Since the local holonomy associated to the local separatrix of q contained in D_j is linearizable, it follows that the strong manifold of q is transverse to D_j. This proves the lemma. □

LEMMA 5. *Assume that $G_j^{\text{sing}} \subset \mathbb{H}_{k_j}$ is non-abelian. There exists a covering of D_j by a collection $(U_\alpha)_{\alpha\in A}$ of open connected subsets of $\tilde{M}$ with holomorphic coordinates (x_α, y_α) in U_α such that:*

(1) *$D \cap U_\alpha = (y_\alpha = 0)$;*
(2) *for any $\alpha \in A$ such that $U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$, $\tilde{\mathcal{F}}|_{U_\alpha}$ is given by $dy_\alpha = 0$ and for any $\varphi \in \operatorname{Hol}(\tilde{\mathcal{F}}, D_j, \Sigma^\alpha)$ where $\Sigma^\alpha = (x_\alpha = p_\alpha)$, $\varphi(y_\alpha)^{k_j} = \lambda_\varphi y_\alpha^{k_j}/(1 + a_\varphi y_\alpha^{k_j})$;*
(3) *if $U_\alpha \cap U_\beta \neq \varnothing$ and $\operatorname{sing}\tilde{\mathcal{F}} \cap (U_\alpha \cup U_\beta) = \varnothing$, then $y_\alpha^{k_j} = H_{\alpha\beta}(y_\beta^{k_j})$ for some $H_{\alpha\beta} \in \mathbb{H}_1$;*
(4) *if $q_\alpha = U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} \neq \varnothing$ then q_α is one point, $x_\alpha(q_\alpha) = y_\alpha(q_\alpha) = 0$ and if $U_\beta \cap U_\alpha \neq \varnothing$, then $U_\beta \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$ and $U_\beta \cap U_\alpha$ is simply connected. Moreover, $\tilde{\mathcal{F}}|_{U\alpha}$ is given by one of the normal forms (a), (b), (c) or (d) of Lemma 3 above.*

Proof. Properties (1) and (2) are immediate. Property (3) follows from Lemma 2 and the fact that there is at least one singular point in D_j whose local holonomy is of the form $f(z) = \lambda z$, with λ not a root of unity. Finally (4) follows from Lemma 3. □

LEMMA 6. *Assume that G_j^{sing} is solvable (perhaps abelian). There exists a meromorphic logarithmic derivative $\tilde{\eta}_j$ for $\tilde{\Omega}$ defined in a neighborhood of D_j and adapted to D_j.*

Proof. First we assume that G_j^{sing} is non-abelian and $G_j^{\text{sing}} \subset \mathbb{H}_{k_j}$. First we consider the case where G_j is non-abelian, $G_j \subset \mathbb{H}_{k_j}$ and take the covering $\{(x_\alpha, y_\alpha) \in (U_\alpha)\}_{\alpha\in A}$ given by Lemma 4. Assume first that $U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$ and let $\tilde{\Omega}|_{U_\alpha} = g_\alpha\, dy_\alpha$. Then we define

$$\tilde{\eta}_\alpha^j = (k_j+1)\frac{dy_\alpha}{y_\alpha} + \frac{dg_\alpha}{g_\alpha}.$$

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Whenever $U_\alpha \cap U_\beta \neq \varnothing$ and $(U_\alpha \cup U_\beta) \cap \operatorname{sing} \tilde{\mathcal{F}} = \varnothing$ we have $y_\alpha^{k_j} = H_{\alpha\beta}(y_\beta^{k_j})$ for some $H_{\alpha\beta}(z) = \lambda_{\alpha\beta} z/(1 + a_{\alpha\beta} z) \in \mathbb{H}_1$, so we conclude that $dy_\alpha/y_\alpha^{k_j+1} = dy_\beta/\lambda_{\alpha\beta} y_\beta^{k_j+1}$ and $\tilde{\eta}_\beta^j = \tilde{\eta}_\alpha^j$ in $U_\alpha \cap U_\beta$. Clearly $\tilde{\eta}_j := \tilde{\eta}_\alpha^j$ in U_α defines a logarithmic derivative of $\tilde{\Omega}$ adapted to D_j in a neighborhood $\mathcal{U}_j'$ of $D_j \setminus \operatorname{sing} \tilde{\mathcal{F}}$. Now we observe that $\tilde{\eta}_j$ is holomorphic outside the poles of $\tilde{\Omega}$ and that it has simple poles along $(\tilde{\Omega})_\infty$. Moreover, given any irreducible component $L \subset (\tilde{\Omega})_\infty$ with $L \cap D_j \neq \varnothing$, then by construction we have that $\operatorname{Res}_L \tilde{\eta}_j = -$order of $(\tilde{\Omega})_\infty$ along L: indeed, let m be the order of L as a pole of $\tilde{\Omega}$. Choose local holomorphic coordinates $(x, y) \in U$ for M, such that $U \cap L = (x = 0)$ and we have $\tilde{\Omega} = g\, dy$ as above. We have that $g = g_1/x^m$ for some holomorphic non-vanishing function g_1 in U. Therefore we have

$$\tilde{\eta}_j|_U = \frac{dg_1}{g_1} - m\frac{dx}{x} + \frac{dy}{y}.$$

Now we must show that $\tilde{\eta}_j$ extends meromorphically to $\operatorname{sing} \tilde{\mathcal{F}} \cap D_j$. Indeed, given a singularity $q_0 \in \operatorname{sing} \tilde{\mathcal{F}} \cap D_j \cap U_\alpha$ we have four possibilities according to Lemma 3.

(a) q_0 is a linearizable singularity of the form $\Omega_{\lambda_\alpha} = 0$. The local coordinate (defined on the transverse section Σ_j) $\widehat{y}_\alpha = y_\alpha x_\alpha^{-\lambda_\alpha}$ linearizes the local holonomy of D_j. Now, as in the proof of Lemma 3 we may write

$$\tilde{\Omega}|_{U_\alpha} = g_\alpha(x_\alpha\, dy_\alpha - \lambda_\alpha y_\alpha\, dx_\alpha) = \widehat{g}_\alpha\, d\widehat{y}_\alpha,$$

where g_α is a meromorphic function, $\widehat{g}_\alpha = g_\alpha x_\alpha^{1+\lambda_\alpha}$ and $\widehat{y}_\alpha = y_\alpha x_\alpha^{-\lambda_\alpha}$. We define

$$\eta_{j,\alpha} := (k_j + 1)\frac{d\widehat{y}_\alpha}{\widehat{y}_\alpha} + \frac{d\widehat{g}_\alpha}{\widehat{g}_\alpha} = (k_j + 1)\frac{dy_\alpha}{y_\alpha} + (1 - \lambda_\alpha k_j)\frac{dx_\alpha}{x_\alpha} + \frac{dg_\alpha}{g_\alpha}.$$

The difference $\tilde{\eta}_j - \eta_{j,\alpha}$ can be written as $\tilde{\eta}_j - \eta_{j,\alpha} = h_{q_0}\tilde{\Omega}$, for some meromorphic function h_{q_0} defined over $(x_\alpha \neq 0)$ and which satisfies $d(h_{q_0}\tilde{\Omega}) = 0$. Clearly the poles of h_{q_0} are contained in D_j (notice that the forms $\tilde{\eta}$ and $\eta_{j,\alpha}$ have poles contained in the divisor D). On the other hand, since $\operatorname{Res}_{D_j} \tilde{\eta}_j = \operatorname{Res}_{D_j} \eta_{j,\alpha}$, it follows† that $h_{q_0}\tilde{\Omega}$ is *holomorphic* along D_j and therefore $h_{q_0}\tilde{\Omega}$ is a closed *holomorphic* 1-form which defines $\tilde{\mathcal{F}}$ in $(x_\alpha \neq 0)$. Thus we have two possibilities.

(a.1) If $\lambda_\alpha \notin \mathbb{Q}$. In this case we must have $h_{q_0}\tilde{\Omega} = 0$. In fact, as it follows from [**14**], if we have $df \wedge (x\, dy - \lambda y\, dx) = 0$, $\lambda \notin \mathbb{Q}$, for some holomorphic function f defined on $(x \neq 0)$, then f must be constant (this can be seen using Laurent series for $f(x, y)$). Thus we conclude that $h_{q_0}\tilde{\Omega} = 0$.

(a.2) If $\lambda_\alpha = -p/q \in \mathbb{Q}_-$, $p, q \in \mathbb{N}\langle p, q\rangle = 1$. In this case we obtain (again using Laurent series) that $h_{q_0}\tilde{\Omega} = d\varphi(x^p y^q)$ for some one-variable *holomorphic* function $\varphi \in \mathcal{O}_1$. Thus in this case we may extend $\tilde{\eta}_j$ to the singularity q_0 as $\tilde{\eta}_j = \eta_{j,\alpha} + d\varphi(x^p y^q)$.

(b) The singularity q_0 is resonant nonlinearizable of the form $\Omega_{k_j,\ell} = 0$. We define

$$\eta_{j,\alpha} = (k_j + 1)\frac{d\,y}{y} + (\ell + 1)\frac{dx}{x} + \frac{d\tilde{g}}{\tilde{g}}$$

as the 1-form $\eta_{j,q}$ in (the proof of) Lemma 3. Now we consider the 1-form $\tilde{\eta}_j$ defined in $\mathcal{U}_j'$ and we have $\tilde{\eta}^j - \eta_{j,\alpha} = f_{\alpha,j}\tilde{\Omega}$ for some meromorphic function $f_{\alpha,j}$ in $U_\alpha \cap \mathcal{U}_j'$. Clearly $d(f_{\alpha,j}\tilde{\Omega}) = 0$.

† Notice that dg_α/g_α and dg_β/g_β have the same residue along D_j, given by the order of $(\tilde{\Omega})_0$ along D_j.

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CLAIM. *The function $f_{\alpha,j}$ extends meromorphically to U_α.*

Proof. Indeed, we have an integrating factor h for $\tilde{\Omega}$, which is meromorphic in a neighborhood V of q_0 (see Remark 7). We can assume that $V \supset U_\alpha$. The quotient $f_{\alpha,j}/h$ is therefore a meromorphic first integral for Ω in $U_\alpha \cap \mathcal{U}'_j$. If this first integral is not constant then the local holonomy of D_j around q_0 is finite and therefore linearizable, and this implies that $\tilde{\mathcal{F}}$ is linearizable in a neighborhood of q_0, a contradiction. Thus in $U_\alpha \cap \mathcal{U}'_j$ we have $f_{\alpha,j} = c.h$ for some constant $c \in \mathbb{C}$, and therefore $f_{\alpha,j}$ extends meromorphically to U_α as $c.h$. □

Comparing the poles of $\tilde{\eta}_j$ and $\eta_{j,\alpha}$ we conclude that the poles of $f_{\alpha,j}$ are contained in $(x_\alpha = 0)$. On the other hand (as it follows from the proof of the claim above) the integrating factors of the 1-form $\Omega_{k_j,l}$ are of the form $c/(x^{l+1}y^{k_j+1}), c \in \mathbb{C}$. Thus $f_{\alpha,j} \equiv 0$. We conclude that $\tilde{\eta}^j \equiv \eta_{j,\alpha}$ in $U_\alpha \cap \mathcal{U}'_j$.

(c) The singularity q_0 is a saddle node in good position, given by $\Omega_{k_j} = 0$. We define

$$\eta_{j,\alpha} = (k+1)\frac{dy}{y} + \frac{dx}{x} + \frac{dg}{g}$$

as the 1-form $\eta_{j,q}$ in (the proof of) Lemma 3. Proceeding as in (b) one can show that $\tilde{\eta}^j \equiv \eta_{j,\alpha}$ in $U_\alpha \cap \mathcal{U}'_j$. In fact, once again one can use Laurent series (and the fact that a saddle node does not admit even a formal meromorphic first integral defined over its strong manifold).

(d) q_0 is a saddle node Ω^{k_i}, whose strong manifold is transverse to D_j. In particular, G_j is abelian. First we remark that sing $\tilde{\mathcal{F}} \cap D_j$ *contains no saddle-nodes in good position with respect to D_j and that the non-degenerate singularities are linearizable* (indeed, G_j is linearizable). Using the techniques of [**2**] one constructs a closed meromorphic 1-form $\tilde{\Omega}_j$ in a neighborhood V_j of D_j minus the saddle nodes in sing $\tilde{\mathcal{F}} \cap D_j$. This form satisfies $\tilde{\Omega}_j \wedge \tilde{\Omega} = 0$, and $\tilde{\Omega}_j$ has simple poles, all of them contained in the divisor D. We have $\tilde{\Omega} = h_j\tilde{\Omega}_j$ for some meromorphic function h_j in V_j. Define now the 1-form $\tilde{\eta}'_j = dh_j/h_j$. Clearly $\tilde{\eta}'_j$ is a candidate for a logarithmic derivative for $\tilde{\Omega}$ adapted to D_j. Notice that $\tilde{\eta}'_j$ *has residue one along D_j*. We must show that $\tilde{\eta}'_j$ extends to the saddle-node singularities in D_j. Fix such a singularity $q_0 \in D_j$. We must have $q_0 = D_l \cap D_j$ for some D_l. We will show the extension of $\tilde{\eta}'_j$ to q_0 using the fact that the adjunction holonomy group $G_j * \mathcal{D}_*(G_l)$ is solvable. Choose local coordinates $(x, y) \in U$ centered at q_0, such that $D_l \cap U = (x = 0)$, and $D_j = (y = 0)$, and such that $\tilde{\Omega}|_U(x, y) = g(x^{p+1}\,dy - y\,dx)$ for some meromorphic function g and some $p \in \mathbb{N}$. As we know, the holonomy group G_l of the component D_l must be solvable non-abelian $G_l \subset \mathbb{H}_p$. Moreover, we can assume that the coordinate $x|_{\Sigma_l}$, where $\Sigma_l = (y = 1)$, normalizes the group $\mathrm{Hol}(\tilde{\mathcal{F}}, D_l, \Sigma_l) \cong G_l$. Take an element tangent to the identity $h \in G_l$, say $h(x) = x/(1 + ax^p)^{1/p}$. Then the corresponding element $h^{\mathcal{D}} \in G_j * \mathcal{D}_*(G_l)$ is given by $h^{\mathcal{D}}(y) = e^{a/p}y$. Now let $z \in (\Sigma_j, q_j)$ be a local coordinate, where $\Sigma_j = (x = 1)$ and $q_j = \Sigma_j \cap D_j$, which linearizes the holonomy group $G_j \cong \mathrm{Hol}(\tilde{\mathcal{F}}, D_j, \Sigma_j)$. Then the fact that the singular holonomy group G_j^{sing} is solvable implies that either $y = \varphi(z)$ for some diffeomorphism $\varphi \in \mathbb{H}_k$ when $G_j^{\mathrm{sing}} \subset \mathbb{H}_k$, or $y = \mu z$ for some $\mu \in \mathbb{C}^*$ when G_j^{sing} is abelian linearizable. According to this we can assume that the coordinate $y|_{\Sigma_j}$ normalizes G_j. This implies that

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the 1-forms $\tilde{\eta}'_j$ and

$$\eta_{q_0} := \frac{dy}{y} + (p+1)\frac{dx}{x} + \frac{dg}{g}$$

coincide over the transverse section Σ_j. This same argument shows that they coincide over an open neighborhood of q_j. Therefore, $\tilde{\eta}'_j$ extends as η_{q_0} to the singularity q_0. We rename $\tilde{\eta}'_j$ as $\tilde{\eta}_j$. This is the only case where we use the fact that the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ is solvable. The solvability will be used for the other cases during the glueing process for the 1-forms $\tilde{\eta}_j$ (see the proof of Proposition 3 below). Thus $\tilde{\eta}^j_\alpha$ extends meromorphically to a neighborhood $\tilde{\mathcal{U}}_j$ of D_j.

Now we assume that G_j^{sing} is abelian. In particular, G_j is abelian linearizable. Using the same arguments of (d) above we can conclude that there exists a logarithmic derivative $\tilde{\eta}_j$ of $\tilde{\Omega}$ adapted to D_j in a neighborhood of D_j. □

Proof of Proposition 3. We start by defining $\tilde{\eta}$ in a neighborhood of D_j as $\tilde{\eta}_j$ given by Lemma 6 above. Now we show that given a corner $q_0 = D_i \cap D_j$ then in a neighborhood of q_0, $\tilde{\eta}_i = \tilde{\eta}_j$. First we assume that q_0 is a saddle node $\Omega_{k_j} = 0$ as in (c) of Lemma 3. In this case we have $\tilde{\eta}_i - \tilde{\eta}_j = f_{ij}\Omega_{k_j}$ for some meromorphic function f_{ij} in a neighborhood of q_0. Clearly we have $d(f_{ij}\Omega_{k_j}) = 0$ so that $f_{ij} = \mu/xy^{k_j+1}$ for some $\mu \in \mathbb{C}$ (see the proof of Theorem 2). Therefore,

$$\tilde{\eta}_i - \tilde{\eta}_j = \mu\left(\frac{dx}{x} - \frac{dy}{y^{k_j+1}}\right).$$

The 1-form $\tilde{\eta}_i - \tilde{\eta}_j$ has only simple poles along D_j so that $\mu = 0$ and therefore $\tilde{\eta}_i \equiv \tilde{\eta}_j$ in a neighborhood of q_0. In the same way we have $\tilde{\eta}_i = \eta^{k_j} = \tilde{\eta}_j$ if q_0 is a saddle node of the form $\Omega_{k_j} = 0$ as in (d) of Lemma 3.

Now we assume that q_0 is not a saddle node. We have that q_0 is a singularity of the form $x\,dy - \lambda y\,dx + \text{h.o.t.} = 0$ where $D_i : (x = 0)$ and $D_j : (y = 0)$. As above $\tilde{\eta}_i - \tilde{\eta}_j = f_{ij}\tilde{\Omega}$. Moreover, $d(f_{ij}\tilde{\Omega}) = 0$. First we consider the case where $\lambda \notin \mathbb{Q}$. Assume that $f_{ij} \not\equiv 0$. In this case, $f_{ij}\tilde{\Omega}$ is a closed meromorphic 1-form which defines $\tilde{\mathcal{F}}$ in a neighborhood of q_0, so that q_0 must be linearizable [**3**]. Moreover, according to Theorem 2(iii), at least one of these holonomy groups is abelian, say G_i is abelian linearizable. Thus the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ is defined and solvable. According to the Compatibility lemma, we can find normalizing coordinates for the singularity and for the holonomy groups G_j and G_i at the same time. Now, the same arguments used in parts (a) and (b) in the proof of Lemma 6 apply to show that $\tilde{\eta}_i$ and $\tilde{\eta}_j$ coincide in a neighborhood of q_0.

Assume now that $\lambda = -p/q \in \mathbb{Q}_-$. If q_0 is nonlinearizable then the glueing of $\tilde{\eta}_i$ and $\tilde{\eta}_j$ is immediate (see Remark 7). Thus we will assume that q_0 is linearizable with a holomorphic first integral. Therefore, once again the adjunction holonomy group is defined and solvable, so that as in the case considered above the simultaneous normalization for q_0 and the holonomy groups gives the glueing.

Thus the collection of 1-forms $(\tilde{\eta}_i)_{i=1}^m$ defines a 1-form $\tilde{\eta}$ in a neighborhood of $D = \pi^{-1}(S)$. The 1-form $\tilde{\eta}$ is clearly a logarithmic derivative of $\tilde{\Omega}$ adapted to D. Blowing down this 1-form, we obtain a 1-form $\pi_*(\tilde{\eta})$ in a neighborhood of S such that it is a logarithmic derivative of $\pi_*(\tilde{\Omega})$ adapted to S.

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Assume now that $M = \mathbb{C}P(2)$ and that S is an algebraic curve. According to Levi's extension theorem (see [**2**]) the 1-form η defined in a neighborhood of S extends to a rational 1-form on $\mathbb{C}P(2)$. Clearly this 1-form defines a Liouvillian function H by $\eta = dH/H$ and we have $d(\Omega/H) = 0$. Therefore, there exists a Liouvillian first integral F such $dF = \Omega/H$. □

Now we prove the existence of transverse logarithmic derivatives associated to a given closed logarithmic derivative. First we prove the following result which deals with the abelian case for the singular holonomy groups.

PROPOSITION 4. *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, with an algebraic leaf S such that* $\operatorname{sing}\mathcal{F} \cap S$ *contains only infinitesimally hyperbolic singularities. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is a Darboux foliation;*

(ii) *$\mathcal{F}$ is given by a closed rational 1-form;*

(iii) *all the singular holonomy groups arising in the reduction of the singularities of* $\operatorname{sing}\mathcal{F} \cap S$ *are abelian.*

Proof. First we prove (iii) $\Longrightarrow$ (i). We fix $\pi : (\tilde{\mathbb{C}}P(2), \tilde{\mathcal{F}}, D) \to (\mathbb{C}P(2), \mathcal{F}, S)$, the reduction of singularities of $\operatorname{sing}\mathcal{F} \cap S$ as in §1. Let us fix a component D_j of D. By hypothesis the singular holonomy group G_j^{sing} is abelian, and since D_j contains a non-resonant linearizable singularity, G_j^{sing} is in fact analytically linearizable. Now we fix a singularity $q \in \operatorname{sing}\tilde{\mathcal{F}} \cap D_j$.

CLAIM. *q is a linearizable singularity.*

Proof. This is clear from [**10, 11**] if q is not a saddle node. We assume therefore that q is a saddle node. The local holonomy associated to the strong manifold of q is not linearizable. Therefore, we conclude that the strong manifold of q is not contained in (any component of) D. This contradicts the fact that by the hypothesis q is in good position with respect to D. □

Thus any singularity $q \in \operatorname{sing}\tilde{\mathcal{F}} \cap D$ is in fact linearizable. Using [**2**] we conclude that, for any fixed component D_j of D, there exists an open covering of D_j by a collection $(U_\alpha)_{\alpha\in A}$ of open connected subsets of $\tilde{\mathbb{C}}P(2)$ with holomorphic coordinates (x_α, y_α) in U_α such that:

(1) $D \cap U_\alpha = \{y_\alpha = 0\}$;

(2) for any $\alpha \in A$ such that $U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$, $\tilde{\mathcal{F}}|_{U_\alpha}$ is given by $dy_\alpha = 0$ and for any $\varphi \in \operatorname{Hol}(\tilde{\mathcal{F}}, D_j, \Sigma^\alpha)$ where $\Sigma^\alpha = (x_\alpha = p_\alpha)$, $\varphi(y_\alpha) = \lambda_\varphi y_\alpha$;

(3) if $U_\alpha \cap U_\beta \neq \varnothing$ and $\operatorname{sing}\tilde{\mathcal{F}} \cap (U_\alpha \cup U_\beta) = \varnothing$, then $y_\alpha = \lambda_{\alpha\beta} y_\beta$ for some $\lambda_{\alpha\beta} \in \mathbb{C}^*$;

(4) if $q_\alpha = U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} \neq \varnothing$ then q_α is one point, $x_\alpha(q_\alpha) = y_\alpha(q_\alpha) = 0$ and if $U_\beta \cap U_\alpha \neq \varnothing$, then $U_\beta \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$ and $U_\beta \cap U_\alpha$ is simply connected. Moreover, $\tilde{\mathcal{F}}|_{U\alpha}$ is in the linear form $x_\alpha\, dy_a - \lambda_q y_\alpha\, dx_\alpha = 0$, for some $\lambda_q \in \mathbb{C}^*$.

In the open sets U_α such that $U_\alpha \cap \operatorname{sing}\tilde{\mathcal{F}} = \varnothing$, we define the local 1-forms $\Omega_\alpha = dy_\alpha/y_\alpha$. In those U_β for which $q = U_\beta \cap \operatorname{sing}\tilde{\mathcal{F}} \neq \varnothing$, we define $\Omega_\beta = dy_\beta/y_\beta - \lambda_q . dx_\beta/x_\beta$. These 1-forms glue into a closed meromorphic 1-form Ω_j with simple poles, which defines $\tilde{\mathcal{F}}$ in a neighborhood $\mathcal{U}_j$ of D_j. We proceed to prove that

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if $q = D_i \cap D_j$ is a corner then $\Omega_i = \Omega_j$ in a neighborhood of q. Indeed, since both 1-forms are closed and we define $\tilde{\mathcal{F}}$ in a neighborhood of q, it follows that $\Omega_j = f\Omega_i$ for some meromorphic function which is a first integral for $\tilde{\mathcal{F}}$ around q. On the other hand, we know that $\tilde{\mathcal{F}}$ is locally linearizable around q, say $\tilde{\mathcal{F}} : x\,dy - \lambda_q y\,dx = 0$, for some local holomorphic coordinates $(x, y) \in U$, centered at q, and such that $D_j \cap U = (y = 0)$, $D_i \cap U = (x = 0)$. Therefore, it follows that either f is constant, or $\lambda = m/n$ for some $n, m \in \mathbb{N}$ relatively prime. We consider the case $\lambda = m/n$ and will also prove that in this case f is constant. In fact, we may assume that the coordinate $(xy^{-1/\lambda})|_{(y=1)}$ is a linearizing coordinate for the singular holonomy group G_i^{sing}. Now, according to the Compatibility lemma, the fact that the singular holonomy groups of D_i and D_j are abelian implies that the coordinate $(yx^{-\lambda})|_{(x=1)}$ also linearizes the singular holonomy group G_j^{sing}. Finally, by construction, we know that the 1-form Ω_i, respectively Ω_j, is constructed in such a way that if $z \in (x = 1)$, respectively $w \in (y = 1)$, is a linearizing coordinate for the singular holonomy group of D_j, respectively of D_i, then we must have $\Omega_j|_{(x=1)} = dz/z$, respectively $\Omega_i|_{(y=1)} = dw/w$. On the other hand, we must have $w = \mu(xy^{-1/\lambda})|_{(y=1)}$, respectively $z = \mu(yx^{-\lambda})|_{(x=1)}$, for some $\mu \in \mathbb{C}^*$. Therefore, we obtain

$$\Omega_j|_U = \frac{d(xy^{-1/\lambda})}{(xy^{-1/\lambda})} = \frac{dx}{x} - \frac{1}{\lambda}\frac{dy}{y} = \frac{1}{\lambda}\left(\frac{dy}{y} - \lambda\frac{dx}{x}\right).$$

Therefore, f is constant. Proceeding as in [**2**] one can easily conclude that there exists a closed rational 1-form Ω in all $\mathbb{C}P(2)$ which is logarithmic, and defines $\mathcal{F}$.

Finally, implication (i) $\Longrightarrow$ (ii) is clear. For (ii) $\Longrightarrow$ (iii) we fix a closed rational 1-form $\tilde{\Omega}$ which defines $\tilde{\mathcal{F}}$ in the blow-up manifold M. First we remark that all the holonomy groups G_j are abelian. This is a consequence of the following lemma and of [**13**].

LEMMA 7. *Given any point $q \in D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$ there are local coordinates $(x, y) \in U$ centered at q, such that $D_j \cap U = (y = 0)$, and the 1-form $\tilde{\Omega}$ can be written in these coordinates as*

$$\tilde{\Omega}(x, y) = \frac{ay^n + 1}{y^{n+1}}\,dy,$$

where $n \in \mathbb{N}$ and $a \in \mathbb{C}$ are fixed.

Therefore, the holonomy group G_j leaves invariant a germ of the holomorphic singular vector field

$$X(z) = \frac{z^{n+1}}{1 + az^n}\frac{d}{dz},$$

and according to [**13**] this is an abelian subgroup of $\operatorname{Diff}(\mathbb{C}, 0)$.

Now we notice that since G_j contains (by hypothesis) a non-resonant linearizable diffeomorphism, G_j is analytically linearizable. This implies (as above) that all the singularities $q \in D \cap \operatorname{sing}\tilde{\mathcal{F}}$ are linearizable. Using the same arguments as for the proof of (iii) $\Longrightarrow$ (i) we conclude that $\tilde{\mathcal{F}}$ is given by a closed meromorphic 1-form $\tilde{\Omega}_j$ with simple poles, in a neighborhood V_j of D_j in $\tilde{M}$. We have $\tilde{\Omega} \wedge \tilde{\Omega}_j = 0$ in V_j, so that $\tilde{\Omega} = f\tilde{\Omega}_j$ for some meromorphic function f which is a meromorphic first integral for $\tilde{\mathcal{F}}$ in V_j. Now, by the hypothesis of infinitesimal hyperbolicity, f must be constant. This gives that $\tilde{\Omega}$ has only simple poles in $\tilde{M}$. In particular, Ω is already a logarithmic 1-form. □

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The following proposition assures the existence of a transverse logarithmic derivative after the resolution process.

PROPOSITION 5. *Suppose that $M = \mathbb{C}P(2)$, and that $\mathcal{F}$ is given by a polynomial 1-form Ω, with $\mathbb{C}P(2)\backslash\mathbb{C}^2$ in general position with respect to $\mathcal{F}$. Let S be an irreducible algebraic invariant curve. Denote by $\pi : (\tilde{M}, \tilde{\mathcal{F}}, \tilde{S}) \to (M, \mathcal{F}, S)$ the resolution of singularities of $\mathcal{F}|_S$. Assume that $\tilde{\eta}$ is a (simple poles) logarithmic derivative of $\tilde{\Omega}$ adapted to D. Then:*

(i) *there exists a logarithmic derivative $\tilde{\eta}_1$ of $\tilde{\Omega}$ transverse to D (such that $\tilde{\Omega}$ divides $\tilde{\eta} - \tilde{\eta}_1$);*

(ii) *if $\tilde{\eta}_1$ is also closed then $\mathcal{F}$ is given by a closed meromorphic form on M.*

Proof. For the sake of simplicity of the notation, we will consider the case when $\mathcal{F}|_S$ is already desingularized, hence we will be placed on $M = \mathbb{C}P(2)$, so that $\tilde{M} = M$, $\tilde{\Omega} = \Omega$, $\tilde{\eta} = \eta$ and $D = S$. The proof we give applies to the general case. By the hypothesis, S is given by $\{f = 0\}$ for some irreducible polynomial f defined on $\mathbb{C}^2$. Since $(\eta)_\infty$ has order one along S, this 1-form can be written as $\eta = a\,df/f + \eta_2$ for some closed 1-form η_2 and some constant $a \in \mathbb{C}^*$. Moreover, the 1-form η_2 has polar divisor which does not contain† S. We assume that $\mathcal{F}$ is not given by a closed rational 1-form. Thus, according to Proposition 4 above we may assume some of the singular holonomy groups associated to some of the resolutions of $\mathcal{F}|_S$ are not abelian. (The strict transform of) S is non-abelian, contained in some $\mathbb{H}_k$, $k \in \mathbb{N}$. We claim that $df \wedge \eta_2 \not\equiv 0$. Indeed η is a closed 1-form whose polar divisor contains (and is simple at) $(\Omega)_\infty$ which in turn does not contain S. Since $(\eta_2)_\infty = (\eta)_\infty\backslash S$, we still have that $(\Omega)_\infty$ is contained in $(\eta_2)_\infty$. As the polar divisor of η_2 is invariant for the foliation defined by η_2, this implies that $df \wedge \eta_2 \not\equiv 0$. Thus we can write $\Omega = p\,df - f\Omega_1$ for some meromorphic 1-form $\Omega_1 := h\eta_2$ and some meromorphic functions p, h. Since S is invariant we have that $\Omega_1 \wedge df$ is a holomorphic 2-form outside $(\Omega)_\infty$. In particular, S is not a component of $(h)_\infty$. We define the 1-form $\eta_1 := \eta + ag\Omega_1$ where $g := -1/pf$. Replacing these expressions we obtain

$$\eta_1 = \left(1 + \frac{ah}{p}\right)\eta_2.$$

We claim that η_1 is a logarithmic derivative of Ω and is transverse to S. It is necessary to prove that (1) $(h)_\infty\backslash(\Omega)_\infty$ is invariant by $\mathcal{F}$; (2) $\{p = 0\}$ is invariant and S is not one of its components.

Proof of (i). Let L be any component of $(h)_\infty\backslash(\Omega)_\infty$. We have $d\Omega = \eta \wedge \Omega$, $\eta = 2\,df/f + \eta_2$, so that $d(\Omega/f^2) = \eta_2 \wedge (\Omega/f^2)$. Suppose that L is not contained in $\{f = 0\} \cup (\eta_2)_\infty$, which is invariant. Thus we choose a point $q \in L\backslash(\{f = 0\} \cup (\eta_2)_\infty)$ and a local holomorphic function $\varphi : U \to \mathbb{C}^*$ defined in $U \ni q$, such that $\eta_2 = d\varphi/\varphi$. Then we have $d((1/\varphi f^2)\Omega) = 0$. On the other hand, we have

$$\frac{-p}{\varphi}d\left(\frac{1}{f}\right) + \frac{h}{f}d\left(\frac{1}{\varphi}\right) = \frac{1}{\varphi f^2}\Omega.$$

Thus we conclude that L is contained in the polar set of the closed meromorphic 1-form $(1/\varphi f^2)\Omega$ and therefore (as a consequence of a general fact) it is invariant by $\mathcal{F}$.

† Notice that if η_2 is holomorphic then $\eta_2 \equiv 0$.

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Proof of (ii). If there is some component L of $\{p = 0\}$ which is contained in S then we choose local coordinates (x, y) centered at a generic point $p \in L$ such that $L = (y = 0)$. Then we can write $p(x, y) = y^m u(x, y)$ for some unity u and some $m \in \mathbb{N}$. Thus we have $\Omega = y^m u\, dy - y\Omega_1$, and $\Omega_1 \wedge dy$ is holomorphic. But since $\Omega_1 = h\eta_2$ and η_2 is closed with poles not contained in $S = (y = 0)$, it follows that $\eta_2 = d\psi$ for some local *holomorphic* function $\psi(x, y)$. Since the singularities of Ω are isolated there is a meromorphic local function H such that $h = H/y^r, r \geq 1$, H not identically zero at $(y = 0)$. Therefore $(H/y^r)\, d\psi \wedge dy$ is holomorphic, which implies that $\psi = y^{r+1}\theta$ for some holomorphic θ. Therefore we have

$$\Omega = y^m u\, dy - (r + 1)\theta y H\, dy - y^2 H\, d\theta$$

which is a clear contradiction with the fact that Ω has isolated singularities outside $(\Omega)_\infty$. Assume now that $\{p = 0\}$ has some component L which is not invariant. Then L is not contained in $(\eta_2)_\infty$. First we remark that $L \cap \{f = 0\} \subset \operatorname{sing} \mathcal{F}$. In fact fixed any point $q \in L \cap \{f = 0\}$, since η_2 is closed we can assume that $\eta_2 = d\varphi$ for some local holomorphic function φ defined around q. Thus $\Omega = p\, df - fh\, d\varphi$ and then clearly $q \in \{p = 0\} \cap \{f = 0\}$ is a singularity of $\mathcal{F}$, which gives a contradiction. Now we conclude. Let us assume that $(\eta_2)_\infty \cup \{f = 0\}$ is a divisor with normal crossings (this is the final picture after a resolution process so that we can assume, without loss of generality, that $\tilde{M} = \mathbb{C}P(2)$ as announced above). Let us write $(\eta_2)_\infty = \cup_{j=1}^m \{f_j = 0\}$ in irreducible components where f_j is a polynomial for all j. Then given a point $q \in \{f = 0\} \cap \{f_j = 0\} \cap \{p = 0\}$ we choose local coordinates $(x, y) \in U, x(q) = y(q) = 0$, such that $f(x, y) = y$ and $\{f_j = 0\} \equiv (x = 0)$. In this local chart we have $\eta_2 = 2\, dx/x + d\varphi$ for some local holomorphic function $\varphi(x, y)$. Therefore,

$$\Omega = p\, df - fh\eta_2 = (p - yh\varphi_y)\, dy - yh\left(\frac{2}{x} + \varphi_x\right) dx.$$

Since Ω is holomorphic around q it follows that $x | yh$, so $x | h$. Now since $(x = 0)$ is invariant it follows that $x | (p - yh\varphi_y)$, then $x | p$ (because $x | yh\varphi_y$). Thus $\{p = 0\} \equiv (x = 0)$ is invariant.

Finally, we remark that since $\eta - \eta_2$ divides Ω and η is closed, it follows that if η_1 is still closed then necessarily Ω admits a meromorphic integrating factor, so that $\mathcal{F}$ must be given by a closed meromorphic 1-form. □

5. *Proof of Theorem 3 in dimension two*

In this section we prove Theorem 3 in dimension two. The implication (b) $\Longrightarrow$ (a) is in Proposition 3. Now we proceed to prove (a) $\Longrightarrow$ (b). Let, as usual, $D = \bigcup_{j=1}^m D_j$ denote the resolution divisor of $\operatorname{sing} \mathcal{F} \cap S$. Let us fix a meromorphic integrable 1-form Ω with $\operatorname{codim} \operatorname{sing} \Omega \geq 2$ and $(\Omega)_\infty$ transverse to S, which defines $\mathcal{F}$. According to [**16**], the existence of a Liouvillian first integral for $\mathcal{F}$ implies that Ω admits a (rational) logarithmic derivative, say η. We have two possibilities: either η is unique or $\mathcal{F}$ is given by a closed rational 1-form. We will assume that $\mathcal{F}$ is not given by a closed rational 1-form. In this case some of the projective singular holonomy groups G_j^{sing} must be non-abelian (Proposition 4). The 1-form η induces a transversely affine structure for $\mathcal{F}$ outside

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the polar set $(\eta)_\infty$. Now, since each component D_j contains a non-resonant linearizable singularity, we can apply [**14**] in order to conclude that: (i) each projective holonomy group G_j appearing in the resolution of sing $\mathcal{F} \cap S$ is solvable analytically normalizable (possibly abelian); (ii) the 1-form $\tilde{\eta}$ which denotes the logarithmic derivative associated to the pull-back $\tilde{\Omega}$ of Ω must have simple poles along each component D_j of the resolution divisor $D = \pi^{-1}(S)$ of sing $\mathcal{F} \cap S$. In particular, η has simple poles along S.

Now we prove that the singular holonomy groups associated to the components D_j are solvable. Indeed, according to the proofs of Proposition 3, Lemma 3 and Lemma 6 we have that, for any fixed singularity $q = D_i \cap D_j \in \text{sing}\,\tilde{\mathcal{F}} \cap D$, the 1-form $\tilde{\eta}$ has suitable local expressions as follows. Choose local coordinates $(x, y) \in U$ centered at q, such that $\tilde{\mathcal{F}}$ has one of the local forms (a), (b), (c) or (d) of Lemma 3. Define as usual the local transverse sections $\Sigma_i = (y = 1)$ and $\Sigma_j = (x = 1)$. We can assume that if q is a saddle node then it has its strong manifold contained in D_i. We may also assume that $x|_{\Sigma_i}$ normalizes the holonomy group $\text{Hol}(\tilde{\mathcal{F}}, D_i, \Sigma_i)$. Now, choose a system of coordinates $(u, z) \in W$, centered at the point $q_j = D_j \cap \Sigma_j$, which satisfies $D_j \cap W = (z = 0)$, $\tilde{\Omega}(u, z) = g_1.dz$ (for some meromorphic function g_1) and such that we have a normalizing coordinate $z \in \Sigma_j$ for the holonomy group $\text{Hol}(\tilde{\mathcal{F}}, D_j, \Sigma_j)$ (see Lemma 5). First we consider the case where $\text{Res}_{D_j} \tilde{\eta} = k_j + 1$, for some $k_j \in \mathbb{N}$. We know that $\tilde{\eta} = (k_j + 1)\, d \log z + d \log(g_1)$. If q is of the form (a) in Lemma 3, then we have

$$\tilde{\Omega}(x, y) = g(x\, dy - \lambda y\, dx),$$

and

$$\tilde{\eta} = \frac{dg}{g} + a\frac{dx}{x} + (k_j + 1)\frac{dy}{y} + d\varphi(x, y)$$

where $1 + \lambda = a + (k_j + 1)\lambda$ (see the proof of Theorem 2), where g is meromorphic in U and φ is a holomorphic first integral for $\mathcal{F}|_U$. Thus if $\lambda \notin \mathbb{Q}$ then φ must be constant and we have

$$\tilde{\eta} = \frac{dg}{g} + (1 - k_j\lambda)\frac{dx}{x} + (k_j + 1)\frac{dy}{y},$$

so that

$$\tilde{\Omega} = g x^{1+\lambda}\, d \log(y x^{-\lambda}),$$

and

$$\tilde{\eta} = d \log(g x^{1+\lambda}) + (k_j + 1)\, d \log(y x^{-\lambda}).$$

These expressions we consider in a simply-connected neighborhood $V \subset U \cap W$ of q_j, which does not intersect $(x = 0)$. Now from the equalities

$$(k_j + 1)\, d \log z + d \log(g_1) = \tilde{\eta} = d \log(g x^{1+\lambda}) + (k_j + 1)\, d \log(y x^{-\lambda}),$$

and from the equalities

$$g_1\, dz = \tilde{\Omega} = \hat{g}\, d\hat{y}, \qquad \text{for } \hat{g} = g x^{1+\lambda}, \hat{y} = y x^{-\lambda},$$

we have $y|_{\Sigma_j} = \hat{y}|_{\Sigma_j} = \psi(z|_{\Sigma_j})$ for some $\psi \in \mathbb{H}_{k_j}$ (see [**14**, ch. I]). Therefore, according to the Compatibility lemma, the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ is solvable. The case $\lambda \in \mathbb{Q}_-$ as well as the other cases are done in a similar way. We use the local

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logarithmic derivatives $\eta_{j,q}$ and the induced local Liouvillian first integrals $\hat{y}$ introduced in Lemma 3. We leave these other cases to the reader. Thus the singular holonomy groups are solvable†. The Structure theorem is therefore proved for $n = 2$. □

6. *Construction of projective triples*

In this section we prove Theorem 4 in the case $n = 2$. Therefore, let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, and let S and $\operatorname{sing}\mathcal{F} \cap S$ be as in the statement, such that $\mathcal{F}$ has a Liouvillian resolution relative to S. We consider $\tilde{\Omega}$ and $\tilde{\eta}$ given by Proposition 3, and denote by $\operatorname{Sep}(\tilde{\mathcal{F}}, \tilde{S})$ the invariant part of the polar divisor $(\tilde{\eta})_\infty$ *minus* $\tilde{S}$. According to the construction of $\tilde{\eta}$ in Lemma 6 and Proposition 3, we know that $\operatorname{Sep}(\tilde{\mathcal{F}}, \tilde{S})$ consists, in a neighborhood of D, of the separatrices of $\tilde{\mathcal{F}}$ that are transverse to the divisor D, and which (as a consequence of the fact that $\tilde{\eta}$ extends meromorphically to $\tilde{M}$) are contained in algebraic invariant curves.

From now on we will adopt the following notation. We denote by $\tilde{\eta}$ (instead of $\tilde{\eta}_1$) the *transverse* to D logarithmic derivative of $\tilde{\Omega}$ on $\tilde{M}$, given by Propositions 3 and 5. We proceed to construct a transversely projective structure for $\tilde{\mathcal{F}}$ in $\tilde{V} \setminus \operatorname{Sep}(\tilde{\mathcal{F}}, \tilde{S})$, where $\tilde{V}$ is a neighborhood of $D = \pi^{-1}(S)$ in $\tilde{M}$, and $\tilde{\xi}$ is a meromorphic 1-form in $\tilde{V}$. We recall the following definition [**14**].

Definition 6. (Projective triples) Let $\mathcal{F}$ be a foliation given by a meromorphic 1-form Ω on M (not necessarily with isolated singularities). We assume that $(\Omega)_0 \cup (\Omega)_\infty$ has no $\mathcal{F}$-invariant component. A 1-form ξ is such that (Ω, η, ξ) defines a *projective triple* for $\mathcal{F}$, if:

(a) $d\Omega = \eta \wedge \Omega$, with $(\eta)_\infty = (\Omega)_\infty \cup (\Omega)_0$, $(\eta)_\infty$ of order one and such that for any irreducible component $C \subset (\Omega)_\infty \cup (\Omega)_0$ we have either $\operatorname{Res}_C \eta = -\operatorname{ord}((\Omega)_\infty, C)$, or $\operatorname{Res}_C \eta = \operatorname{ord}((\Omega)_0, C)$, respectively;

(b) $d\eta = \Omega \wedge \xi$;

(c) $d\xi = \xi \wedge \eta$, $(\xi)_0 = (\Omega)_\infty$ and $(\xi)_\infty = (\Omega)_0$ and for each irreducible component $C \subset (\Omega)_\infty \cup (\Omega)_0$ we have either $\operatorname{ord}((\xi)_0, C) = \operatorname{ord}((\Omega)_\infty, C)$ or $\operatorname{ord}((\xi)_\infty, C) = \operatorname{ord}((\Omega)_0, C)$, respectively.

The existence of a projective triple is equivalent to the existence of a projective transverse structure for a given foliation as we will see in what follows. In what follows we construct a projective triple $(\tilde{\Omega}, \tilde{\eta}, \tilde{\xi})$ for $\tilde{\mathcal{F}}$ in $\tilde{M} \setminus \operatorname{Sep}(\tilde{\mathcal{F}}, \tilde{S})$.

LEMMA 8. *Suppose that* $\dim M = 2$ *and that* $S \subset M$ *is a compact irreducible divisor invariant by* $\mathcal{F}$ *such that the set* $\operatorname{sing}\mathcal{F} \cap S$ *consists of non-dicritical infinitesimally hyperbolic singularities. Assume that* $\mathcal{F}$ *has a Liouvillian resolution relative to* S*, and let* $\tilde{\Omega}, \tilde{\eta}, D_j$ *be as above. There exists a 1-form* $\tilde{\xi}_j$ *defined in a neighborhood* $\mathcal{U}_j$ *of* D_j *such that* $(\tilde{\Omega}, \tilde{\eta}, \tilde{\xi}_j)$ *is a projective triple in* $\mathcal{U}_j \setminus \operatorname{Sep}(\tilde{\mathcal{F}}, \tilde{S})$.

Proof. First we construct the 1-form $\tilde{\xi}_j$ in a neighborhood $\mathcal{U}'$ of $D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$. We consider two distinct cases:

† A proof of the solvability of the singular holonomy groups may also follow from a geometric interpretation of [**13**].

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(i) $G_j^{\text{sing}} \subset \mathbb{H}_{k_j}$ is non-abelian. In this case we consider the covering $\{(x_\alpha, y_\alpha) \in U_\alpha\}$ of D_j given by Lemma 3. In the regular part and also around the singularities of $\tilde{\mathcal{F}}$ in D_j we can write in U_α, $\tilde{\Omega}|_{U_\alpha} = g_\alpha\, df_\alpha$ where $(g_\alpha, f_\alpha) = (g_\alpha, y_\alpha)$ or $(\hat{g}_\alpha, \hat{y}_\alpha)$ depending on whether $U_\alpha \cap \operatorname{sing} \tilde{\mathcal{F}} = \varnothing$ or not and, moreover, if $U_\alpha \cap U_\beta \neq \varnothing$ then $f_\alpha^{k_j} = H_{\alpha\beta}(f_\beta^{k_j})$ for some $H_{\alpha\beta}(z) = \lambda_{\alpha\beta} z/(1 + a_{\alpha\beta} z) \in \mathbb{H}_1$. We write also $\tilde{\Omega} = g_\alpha\, df_\alpha = u_\alpha\, df_\alpha^{k_j}$ where $u_\alpha = g_\alpha / k_j f_\alpha^{k_j - 1}$. Therefore, $d\tilde{\Omega} = (du_\alpha/u_\alpha) \wedge \tilde{\Omega}$ and since $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$, we obtain that there is h_α meromorphic function defined in U_α such that

$$\tilde{\eta} = \frac{du_\alpha}{u_\alpha} + h_\alpha \tilde{\Omega} = \frac{dg_\alpha}{g_\alpha} - (k_j - 1)\frac{df_\alpha}{f_\alpha} + h_\alpha \tilde{\Omega}.$$

Inspired by [**14**] we define $\tilde{\xi}_\alpha^j$ in U_α as

$$\tilde{\xi}_\alpha^j := \frac{h_\alpha^2}{2}\tilde{\Omega} - h_\alpha \tilde{\eta} - dh_\alpha.$$

We proceed to prove that $\tilde{\xi}_\alpha^j = \tilde{\xi}_\beta^j$ in each of these $U_\alpha \cap U_\beta \neq \varnothing$. In fact, according to the identity principle it is enough to verify this identity where $f_\alpha \neq 0 \neq f_\beta$. Now, in $U_\alpha \cap U_\beta \neq \varnothing$ we can rewrite

$$\tilde{\Omega} = g_\alpha\, df_\alpha = g_\beta\, df_\beta \quad \text{and} \quad f_\alpha^{k_j} = \frac{a_{\alpha\beta} f_\beta^{k_j} + b_{\alpha\beta}}{c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta}},$$

with $a_{\alpha\beta} d_{\alpha\beta} - b_{\alpha\beta} c_{\alpha\beta} = 1$. Thus

$$df_\alpha^{k_j} = \frac{1}{(c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})^2} \cdot df_\beta^{k_j}$$

and since $df_\alpha^{k_j}/f_\alpha^{k_j} = k_j\, df_\alpha/f_\alpha$, it follows that

$$\frac{g_\alpha}{f_\alpha^{k_j - 1}} = (c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})^2 \frac{g_\beta}{f_\beta^{k_j - 1}}$$

or

$$\frac{dg_\alpha}{g_\alpha} - (k_j - 1)\frac{df_\alpha}{f_\alpha} = \frac{dg_\beta}{g_\beta} - (k_j - 1)\frac{df_\beta}{f_\beta} + \frac{2c_{\alpha\beta}\, df_\beta^{k_j}}{(c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})}.$$

Thus

$$\tilde{\eta} - h_\alpha \tilde{\Omega} = \tilde{\eta} - h_\beta \tilde{\Omega} + \frac{2c_{\alpha\beta}}{(c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})} df_\beta^{k_j}, \quad (h_\beta - h_\alpha)\tilde{\Omega} = \frac{2c_{\alpha\beta}}{(c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})} df_\beta^{k_j}.$$

From this we obtain

$$(h_\beta - h_\alpha)\frac{1}{k_j}\frac{g_\beta}{f_\beta^{k_j - 1}} = \frac{2c_{\alpha\beta}}{(c_{\alpha\beta} f_\beta^{k_j} + d_{\alpha\beta})} = 2c_{\alpha\beta}\left(\frac{g_\beta}{g_\alpha}\right)^{1/2}\left(\frac{f_\alpha}{f_\beta}\right)^{(k_j - 1)/2},$$

or

$$h_\beta - h_\alpha = 2k_j c_{\alpha\beta} \frac{(f_\alpha f_\beta)^{(k_j - 1)/2}}{(g_\alpha g_\beta)^{1/2}}$$

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and so

$$\frac{d(h_\beta - h_\alpha)}{h_\beta - h_\alpha} = +\frac{k_j - 1}{2}\left(\frac{df_\alpha}{f_\alpha} + \frac{df_\beta}{f_\beta}\right) - \frac{1}{2}\left(\frac{dg_\alpha}{g_\alpha} + \frac{dg_\beta}{g_\beta}\right) = -\tilde{\eta} + \frac{h_\alpha + h_\beta}{2}\tilde{\Omega}.$$

Finally,

$$\tilde{\xi}^j_\beta - \tilde{\xi}^j_\alpha = (h_\beta - h_\alpha)\left[\frac{h_\beta + h_\alpha}{2}\tilde{\Omega} - \tilde{\eta} - \frac{d(h_\beta - h_\beta)}{h_\beta - h_\alpha}\right] = 0.$$

Thus we have constructed $\tilde{\xi}_j$ in a neighborhood $\mathcal{U}'_j$ of $D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$.

(ii) $G^{\operatorname{sing}}_j$ is abelian linearizable. Proceeding as in [**2**] one can find a family of coordinate charts $\{(x_\alpha, y_\alpha) \in U_\alpha\}_{\alpha\in A}$, such that:

(a) $\bigcup_{\alpha\in A} U_\alpha = D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$, and $\tilde{\mathcal{F}}|_{U_\alpha}$ is given by $dy_\alpha = 0$;
(b) if $U_\alpha \cap U_\beta \neq \varnothing$ then $y_\alpha = \lambda_{\alpha\beta} y_\beta$ for some constant $\lambda_{\alpha\beta} \neq 0$;
(c) the restriction $y_\alpha|_{\Sigma_\alpha}$, where Σ_α is the local transverse section given by $(x_\alpha = 1)$, linearizes the singular holonomy group $G^{\operatorname{sing}}_j$.

We write $\tilde{\Omega}|_{U_\alpha} = g_\alpha\, dy_\alpha$ and $\tilde{\eta}|_{U_\alpha} = dg_\alpha/g_\alpha + h_\alpha\tilde{\Omega}$. Now, if $U_\alpha \cap U_\beta \neq \varnothing$, then we have $g_\alpha\, dy_\alpha = g_\beta\, dy_\beta$ so that $g_\beta = \lambda_{\alpha\beta}.g_\alpha$. Therefore, $dg_\alpha/g_\alpha = dg_\beta/g_\beta$ so that $h_\alpha = h_\beta$ in $U_\alpha \cap U_\beta$. As above we define the forms $\tilde{\xi}^j_\alpha = \frac{1}{2}h^2_\alpha\tilde{\Omega} - h_\alpha\tilde{\eta} - dh_\alpha$. We conclude that $\tilde{\xi}^j_\alpha = \tilde{\xi}^j_\beta$ in each $U_\alpha \cap U_\beta \neq \varnothing$. Thus once again we have constructed $\tilde{\xi}_j$ in a neighborhood $\mathcal{U}'$ of $D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$ (this neighborhood may contain no punctured polydisk centered at some singularity).

Now we will extend the 1-form $\tilde{\xi}_j$ to the singularities of $\tilde{\mathcal{F}}$ in D_j. Again taking the covering given by Lemma 5 and the proof of Proposition 4, we consider a singular point $q \in U_\alpha \cap D_j$ and the closed 1-form $\eta_{j,\alpha}$ defined in U_α and adapted to D_j. We use the following lemma of straightforward validation.

LEMMA 9. *Let $0 \in \mathbb{C}^2$ be a germ of singularity for a foliation $\mathcal{F} : \Omega = 0$, of the form (a), (b), (c) or (d) in Lemma 3, where the germ Ω has an isolated singularity at zero. Denote by $D \subset \mathbb{C}^2$ the germ of the analytic set of separatrices of Ω through zero. Assume that Ω admits a simple pole closed logarithmic derivative say, η_0, in a neighborhood of zero and let η be a logarithmic derivative transverse to D, given by $\eta = \eta_0 + h\Omega$, for some suitable meromorphic function h. Then the 1-form $\xi := \frac{1}{2}h^2\Omega - h\eta - dh$ defines a projective triple (Ω, η, ξ) in a neighborhood of $0 \in \mathbb{C}^2$.*

Using this lemma we write that in U_α, $\tilde{\eta}|_{U_\alpha} = \eta_{j,\alpha} + \tilde{h}_\alpha\tilde{\Omega}$ for some meromorphic function $\tilde{h}_\alpha$ which is holomorphic in a neighborhood of $D_j \setminus \operatorname{sing}\tilde{\mathcal{F}}$, minus the polar set of $\tilde{\Omega}$, and define the 1-form $\xi_{j,\alpha}$ in U_α by $\xi_{j,\alpha} := \frac{1}{2}\tilde{h}^2_\alpha\tilde{\Omega} - \tilde{h}_\alpha\tilde{\eta} - d\tilde{h}_\alpha$. It is easy to see that $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$, $d\tilde{\eta} = \xi_{j,\alpha} \wedge \tilde{\Omega}$, $d\xi_{j,\alpha} = \xi_{j,\alpha} \wedge \tilde{\eta}$. Thus $(\tilde{\Omega}, \tilde{\eta}, \xi_{j,\alpha})$ defines a projective triple for $\tilde{\mathcal{F}}$ wherever $\tilde{\Omega}$, $\tilde{\eta}$, $\tilde{\xi}$ are holomorphic. Now we remark that in $\mathcal{U}'_j \cap U_\alpha$ we have $\tilde{\xi}_j = \xi_{j,\alpha} + \ell_\alpha\tilde{\Omega}$ for some meromorphic function ℓ_α such that $d(\ell^{1/2}_\alpha\tilde{\Omega}) = 0$ (see [**14**], and also the proof of Proposition 3 for the possibilities for $\tilde{h}_\alpha$). Since $\tilde{\xi}_j|_{\mathcal{U}'_j \cap U_\alpha}$ is holomorphic and $\xi_{j,\alpha}$ is holomorphic in $U_\alpha \setminus (x_\alpha = 0)$ we have that ℓ_α is holomorphic in $U_\alpha \setminus (x_\alpha = 0)$ and thus using the normal forms (a), (b), (c) and (d) of Lemma 3 for $\tilde{\Omega}|_{U_\alpha}$ it follows that $\ell \equiv 0$ and therefore $\tilde{\xi}_j$ extends to U_α as $\xi_{j,\alpha}$. Therefore, $\tilde{\xi}_j$ is holomorphic along $D_j \setminus (\operatorname{sing}\tilde{\mathcal{F}} \cup (\Omega)_\infty)$. This completes the proof of Lemma 8. □

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Now it remains to show that if $q_0 = D_i \cap D_j$ then, in a neighborhood of q_0, $\tilde{\xi}_i = \tilde{\xi}_j$. Here we use the same arguments in the proof of Proposition 3. We give a proof of one case: assume that both singular holonomy groups are non-commutative, say $G_i^{\text{sing}} \subset \mathbb{H}_{k_i}$ and $G_j^{\text{sing}} \subset \mathbb{H}_{k_j}$, and that q_0 is linearizable resonant. We choose normalizing coordinates $f_i \in (\Sigma_i, q_i)$ and $f_j \in (\Sigma_j, q_j)$ for these groups such that, in a neighborhood of q_i, respectively q_j, we have $\tilde{\Omega} = g_i\, df_i$, respectively $\tilde{\Omega} = g_j\, df_j$, for some meromorphic function g_i, respectively g_j. Moreover, we can assume that we have

$$\tilde{\eta} = \frac{dg_i}{g_i} - (k_i - 1)\frac{df_i}{f_i} + h_i\tilde{\Omega}$$

and

$$\tilde{\eta} = \frac{dg_j}{g_j} - (k_j - 1)\frac{df_j}{f_j} + h_j\tilde{\Omega}$$

respectively. Using the Dulac correspondence associated to the corner q_0 we can interpret the solvability of the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ as the following compatibility condition for f_i and f_j;

$$f_j^{k_j} = \frac{f_i^{k_i}}{1 + bf_i^{k_i}}.$$

Now, by definition

$$\tilde{\xi}_i = \frac{h_i^2}{2}\tilde{\Omega} - h_i\tilde{\eta} - dh_i$$

and

$$\tilde{\xi}_j = \frac{h_j^2}{2}\tilde{\Omega} - h_j\tilde{\eta} - dh_j.$$

Therefore a similar computation as above shows that

$$\frac{d(h_i - h_j)}{h_i - h_j} = \frac{k_j - 1}{2}\frac{df_j}{f_j} + \frac{k_i - 1}{2}\frac{df_i}{f_i} - \frac{1}{2}\frac{dg_i}{g_i} - \frac{1}{2}\frac{dg_j}{g_j} = -\tilde{\eta} + \frac{h_i + h_j}{2}\tilde{\Omega},$$

therefore

$$\tilde{\xi}_i - \tilde{\xi}_j = (h_i - h_j)\left[\frac{h_i + h_j}{2}\tilde{\Omega}|_{U_\alpha} - \tilde{\eta} - \frac{d(h_i - h_j)}{h_i - h_j}\right] = 0.$$

The other cases are treated in a similar way. Thus the collection of 1-forms $(\tilde{\xi}_i)_{i=1}^m$ defines a 1-form $\tilde{\xi}$ in a neighborhood $\tilde{\mathcal{U}}$ of $\pi^{-1}S$. Moreover, it is evident that $d\tilde{\eta} = \tilde{\xi} \wedge \tilde{\Omega}$ and $d\tilde{\xi} = \tilde{\xi} \wedge \tilde{\eta}$.

Using the construction of $\tilde{\eta}$, $\tilde{\xi}$, the properties required for the divisor $(\tilde{\Omega})_\infty \cup D$ and techniques from [**14**] we will prove the following.

PROPOSITION 6. *$\tilde{\xi}$ defines a transversely projective foliation $\mathcal{F}^\perp$ in a neighborhood of D.*

Proof. We use the fact that $\tilde{\xi}$ is holomorphic outside $D \cup (\tilde{\Omega})_\infty \cup \text{Sep}(\tilde{\mathcal{F}}, D)$, that $\tilde{\eta}$ is transverse to D and that $(\tilde{\Omega})_\infty$ is in general position with respect to $\tilde{\mathcal{F}}$. Fix a component $D_j \subset D$, $j \neq 0$, and take the open covering $(x_\alpha, y_\alpha) \in U_\alpha$ of D_j, given by Lemmas 5 and 6.

First we prove the following.

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LEMMA 10. [**14**] *Let* $U_\alpha \cap (\operatorname{sing} \tilde{\mathcal{F}} \cup (\tilde{\Omega})_\infty \cup (\tilde{\Omega})_0) = \varnothing$. *We may write locally*

$$\tilde{\Omega}|_{U_\alpha} = X_\alpha\, dY_\alpha - Y_\alpha\, dX_\alpha = Y_\alpha^2\, d\left(-\frac{X_\alpha}{Y_\alpha}\right),$$

$$\tilde{\eta} = 2\frac{dY_\alpha}{Y_\alpha} + h_\alpha(X_\alpha\, dY_\alpha - Y_\alpha\, dX_\alpha) = 2(W_\alpha\, dX_\alpha - V_\alpha\, dY_\alpha)$$

and

$$\tilde{\xi} = 2W_\alpha^2\, d\left(\frac{V_\alpha}{W_\alpha}\right)$$

where

$$\begin{pmatrix} X_\alpha & V_\alpha \\ Y_\alpha & W_\alpha \end{pmatrix} : U_\alpha \to \mathbb{P}\,\mathrm{SL}(2, \mathbb{C})$$

is a holomorphic mapping.

Proof. We write $\tilde{\Omega}(x_\alpha, y_\alpha) = g_\alpha\, dy_\alpha$. Assume that $(\tilde{\Omega})_\infty \cap U_\alpha = \varnothing$. Then g_α is holomorphic and non-vanishing so that we may define Y_α by $Y_\alpha^2 := g_\alpha$ and $-X_\alpha/Y_\alpha := y_\alpha$ defines X_α. Thus we have

$$\tilde{\Omega}(x_\alpha, y_\alpha) = Y_\alpha^2\, d\left(-\frac{X_\alpha}{Y_\alpha}\right) \Rightarrow \tilde{\Omega}(x_\alpha, y_\alpha) = X_\alpha\, dY_\alpha - Y_\alpha\, dX_\alpha.$$

Therefore we have

$$\tilde{\eta}(x_\alpha, y_\alpha) = 2\frac{dY_\alpha}{Y_\alpha} + h_\alpha(X_\alpha\, dY_\alpha - Y_\alpha\, dX_\alpha)$$

so that we define

$$V_\alpha := \frac{-1}{Y_\alpha} - h_\alpha \frac{X_\alpha}{2}$$

so that if $W_\alpha := -Y_\alpha h_\alpha/2$ then we have

$$X_\alpha W_\alpha - V_\alpha Y_\alpha = X_\alpha\left(-\frac{Y_\alpha h_\alpha}{2}\right) - \left(-\frac{1}{Y_\alpha} - h_\alpha\frac{X_\alpha}{2}\right) Y_\alpha = 1.$$

Also we have

$$\tilde{\eta}(x_\alpha, y_\alpha) = 2(W_\alpha\, dX_\alpha - V_\alpha\, dY_\alpha)$$

and

$$\tilde{\xi}(x_\alpha, y_\alpha) = \frac{1}{2}h_\alpha^2\Omega - h_\alpha\eta - dh_\alpha = W_\alpha\, d\left(\frac{V_\alpha}{W_\alpha}\right). \qquad \square$$

We will also use the following lemma.

LEMMA 11. [**14**] *Let*

$$\begin{pmatrix} x & v \\ y & \Omega \end{pmatrix}, \begin{pmatrix} \bar{x} & \bar{v} \\ \bar{y} & \bar{\Omega} \end{pmatrix} : U \to \mathbb{P}\,\mathrm{SL}(2, \mathbb{C})$$

be holomorphic mappings where $U \subset \mathbb{C}^n$ *is a connected open subset. Assume that we have:*

(a) $x^2\, d(y/x) = \bar{x}^2\, d(\bar{y}/\bar{x})$;

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(b) $\Omega\,dx - v\,dy = \bar{\Omega}\,d\bar{x} - \bar{v}\,d\bar{y}$;
(c) $\Omega^2\,d(v/\Omega) = \bar{\Omega}^2\,d(\bar{v}/\bar{\Omega})$.
Then there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ *such that* $\left(\begin{smallmatrix} \bar{x} & \bar{v} \\ \bar{y} & \bar{\Omega} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & v \\ y & \Omega \end{smallmatrix}\right)$.

Now we consider two holomorphic mappings

$$\begin{pmatrix} X_\alpha & V_\alpha \\ Y_\alpha & W_\alpha \end{pmatrix} : U_\alpha \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}), \quad \begin{pmatrix} X_\beta & V_\beta \\ Y_\beta & W_\beta \end{pmatrix} : U_\beta \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$$

as in Lemma 10, such that $U_\alpha \cap U_\beta \neq \varnothing$. Then, in the intersection $U_\alpha \cap U_\beta$, we have

$$\begin{pmatrix} X_\beta & V_\beta \\ Y_\beta & W_\beta \end{pmatrix} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \begin{pmatrix} X_\alpha & V_\alpha \\ Y_\alpha & W_\alpha \end{pmatrix}$$

for some

$$\begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \in \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}).$$

In particular,

$$\frac{V_\alpha}{W_\beta} = \frac{a_{\alpha\beta}V_\alpha + b_{\alpha\beta}W_\alpha}{c_{\alpha\beta}V_\alpha + d_{\alpha\beta}W_\alpha} = \frac{a_{\alpha\beta}V_\alpha/W_\alpha + b_{\alpha\beta}}{c_{\alpha\beta}V_\alpha/W_\alpha + d_{\alpha\beta}}.$$

This already shows that $\mathcal{F}^\perp$ is transversely projective in a neighborhood of $D \setminus ((\tilde{\Omega})_\infty \cup (\tilde{\Omega})_0 \cup \operatorname{sing}\tilde{\mathcal{F}})$.

Let us consider the case where $U_\alpha \cap (\tilde{\Omega})_\infty \neq \varnothing$. In this case we may assume that $(\tilde{\Omega})_\infty \cap U_\alpha = f^{-1}(0)$ for some holomorphic reduced equation $f : U_\alpha \to \mathbb{C}$.

We write, therefore, $\tilde{\Omega}(x_\alpha, y_\alpha) = f^{-n}\tilde{\Omega}'(x_\alpha, y_\alpha)$ where $n = \operatorname{ord}((\tilde{\Omega})_\infty, (f = 0)) \in \mathbb{N}$.

By the definition of projective triple we have $\tilde{\eta} = \tilde{\eta'} + (-n)\cdot df/f$, where $\tilde{\eta'}$ is holomorphic in U_α and is a logarithmic derivative of $\tilde{\Omega}'$, transverse to D_j. We define

$$\tilde{\xi'} := \frac{1}{f^{-n}}\tilde{\xi} = f^n\tilde{\xi}$$

in U_α and obtain a projective triple $(\tilde{\Omega}', \tilde{\eta}', \tilde{\xi}')$ in U_α which is holomorphic and by the preceding case we may find a holomorphic mapping

$$\begin{pmatrix} X_\alpha & V_\alpha \\ Y_\alpha & W_\alpha \end{pmatrix} : U_\alpha \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$$

such that

$$\tilde{\Omega}'|_{U_\alpha} = X_\alpha\,dY_\alpha - Y_\alpha\,dX_\alpha, \quad \tilde{\eta}'|_{U_\alpha} = 2(W_\alpha\,dX_\alpha - V_\alpha\,dY_\alpha),$$
$$\tilde{\xi}'|_{U_\alpha} = 2W_\alpha\,d\left(\frac{V_\alpha}{W_\alpha}\right) = 2(W_\alpha\,dV_\alpha - V_\alpha\,dW_\alpha).$$

Therefore, we obtain

$$\tilde{\Omega}|_{U_\alpha} = \frac{X_\alpha\,dY_\alpha - Y_\alpha\,dX_\alpha}{f^n}, \quad \tilde{\eta}|_{U_\alpha} = 2(W_\alpha\,dX_\alpha - V_\alpha\,dY_\alpha) - \frac{n\,df}{f},$$
$$\tilde{\xi}|_{U_\alpha} = 2f^n(W_\alpha\,dV_\alpha - V_\alpha\,dW_\alpha).$$

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Then we may assume that $U_\beta \cap U_\alpha \neq \varnothing \Rightarrow U_\beta \cap (\tilde{\Omega})_\infty = \varnothing$ and therefore in $U_\alpha \cap U_\beta$ we have $f \neq 0$, so that we may consider a holomorphic branch† of $f^{n/2}$, and obtain:

$$\frac{X_\alpha^2}{f^n}\, d\left(\frac{Y_\alpha}{X_\alpha}\right) = \left(\frac{X_\alpha}{f^{n/2}}\right)^2 \cdot d\left(\frac{Y_\alpha \cdot f^{-n/2}}{X_\alpha \cdot f^{-n/2}}\right)$$
$$= X_\beta^2 \cdot d\left(\frac{Y_\beta}{X_\beta}\right), \tag{a}$$

$$2(W_\alpha\, dX_\alpha - V_\alpha\, dY_\alpha) - \frac{n\, df}{f} = 2[(W_\alpha \cdot f^{n/2}) \cdot d(X_\alpha f^{-n/2}) - V_\alpha f^{n/2} \cdot d(Y_\alpha f^{-n/2})]$$
$$= 2(W_\beta\, dX_\beta - V_\beta\, dY_\beta), \tag{b}$$

$$2f^n \left(W_\alpha^2\, d\left(\frac{V_\alpha}{W_\alpha}\right)\right) = 2(W_\alpha f^{n/2}) \cdot d\left(\frac{V_\alpha f^{n/2}}{W_\alpha f^{n/2}}\right) = 2W_\beta^2\, d\left(\frac{V_\beta}{W_\beta}\right). \tag{c}$$

Thus according to Lemma 11 above we obtain:

$$\begin{pmatrix} X_\beta & V_\beta \\ Y_\beta & W_\beta \end{pmatrix} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \cdot \begin{pmatrix} X_\alpha f^{-n/2} & V_\alpha f^{n/2} \\ Y_\alpha f^{-n/2} & W_\alpha f^{n/2} \end{pmatrix}$$

and therefore

$$\frac{V_\beta}{W_\beta} = \left[a_{\alpha\beta}\left(\frac{V_\alpha f^{n/2}}{W_\alpha f^{n/2}}\right) + b_{\alpha\beta}\right] \cdot \left[c_{\alpha\beta}\left(\frac{V_\alpha f^{n/2}}{W_\alpha f^{n/2}}\right) + d_{\alpha\beta}\right]^{-1}$$
$$= \frac{a_{\alpha\beta} V_\alpha / W_\alpha + b_{\alpha\beta}}{c_{\alpha\beta} V_\alpha / W_\alpha + d_{\alpha\beta}}\ .$$

The argument above extends the projective transverse structure of $\tilde{\mathcal{F}}^\perp$ to the polar divisor $(\tilde{\Omega})_\infty$.

Finally, we consider the case where $D_j \subset (\tilde{\Omega})_0$. We write, therefore, $\tilde{\Omega}(x_\alpha, y_\alpha) = y_\alpha^{-n} \tilde{\Omega}'(x_\alpha, y_\alpha)$ where $n = \mathrm{ord}((\tilde{\Omega})_\infty, D_j) \in \mathbb{N}$.

By the definition of projective triple we have $\tilde{\eta} = \tilde{\eta'} + (-n) \cdot dy_\alpha / y_\alpha$ where $\tilde{\eta'}$ is a logarithmic derivative of $\tilde{\Omega}'$. We define

$$\tilde{\xi'} := \frac{1}{y_\alpha^{-n}} \cdot \tilde{\xi} = y_\alpha^n \tilde{\xi}$$

in U_α and obtain a projective triple $(\tilde{\Omega}', \tilde{\eta'}, \tilde{\xi'})$ in U_α, and by Lemma 10 we may find a holomorphic mapping

$$\begin{pmatrix} X_\alpha & V_\alpha \\ Y_\alpha & W_\alpha \end{pmatrix} : U_\alpha \to \mathbb{P}\,\mathrm{SL}(2, \mathbb{C})$$

such that

$$\tilde{\Omega}'|_{U_\alpha} = X_\alpha\, dY_\alpha - Y_\alpha\, dX_\alpha, \quad \tilde{\eta'}|_{U_\alpha} = 2(W_\alpha\, dX_\alpha - V_\alpha\, dY_\alpha),$$
$$\tilde{\xi'}|_{U_\alpha} = 2W_\alpha\, d\left(\frac{V_\alpha}{W_\alpha}\right) = 2(W_\alpha\, dV_\alpha - V_\alpha\, dW_\alpha)$$

† We consider a fixed holomorphic and non-vanishing determination of $f^{n/2}$ on $U_\alpha \cap U_\beta$, which we assume to be simply connected.

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Now in the intersection $U_\alpha \cap U_\beta \neq \varnothing$ we have $y_\alpha = u_{\alpha\beta} y_\beta$ for some unity $u_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)^*$. Thus we obtain

$$\frac{X_\alpha^2}{u_{\alpha\beta}^n}\, d\left(\frac{Y_\alpha}{X_\alpha}\right) = \left(\frac{X_\alpha}{u_{\alpha\beta}^{n/2}}\right)^2 \cdot d\left(\frac{Y_\alpha \cdot u_{\alpha\beta}^{-n/2}}{X_\alpha \cdot u_{\alpha\beta}^{-n/2}}\right) = X_\beta^2 \cdot d\left(\frac{Y_\beta}{X_\beta}\right), \tag{a}$$

$$\begin{aligned} 2(W_\alpha\, dX_\alpha - V_\alpha\, dY_\alpha) - \frac{n\, du_{\alpha\beta}}{u_{\alpha\beta}} &= 2[(W_\alpha \cdot u_{\alpha\beta}^{n/2}) \cdot d(X_\alpha u_{\alpha\beta}^{-n/2}) - V_\alpha u_{\alpha\beta}^{n/2} \cdot d(Y_\alpha u_{\alpha\beta}^{-n/2})] \\ &= 2(W_\beta\, dX_\beta - V_\beta\, dY_\beta), \end{aligned} \tag{b}$$

$$2u_{\alpha\beta}^n \left(W_\alpha^2\, d\left(\frac{V_\alpha}{W_\alpha}\right)\right) = 2(W_\alpha u_{\alpha\beta}^{n/2}) \cdot d\left(\frac{V_\alpha u_{\alpha\beta}^{n/2}}{W_\alpha u_{\alpha\beta}^{n/2}}\right) = 2W_\beta^2\, d\left(\frac{V_\beta}{W_\beta}\right). \tag{c}$$

Therefore, using Lemma 11 we conclude that in the intersection $U_\alpha \cap U_\beta$, we have

$$\begin{pmatrix} u_{\alpha\beta}^{n/2}.X_\beta & u_{\alpha\beta}^{n/2}.V_\beta \\ u_{\alpha\beta}^{n/2}.Y_\beta & u_{\alpha\beta}^{n/2}.W_\beta \end{pmatrix} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \begin{pmatrix} u_{\alpha\beta}^{n/2}.X_\alpha & u_{\alpha\beta}^{n/2}.V_\alpha \\ u_{\alpha\beta}^{n/2}.Y_\alpha & u_{\alpha\beta}^{n/2}.W_\alpha \end{pmatrix}$$

for some

$$\begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \in \mathbb{P}\,\mathrm{SL}(2, \mathbb{C}).$$

In particular,

$$\frac{V_\alpha}{W_\beta} = \frac{a_{\alpha\beta} V_\alpha + b_{\alpha\beta} W_\alpha}{c_{\alpha\beta} V_\alpha + d_{\alpha\beta} W_\alpha} = \frac{a_{\alpha\beta} V_\alpha / W_\alpha + b_{\alpha\beta}}{c_{\alpha\beta} V_\alpha / W_\alpha + d_{\alpha\beta}}.$$

This proves that $\tilde{\xi}$ defines a transversely projective foliation $\tilde{\mathcal{F}}^\perp$, in a neighborhood of $D_j \setminus (\mathrm{sing}\, \tilde{\mathcal{F}} \cap D_j)$. In order to see that this transverse structure extends to $\mathrm{sing}\, \tilde{\mathcal{F}} \cap D_j$ we proceed as follows. First we notice that, since $\tilde{\Omega}$, $\tilde{\eta}$, $\tilde{\xi}$ are holomorphic outside $(\tilde{\Omega})_\infty \cup D \cup \mathrm{Sep}(\tilde{\mathcal{F}}, \tilde{S})$, it follows that $\tilde{\mathcal{F}}^\perp$ is already transversely projective in $\tilde{M} \setminus (D \cup \mathrm{Sep}(\tilde{\mathcal{F}}, \tilde{S}))$. As we have see above, the fact that $\tilde{\eta}$ is transverse to D implies that this projective transverse structure extends to $D \setminus \mathrm{Sep}(\tilde{\mathcal{F}}, \tilde{S})$. Now, given a component $S_1 \subset \mathrm{Sep}(\tilde{\mathcal{F}}, \tilde{S})$ we have two possibilities.

(a) S_1 is not $\tilde{\mathcal{F}}^\perp$-invariant. In this case the projective transverse structure of $\tilde{\mathcal{F}}^\perp$ extends to S_1 in a natural way [**14**].

(b) S_1 is $\tilde{\mathcal{F}}^\perp$-invariant. Then S_1 meets D at a point $q_0 \in \mathrm{sing}\, \tilde{\mathcal{F}}$ where we have a common local separatrix for $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$.

Analyzing the possibilities for q_0 given by Lemmas 3 and 4 we conclude, using the fact that $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$ have a common separatrix Γ_1 through q_0, that $\tilde{\mathcal{F}}^\perp$ is also transversely projective in a neighborhood of $\Gamma_1 \setminus \{q_0\}$ (see also The Local Case in [**14**]). Thus, $\tilde{\mathcal{F}}^\perp$ is transversely projective in a punctured neighborhood of each singularity $q \in \mathrm{sing}\, \tilde{\mathcal{F}} \cap D$. Since a punctured bidisk is simply-connected, it follows that $\tilde{\mathcal{F}}^\perp$ admits a meromorphic first integral in a neighborhood of each such singularity (perhaps it admits a holomorphic submersion as local first integral). This implies that the projective transverse structure of $\tilde{\mathcal{F}}^\perp$ extends to a neighborhood of D in $\tilde{M}$. This ends the proof of Proposition 6. □

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Proof of Theorem 4 in dimension two. We already know that $\tilde{\mathcal{F}}^{\perp}$ is transversely projective on a neighborhood of D in $\tilde{M}$. Since $\operatorname{sing}\mathcal{F} \cap S$ is non-dicritical $\tilde{\mathcal{F}}^{\perp}$ extends to $\tilde{M}$ as a transversely projective foliation as a consequence of Levi's extension theorem [**2, 14, 17**]. Thus there exists (by blowing-down) a rational projective triple (Ω, η, ξ) where ξ defines a transversely projective foliation $\mathcal{F}^{\perp}$ on $\mathbb{C}P(2)$. Since $\mathbb{C}P(2)$ is simply connected, $\mathcal{F}^{\perp}$ admits a rational first integral [**14**] and therefore by [**14**] $\mathcal{F}$ is either given by a closed rational 1-form or it is a rational Riccati pull-back. In order to conclude that $\mathcal{R}$ is a Bernoulli foliation as in Theorem 1, we use the fact that $\mathcal{F}$ admits a Liouvillian first integral, so that $\mathcal{R}$ is transversely affine outside some algebraic invariant curve (Proposition 1), and this implies that $\mathcal{R}$ has the desired form. □

7. *Theorems 3 and 4 in dimension n*

Now we consider a holomorphic codimension one foliation $\mathcal{F}$ on $M = \mathbb{C}P(n)$, $n \geq 3$, $S \subset M$ an invariant codimension one irreducible divisor. We need the following extension result for adapted logarithmic derivatives.

PROPOSITION 7. *If $\mathcal{F}|_N$ admits a logarithmic derivative adapted to $S \cap N$ for some generic section $N \subset M$ then the same holds for $\mathcal{F}$ and S on M.*

Proof. By induction arguments it is enough to prove the case where $\dim N = \dim M - 1$. Now we recall that for fixed Ω defined on M and defining $\mathcal{F}$, for each regular point $q \in N \setminus \operatorname{sing}\mathcal{F}$, we have $\Omega = g\,df$ for some meromorphic function g and some holomorphic function f, both defined in a neighborhood U of q in M. Furthermore, in $U \cap N$ we have $\eta = a\,df/f + dg/g + d\varphi$, for some holomorphic function $\varphi : U \cap N \to \mathbb{C}$, and therefore using a local flow box for $\mathcal{F}$ around q we can extend φ and η to U, and φ is extended as a holomorphic first integral for $\mathcal{F}$ in U. Two such extensions η and η' are related by $\eta - \eta' = h\Omega$, where h is holomorphic in $U \cap U'$ and $d(h\Omega) = 0$. Therefore, $h\Omega = d\psi$ for some holomorphic first integral ψ of $\mathcal{F}$ in $U \cap U'$, and since $(\eta - \eta')|_{U \cap U' \cap N} \equiv 0$ it follows that ψ is constant in $U \cap U' \cap N$; since it is a transverse section to $\mathcal{F}|_{U \cap U'}$, it follows that $d\psi \equiv 0$ and therefore $\eta \equiv \eta'$ in $U \cap U'$. Thus we can extend η to a neighborhood of $N \setminus (\operatorname{sing}\mathcal{F} \cap N)$ and then using Hartogs' extension theorem we can extend η to a neighborhood of N (recall that $\operatorname{sing}\mathcal{F}$ has codimension two). Then using the fact that $M \setminus N$ is a Stein manifold and the compactness of M, we can use Levi's extension theorem to extend η to M and clearly it is adapted to S on M (see [**2**] for a similar procedure). □

The proof of Theorem 3 in dimension n follows now from the proof in dimension two and from Proposition 7 above.

Proof of Theorem 4. Let $N = N_2 \subset N_3 \subset \cdots N_{n-1} \subset N_n = M$, $\dim_{\mathbb{C}} N_q = q$, be a sequence of subvarieties in general position with respect to $\mathcal{F}$. Let $\mathcal{F}_q = \mathcal{F}|_{N_q}$ and suppose that $S \cap N_2$ has a solvable holonomy for $\mathcal{F}_2$. Then, according to the proof of Proposition 5, there exists a transversely projective form ξ on N which is described locally in a neighborhood V_α of a regular point of the induced foliation $\mathcal{F}_2 = \mathcal{F}|_{N_2}$ by the local

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forms

$$\Omega = -g_\alpha\, df_\alpha, \quad \eta = \frac{dg_\alpha}{g_\alpha} - (k-1)\frac{df_\alpha}{f_\alpha} + h_\alpha\Omega, \quad \xi = \frac{{h_\alpha}^2}{2}\Omega - h_\alpha\eta - dh_\alpha.$$

Now, for such a fixed point, using the local flow box for $\mathcal{F}_3$, we can extend f_α as a constant function along the leaves of $\mathcal{F}_3$ to a neighborhood of this point in N_3 and the same can be done with g_α, h_α, and therefore with ξ (recall that Ω, η are globally defined). Moreover, this extension, say ξ_α, also keeps the local form above so that it satisfies the same projective relations as before. This procedure gives an open covering $(V_\alpha)_\alpha$ of $N_2 - \operatorname{sing}\mathcal{F}_2$ such that in each open set V_α we have defined a ξ_α with $\xi_\alpha|_{V_\alpha\cap N_2} = \xi|_{V_\alpha\cap N_2}$. If $V_\alpha \cap V_\beta \neq \varnothing$, then we have $\xi_\alpha = \xi_\beta + l_{\alpha,\beta}\Omega$ for some meromorphic function $l_{\alpha,\beta}$ on this intersection, satisfying $dl_{\alpha,\beta} \wedge \Omega = -2l_{\alpha,\beta}\, d\Omega$ and $l_{\alpha,\beta} = 0$ in $N \cap V_\alpha \cap V_\beta$ (see [**14**]). But by construction this function is constant along the leaves of $\mathcal{F}$ so that $dl_{\alpha,\beta} \wedge \Omega = 0$ and then if Ω is not closed we have $l_{\alpha,\beta} = 0$. Thus we have defined ξ in a neighborhood of $N_2 \setminus \operatorname{sing}\mathcal{F}_2$ in N_3 and we want to extend this form to a neighborhood of $\operatorname{sing}\mathcal{F}_2$. This is a consequence of Hartogs' extension theorem [**17**], because $\operatorname{sing}\mathcal{F}_3$ is a codimension two analytic set in N_3 and the form ξ is already defined over the base S (see also [**2**]). Once this form is defined in a neighborhood of N_2, it extends to N_3 by the standard argument on Stein spaces, and this extension is also transversely projective. The same argument allows us to pass from $N_q \subset M$ to $N_{q+1} \subset M$ by an induction process. □

Final remarks. Our last comment is the following. Let $\Omega = P\,dx - Q\,dy$ be a polynomial differential 1-form on $\mathbb{C}^2$. Denote by $\mathcal{F}_\Omega$ the corresponding algebraic foliation on $\mathbb{C}P(2)$. The existence of a Liouvillian first integral for $\mathcal{F}_\Omega$ means that the ordinary differential equation $dy/dx = P/Q$, can be integrated by the use of standard differential calculus functions. According to this viewpoint, Theorem 1 assures that (under some hypothesis on the singularities) the highest level of transcendence for the ordinary differential equations which can be integrated by the use of Calculus functions is reached by the Bernoulli equations of the form

$$\frac{dy}{dx} = \alpha(x)y^2 + \beta(x)y,$$

where $\alpha(x)$ and $\beta(x)$ are rational functions.

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References

[1] C. Camacho and A. Lins Neto. *Geometric Theory of Foliations*. Birkhäuser, 1985.

[2] C. Camacho, A. Lins Neto and P. Sad. Foliations with algebraic limit sets. *Ann. Math.* **136** (1992), 429–446.

[3] D. Cerveau and J.-F. Mattei. Formes intégrables holomorphes singulières. *Astérisque* **97** (1982).

https://doi.org/10.1017/S0143385701001353 Published online by Cambridge University Press

[4] D. Cerveau and R. Moussu. Groupes d'automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $u\,dy + \cdots = 0$. *Bull. Soc. Math. France* **116** (1988), 495–488.
[5] D. Cerveau and P. Sad. Solvable holonomy groups and Bernoulli foliations. *Trans. Amer. Math. Soc.* **350**(8) (1998), 3065–3081. To appear.
[6] D. Cerveau and B. Azevedo Scárdua. Complete polynomial vector fields in two complex variables. *Preprint*, Université de Rennes I, 2000.
[7] J. P. Jouanolou. *Équations de Pfaff algèbriques (Lecture Notes in Mathematics, 708)*. Springer, Berlin, 1979.
[8] A. Lins Neto. Construction of singular holomorphic vector fields and foliations in dimension two. *J. Diff. Geometry* **26** (1987), 1–31.
[9] J. Martinet and J.-P. Ramis. Classification analytique des équations différentielles non lineaires resonnants du premier ordre. *Ann. Sc. Ec. Norm. Sup.* **16** (1983), 571–621.
[10] J. Martinet and J.-P. Ramis. Problème de modules pour des équations différentielles non lineaires du premier ordre. *Publ. Math. Inst. Hautes Études Scientifiques* **55** (1982), 63–124.
[11] J. F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sc. Ecole Norm. Sup.* **13** (1980), 469–523.
[12] M. Nicolau and E. Paul. A geometric proof of a Galois differential theory theorem. *Preprint*, 1995.
[13] E. Paul. Formes singuliéres à holonomie résoluble. *J. Reine Angew. Math.* **514** (1999), 9–70.
[14] B. Scárdua. Transversely affine and transversely projective holomorphic foliations. *Ann. Sc. E.N.S.* **30** (1997), 169–204.
[15] A. Seidenberg. Reduction of singularities of the differential equation $A\,dy = B\,dx$. *Amer. J. Math.* **90** (1968), 248–269.
[16] M. F. Singer. Liouvillian first integrals of differential equations. *Trans. Amer. Math. Soc.* **333**(2) (1992), 673–688.
[17] Y. Siu. *Techniques of Extension of Analytic Objects*. Marcel Dekker, New York, 1974.

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