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KAKEYA SETS OVER NON-ARCHIMEDEAN LOCAL RINGS

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Abstract. In a recent paper of Ellenberg, Oberlin, and Tao [The Kakeya set and maximal conjectures for algebraic varieties over finite fields. *Mathematika* **56** (2010), 1–25], the authors asked whether there are Besicovitch phenomena in $\mathbb{F}_q\,[[t]]^n$. In this paper, we answer their question in the affirmative by explicitly constructing a Kakeya set of measure zero in $\mathbb{F}_q\,[[t]]^n$. Furthermore, we prove that any Kakeya set in $\mathbb{F}_q\,[[t]]^2$ or $\mathbb{Z}_p^2$ is of Minkowski dimension 2.

§1. *Introduction.* The classical Kakeya problem asks how small (in measure) a subset of the Euclidean plane can be, if it contains a unit line segment in every possible direction; Besicovitch [1] showed that such sets could have measure zero. Wolff [5] proposed a generalization of the Besicovitch–Kakeya problem to a finite field setting, which was solved by Dvir in [3]. We give a natural generalization of the Besicovitch–Kakeya problem to any infinite ring admitting a finite Haar measure.

Let R be an infinite ring admitting a Haar measure μ such that $\mu(R)$ is finite.

Definition. A *line* with direction vector $v \in R^n$ through the point $x \in R^n$ consists of the elements in R^n of the form $x + tv$ as t runs through the elements of R.

Definition. A *Kakeya set* in R^n is a subset of R^n which contains (all the points on) at least one line with each possible direction vector.

The generalization of the classical Besicovitch–Kakeya problem is then the following.

Question 1. If R is an infinite ring admitting a finite Haar measure and $n > 1$, does there exist a Kakeya set of zero Haar measure in R^n?

If such a Kakeya set of measure zero exists, we refer to it as a Besicovitch set. For such sets, the analogue of the so-called "Kakeya conjecture" is the following.

Question 2. If there exists a Besicovitch set in R^n, is its dimension necessarily equal to n?

In [4], Ellenberg *et al* posed Question 1 for $R = \mathbb{F}_q[[t]]$:

Let E be a subset of $\mathbb{F}_q[[t]]^n$ containing a line in every direction, and write E_k for the image of E under the projection $\mathbb{F}_q[[t]]^n \to (\mathbb{F}_q[[t]]/t^k)^n$. Are there

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Besicovitch phenomena in $\mathbb{F}_q[[t]]^n$*? That is, is it possible that*

$$\lim_{k\to\infty} |E_k||\mathbb{F}_q|^{-nk} = 0?$$

In §2 we show that the answer to Question 1 is "yes" for $R = \mathbb{F}_q[[t]]$ by proving the following theorem.

THEOREM 1.1. *For all* $n > 1$*, there exists a subset* $E \subset \mathbb{F}_q[[t]]^n$ *of measure zero which contains a line in every direction.*

In §3 we show that the answer to Question 2 is "yes" for both $R = \mathbb{F}_q[[t]]$ and $R = \mathbb{Z}_p$, in dimension 2 and with the Minkowski dimension.

THEOREM 1.2. *Let* E *be a Kakeya set in* R^2 *where* $R = \mathbb{Z}_p$ *or* $R = \mathbb{F}_q[[t]]$*. Then* E *has Minkowski dimension* 2*.*

Remark. For a finite ring R, the Minkowski dimension of a set $E \subset R^n$ is defined in [4] as $\log|E|/\log|R|$. The natural analogue of the Minkowski dimension of a compact subset $E \subset \mathbb{F}_q[[t]]^n$ is

$$\lim_{k\to\infty} \frac{\log|E_k|}{\log|\mathbb{F}_q|^k},$$

provided the limit exists. We have an analogous definition of Minkowski dimension for subsets of $\mathbb{Z}_p$ using the projections $\mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$. In the sequel, "Minkowski dimension" will always refer to this definition.

We note that the above statements are analogous to the known results for the original Besicovitch–Kakeya problem over $\mathbb{R}$; cf. [1, 2].

§2. *Existence of a Kakeya set of measure zero for* $R = \mathbb{F}_q[[t]]$. Let $\mathbb{F}_q$ be the finite field of order q. We refer to a non-zero direction vector $v = (a, b)$ in $\mathbb{F}_q[[t]]^2$ as *non-reduced* if t divides both a and b, and as *reduced* otherwise. It is obvious that any line with non-reduced direction vector v passing through (x, y) is contained in the line with direction vector v/t through (x, y); thus, we need only consider reduced direction vectors. Also observe that, since at least one coefficient of a reduced direction vector is a unit, we may rescale to make that coefficient equal to 1.

LEMMA 2.1. *If* H *is a set of measure zero in* $\mathbb{F}_q[[t]]^2$ *containing some line with direction vector* $(1, b)$ *for each* $b \in \mathbb{F}_q[[t]]$*, then* $K := \{(x, y) : (x, y) \in H \text{ or } (y, x) \in H\}$ *is a Besicovitch set in* $\mathbb{F}_q[[t]]^2$*, and* $K^{(n)} := K \times \mathbb{F}_q[[t]]^{n-2}$ *is a Besicovitch set in* $\mathbb{F}_q[[t]]^n$*.*

Proof. Each reduced direction vector in $\mathbb{F}_q[[t]]^2$ is of the form $(a, 1)$ or $(1, a)$ for some $a \in \mathbb{F}_q[[t]]$. Therefore, if H is of measure zero and contains a line with direction vector $(1, b)$ for each $b \in \mathbb{F}_q[[t]]$, then K is also of measure zero and contains a line in every direction; i.e., K is a Besicovitch set in $\mathbb{F}_q[[t]]^2$. Furthermore, $K^{(n)}$ is also of measure zero and will contain a line in every direction; hence, it is a Besicovitch set in $\mathbb{F}_q[[t]]^n$. □

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We employ the following notation:

(1) for any $a \in \mathbb{F}_q[[t]]$, a_i denotes the coefficient of t^i;
(2) for any $a \in \mathbb{F}_q[[t]]$, the element $a^* \in \mathbb{F}_q[[t]]$ is defined by

$$a_i^* = \begin{cases} 0 & \text{if } i = 2^k - 2 \text{ for some natural number } k, \\ a_{i+1} & \text{otherwise.} \end{cases}$$

THEOREM 2.2. *The set*

$$H := \{(x, y) \in \mathbb{F}_q[[t]]^2 : ax + y = a^* \text{ for some } a \in \mathbb{F}_q[[t]]\}$$

contains a line with direction vector $(1, b)$ *for each* $b \in \mathbb{F}_q[[t]]$, *and has measure zero.*

By definition, for any $b \in \mathbb{F}_q[[t]]$, the points $(x, y) = (0, -b^*) + s(1, b)$ are contained in H, as s ranges over $\mathbb{F}_q[[t]]$, since $(-b)s + (-b^* + bs) = (-b)^*$. Therefore, we see that for any $b \in \mathbb{F}_q[[t]]$, a line with direction vector $(1, b)$ is contained in H, and so all that remains to be proved is that $\mu(H) = 0$.

We can see that $(x, y) \in H$ if and only if there exist $a_i \in \mathbb{F}_q$ for all $i \in \mathbb{N}$ such that the coefficients of x and y satisfy the following infinite system:

$$\begin{aligned} &a_0x_0 + y_0 = 0, && [0] \\ &a_1x_0 + a_0x_1 + y_1 = a_2, && [1] \\ &a_2x_0 + a_1x_1 + a_0x_2 + y_2 = 0, && [2] \\ &\qquad\vdots \\ &a_nx_0 + a_{n-1}x_1 + \cdots + a_0x_n + y_n = a_n^*, && [n] \\ &\qquad\vdots \end{aligned}$$

We refer to $a_lx_0 + \cdots + a_0x_l + y_l = a_l^*$ as equation $[l]$ and the system of equations [0] through $[l]$ as the "lth-stage system".

For an arbitrary element $(x, y) \in \mathbb{F}_q[[t]]^2$, define $s_{2^k-2}(x, y)$ to be the number of tuples $(a_0, \ldots, a_{2^k-2}) \in \mathbb{F}_q^{2^k-1}$ satisfying the $(2^k - 2)$nd-stage system; observe that $s_{2^k-2}(x, y)$ only depends on $\{x_0, y_0, \ldots, x_{2^k-2}, y_{2^k-2}\}$. Clearly, if $s_{2^k-2}(x, y) = 0$ for any integer k, then $(x, y) \notin H$, and $s_{2^l-2}(x, y) = 0$ for all $l > k$, so $\mu(\{(x, y) | s_{2^k-2}(x, y) = 0\})$ is non-decreasing as $k \to \infty$. We will prove that $\mu(\{(x, y) \in \mathbb{F}_q[[t]]^2 : s_{2^k-2}(x, y) = 0\}) \to 1$ as $k \to \infty$, from which it follows that $\mu(H) = 0$.

Observe that the equations at any stage are linear in the a_i. Moreover, for i not of the form $2^m - 2$ for some integer m, equation $[i]$ states $a_{i+1} = a_ix_0 + a_{i-1}x_1 + \cdots + a_0x_i + y_i$, and so we may reduce our system of equations by eliminating a_{i+1}. Basic linear algebra then implies that $s_{2^k-2}(x, y)$ is either zero or q^l for some integer $l \leqslant k$.

Notation. In the sequel, C_n will denote (any) polynomial of the variables $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$; thus, for example, $C_n + y_{n+1}C_{n-1} = C_{n+1}$ and $C_n + C_n = C_n$. We additionally will use R_n to denote any rational functions

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of the same variables whose denominator is non-zero everywhere, and C_{-1} to denote (any) element of $\mathbb{F}_q$.

LEMMA 2.3. *Every a_n can be written as*

$$a_n = \sum_{k:2^k-1\leqslant n} C_{n-2^k} a_{2^k-1} + C_{n-1}.$$

Proof. We show the result by strong induction on n. For $n = 2^k - 1$ the statement is immediate since $a_n = a_{2^k-1}$. Now if $n \neq 2^k - 1$, equation $[n-1]$ reads

$$a_{n-1}x_0 + \cdots + a_0 x_{n-1} + y_{n-1} = a_n.$$

By the induction hypothesis, every a_l on the left-hand side is of the form

$$a_l = \sum_{k:2^k-1\leqslant l} C_{l-2^k} a_{2^k-1} + C_{l-1},$$

and so we may write

$$\begin{aligned} a_n &= \sum_{l=0}^{n-1} x_{n-l-1} a_l + y_{n-1} \\ &= \sum_{l=0}^{n-1} x_{n-l-1} \left(\sum_{k:2^k-1\leqslant l} C_{l-2^k} a_{2^k-1} + C_{l-1} \right) + y_{n-1} \\ &= \sum_{k:2^k-1\leqslant n} a_{2^k-1} \left(\sum_{l=2^k-1}^{n-1} x_{n-l-1} C_{l-2^k} \right) + \sum_{l=0}^{n-1} x_{n-l-1} C_{l-1} + y_{n-1}. \end{aligned}$$

By definition we have $\sum_{l=2^k-1}^{n-1} x_{n-l-1} C_{l-2^k} = C_{n-2^k}$ and $\sum_{l=0}^{n-1} x_{n-l-1} C_{l-1} + y_{n-1} = C_{n-1}$. Substituting these expressions yields the result as claimed. □

LEMMA 2.4. *For $n > 0$, the left-hand side of equation $[2^n - 2]$ can be written as*

$$\left[\sum_{0\leqslant k<n} a_{2^k-1} (x_{2^n-2-(2^k-1)} + C_{2^n-2-(2^k-1)-1}) \right] + (y_{2^n-2} + C_{2^n-3}).$$

Proof. The left-hand side of equation $[2^n - 2]$ is

$$\begin{aligned} \sum_{0\leqslant l\leqslant 2^n-2} a_l x_{2^n-2-l} + y_{2^n-2} &= \sum_{0\leqslant k<n} a_{2^k-1} x_{2^n-2-(2^k-1)} \\ &\quad + \sum_{\substack{0\leqslant l\leqslant 2^n-2 \\ l\neq 2^k-1}} a_l x_{2^n-2-l} + y_{2^n-2}. \end{aligned} \tag{2.1}$$

By using Lemma 2.3, (2.1) becomes

$$\begin{aligned} &\sum_{0\leqslant k<n} a_{2^k-1} x_{2^n-2-(2^k-1)} \\ &\quad + \sum_{\substack{0\leqslant l\leqslant 2^n-2 \\ l\neq 2^k-1}} x_{2^n-2-l} \left(\sum_{2^k-1\leqslant l} C_{l-2^k} a_{2^k-1} + C_{l-1} \right) + y_{2^n-2}. \end{aligned} \tag{2.2}$$

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We may rearrange the middle term in (2.2) to obtain

$$\sum_{0\leqslant k<n} a_{2^k-1}\left(\sum_{2^k-1<l\leqslant 2^n-2} C_{l-2^k}x_{2^n-2-l}\right)+\sum_{0\leqslant l\leqslant 2^n-2} x_{2^n-2-l}C_{l-1}. \tag{2.3}$$

Clearly, we have

$$\sum_{2^k-1<l\leqslant 2^n-2} C_{l-2^k}x_{2^n-2-l}=C_{2^n-2^k-2}$$

and

$$\sum_{0\leqslant l\leqslant 2^n-2} x_{2^n-2-l}\left(\sum_{2^k-1\leqslant l} C_{l-1}\right)=C_{2^n-3},$$

so, upon substituting into (2.3), we obtain the desired result. □

Now we use the structure of individual terms in equation $[2^n-2]$, as determined in Lemma 2.4, to compute the distribution of values for the $s_{2^k-2}(x,y)$ as x and y range over all of $\mathbb{F}_q[[t]]$.

LEMMA 2.5. *If, for given x and y, we have $s_{2^n-2}(x,y)=q^l$, and $(x_{2^n-1}, y_{2^n-1},\ldots,x_{2^{n+1}-2}, y_{2^{n+1}-2})$ are randomly and uniformly chosen from $\mathbb{F}_q$, then*

$$s_{2^{n+1}-2}(x,y)=\begin{cases}0 & \text{with probability } \dfrac{q-1}{q^{l+2}},\\ q^l & \text{with probability } 1-\dfrac{1}{q^{l+1}},\\ q^{l+1} & \text{with probability } \dfrac{1}{q^{l+2}}.\end{cases}$$

Proof. Since $s_{2^n-2}(x,y)=q^l$, there are l indices $0<n_1<n_2<\cdots<n_l<n$ such that every a_j $(0\leqslant j\leqslant 2^n-2)$ can be written as

$$a_j=\sum_{n_i:2^{n_i}-1\leqslant j} R_{2^n-2}a_{2^{n_i}-1}+R_{2^n-2}, \tag{2.4}$$

where we have noted that a_{2^j-1} can be expressed in terms of the a_{2^i-1} with $i\leqslant j$. Note that upon eliminating variables, we may introduce rational functions (which may be different for different choices of the parameters x_i and y_i), but by the assumption $s_{2^n-2}(x,y)=q^l$, their denominators are non-zero.

Observe that the coefficients $a_{n_1},\ldots,a_{n_l}$ give a basis for the solution space of the $(2^{n+1}-2)$nd stage system. We will determine the probability that there is a solution to the $(a_{2^{n+1}-1})$st stage system.

Lemma 2.4 states that equation $[2^{n+1}-2]$ reads

$$\begin{aligned}0=\;&a_{2^n-1}(x_{2^{n+1}-2^n-1}+C_{2^{n+1}-2-(2^n-1)-1})\\&+\sum_{0\leqslant k<n} a_{2^k-1}(x_{2^{n+1}-2^k-1}+C_{2^{n+1}-2-(2^k-1)-1})\\&+(y_{2^{n+1}-2}+C_{2^{n+1}-3}).\end{aligned} \tag{2.5}$$

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Using (2.4) for the terms a_{2^k-1} where $k \neq n_i$ in the sum in (2.5), yields

$$\begin{aligned}
&\sum_{\substack{0\leqslant k<n\\ k\neq n_i}} a_{2^k-1}(x_{2^{n+1}-2^k-1} + R_{2^{n+1}-2-(2^k-1)-1})\\
&\quad= \sum_{\substack{0\leqslant k<n\\ k\neq n_i}} \left((x_{2^{n+1}-2^k-1} + R_{2^{n+1}-2-(2^k-1)-1})\left(\sum_{n_i<k} R_{2^n-2}a_{2^{n_i}-1} + R_{2^n-2}\right)\right)\\
&\quad= \sum_{1\leqslant i\leqslant l} a_{2^{n_i}-1}\left(\sum_{n_i<k<n} R_{2^n-2}(x_{2^{n+1}-2^k-1} + R_{2^{n+1}-2-(2^k-1)-1})\right)\\
&\qquad+ \sum_{\substack{0\leqslant k<n\\ k\neq n_i}} R_{2^n-2}(x_{2^{n+1}-2^k-1} + R_{2^{n+1}-2-(2^k-1)-1}).
\end{aligned}$$

Since $k < n$ we have $2^{n+1} - 2^k - 1 < 2^{n+1} - 2^n - 1 = 2^n - 1$, hence the above sum may be written more compactly as

$$\sum_{1\leqslant i\leqslant l} a_{2^{n_i}-1}R_{2^n-2} + R_{2^n-2}.$$

Therefore, again because of $k < n$, (2.5) can be written as

$$\begin{aligned}
0 &= a_{2^n-1}(x_{2^{n+1}-2^n-1} + R_{2^{n+1}-2-(2^n-1)-1})\\
&\quad+ \sum_{n_i} a_{2^{n_i}-1}(x_{2^{n+1}-2^{n_i}-1} + R_{2^{n+1}-2-(2^{n_i}-1)-1})\\
&\quad+ (y_{2^{n+1}-2} + R_{2^{n+1}-3}).
\end{aligned} \tag{2.6}$$

Now we analyze the right-hand side of (2.6),

$$\begin{aligned}
&a_{2^n-1}(x_{2^{n+1}-2^n-1} + R_{2^{n+1}-2-(2^n-1)-1})\\
&\quad+ \sum_{n_i} a_{2^{n_i}-1}(x_{2^{n+1}-2^{n_i}-1} + R_{2^{n+1}-2-(2^{n_i}-1)-1})\\
&\quad+ (y_{2^{n+1}-2} + R_{2^{n+1}-3}),
\end{aligned} \tag{2.7}$$

to determine for which values of the a_{n_i} it takes the value zero.

There are three cases: (2.7) can be "identically zero" (i.e., zero for all values of the a_{n_i}), "never zero" (i.e., zero for no possible values of the a_{n_i}), or "sometimes zero" (i.e., if the coefficient of a_{2^n-1} or one of the $a_{2^{n_i}-1}$ is not zero).

(1) *Expression (2.7) is identically zero.* Since a_{2^n-1} and the $a_{2^{n_i}-1}$ are independent of one another, this case occurs precisely when the coefficients of a_{2^n-1} and the $a_{2^{n_i}-1}$ are zero, and when $y_{2^{n+1}-2} + R_{2^{n+1}-3}$ is zero. The value of $R_{2^{n+1}-2-(2^n-1)-1}$ depends only on $x_0, y_0, \ldots, x_{2^n-2}, y_{2^n-2}$; in particular, it is independent of $x_{2^{n+1}-2^n-1}$. Hence, the coefficient of a_{2^n-1} is zero with probability $1/q$, as this occurs precisely when $x_{2^{n+1}-2^n-1} = -R_{2^{n+1}-2-(2^n-1)-1}$.

Similarly, for each $i \leqslant l$, the value of $R_{2^{n+1}-2-(2^{n_i}-1)-1}$ depends only on $x_0, y_0, \ldots, x_{2^{n+1}-2^{n_i}-2}, y_{2^{n+1}-2^{n_i}-2}$, and so the coefficient of $a_{2^{n_l}-1}$ is zero precisely when $x_{2^{n+1}-2^{n_l}-1} = -R_{2^{n+1}-2-(2^{n_l}-1)-1}$. This also occurs with probability $1/q$ for the same reason as above.

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Finally, $R_{2^{n+1}-3}$ depends only on $x_0, y_0, \ldots, x_{2^{n+1}-3}, y_{2^{n+1}-3}$, and so $y_{2^{n+1}-2} + R_{2^{n+1}-3}$ equals zero with probability $1/q$.

Since each event depends on different x_i or y_i, they are collectively independent. Therefore, expression (2.7) is identically zero with probability $1/q^{l+2}$.

(2) *Expression (2.7) is never zero.* This case occurs precisely when the coefficients of a_{2^n-1} and the $a_{2^{n_i}-1}$ are zero, and when $y_{2^{n+1}-2} + R_{2^{n+1}-3}$ is non-zero. By the same arguments as in case (1), the coefficient of a_{2^n-1} and each $a_{2^{n_i}-1}$ is zero with probability $1/q$, and $y_{2^{n+1}-2} + R_{2^{n+1}-3}$ is non-zero with probability $(q-1)/q$. Again, all events are independent, hence expression (2.7) is never zero with probability $(q-1)/q^{l+2}$.

(3) *In (2.7), the coefficient of a_{2^n-1} or one of the $a_{2^{n_i}-1}$ is not zero.* Since this is the only remaining case, its probability is $1 - (q-1)/q^{l+2} - 1/q^{l+2} = 1 - 1/q^{l+1}$.

Now, if (1) holds, then $s_{2^{n+1}-2}(x, y) = q^{l+1}$, since a_{2^n-1} will be a new free variable, independent of the other a_{2^i-1}. If (2) holds, then $s_{2^{n+1}-2}(x, y) = 0$. Finally, if (3) holds, then $a_{2^{n+1}-2}$ is determined as a linear function of the other a_{2^i-1}, and so $s_{2^{n+1}-2}(x, y) = q^l$. □

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. Observe that the probabilities computed in Lemma 2.5 are compatible with the Haar measure on $\mathbb{F}_q[[t]]^2$, and also that those probabilities depend only on l and not on n. Therefore, we can create a Markov chain on the points $0, 1, q, q^2, \ldots$, and the related random variables described by

$$P(X_n = l) = \mu(\{(x, y) \in \mathbb{F}_q[[t]]^2 : s_{2^n-2}(x, y) = l\}),$$

with transition functions as follows: at time n, from the point q^l we go to one of 0, q^l and q^{l+1}, with probabilities $(q-1)/q^{l+2}$, $1 - 1/q^{l+1}$, $1/q^{l+2}$, respectively.

Observe that the expected value does not change:

$$E(X_n | X_{n-1} = l) = 0 \cdot \frac{q-1}{q^{l+2}} + q^l \cdot \left(1 - \frac{1}{q^{l+1}}\right) + q^{l+1} \cdot \frac{1}{q^{l+2}}$$
$$= q^l - \frac{1}{q} + \frac{1}{q} = q^l.$$

Since from every state a positive proportion (independent of n) is sent to 0, we see by a trivial induction on l that as $n \to \infty$, the proportion of states at q^l tends to zero, for each $l \geqslant 0$. Combined with the observation that the expected value does not change, we see that the limiting distribution is concentrated at 0. Hence $\mu(\{(x, y) | s_{2^n-2}(x, y) = 0\}) \to 1$, and the proof is complete. □

Combining the results of Lemma 2.1 and Theorem 2.2, we obtain Theorem 1.1.

THEOREM 1.1. *For all $n > 1$, there exists a subset $E \subset \mathbb{F}_q[[t]]^n$ of measure zero which contains a line in every direction.*

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Remark 2.6. By using an inductive argument, one may prove that, for $q > 2$, after t iterations of the Markov chain, the total measure $\mu_q(t)$ of the Kakeya set in $\mathbb{F}_q[[t]]^2$ satisfies

$$\mu_q(t) \ll \frac{\ln(t)}{t},$$

where the (explicitly computable) constant depends only on q. In particular, we have

$$|H_k||\mathbb{F}_q|^{-2k} \ll \frac{\ln(\ln(k))}{\ln(k)},$$

where H_k denotes the size of the projection of the Kakeya set to $\mathbb{F}_q[[t]]/t^k$. Taking the limit as $k \to \infty$ yields

$$\lim_{k\to\infty} |H_k||\mathbb{F}_q|^{-2k} = 0,$$

which answers Question 1 as posed in [4].

§3. *Minkowski dimension for* $R = \mathbb{F}_q[[t]]$ *and* $R = \mathbb{Z}_p$. Recall that the analogue of the Minkowski dimension of a subset E of $\mathbb{F}_q[[t]]^n$ is

$$\lim_{k\to\infty} \frac{\log |E_k|}{\log q^k},$$

where E_k is the image of E under the projection $\mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]/t^k$; we have an analogous definition for subsets of $\mathbb{Z}_p$ using the projections $\mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$. In this section, we prove that if E is a Kakeya set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$, then the limit exists and equals two. In order to obtain this result we give a sharper lower bound than [4, Proposition 4.21] for the measure of any Kakeya set in R^2, where $R = \mathbb{F}_q[t]/t^k$ or $\mathbb{Z}/p^k\mathbb{Z}$.

PROPOSITION 3.1. *Let E be a Kakeya set in R^2 where $R = \mathbb{F}_q[t]/t^k$ or $\mathbb{Z}/p^k\mathbb{Z}$. Then*

$$|E| \geqslant \frac{|R|^2}{2k}.$$

Let E be Kakeya; then it contains a line of the form $ax + y = b$ for all $a \in R$. We enumerate the lines by their direction vectors as follows:

(1) if $R = \mathbb{Z}/p^k\mathbb{Z}$ then we take L_i to be (any) of the lines in E of the form $\alpha_i x + y = b_i$, where $\alpha_i = i \pmod{p^k}$;
(2) if $R = \mathbb{F}_q[t]/t^k$ then we take L_i to be (any) of the lines in E of the form $\alpha_i x + y = b_i \pmod{t^k}$, where, for $i = \sum_{j=0}^{k-1} a_j q^j$ in base q, we have $\alpha_i = (\sum_{j=0}^{k-1} a_j t^j)$.

Notation. For non-zero $r \in R$, we denote by $v(r)$ the largest integer satisfying $r \in \mathfrak{m}^{v(r)}$, where $\mathfrak{m}$ is the maximal ideal of R. Equivalently, $v(r)$ is the exact power of t or p dividing r, for r in $\mathbb{F}_q[[t]]$ or $\mathbb{Z}_p$ (respectively). We also let M denote the cardinality of $R/\mathfrak{m}$, which is q or p (respectively).

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For any two lines $L_i : \alpha_i x + y = b_i$ and $L_j : \alpha_j x + y = b_j$ with $i \neq j$, we see that if $(x_0, y_0) \in L_i \cap L_j$ then $(\alpha_i - \alpha_j)x_0 = b_i - b_j$. Hence, if $v(\alpha_i - \alpha_j) = l$, then the number of possible values for x_0 cannot exceed M^l. In particular, since the value of x_0 determines the value of y_0, the size of the intersection $|L_i \cap L_j|$ is at most $M^l = M^{v(\alpha_i - \alpha_j)}$.

LEMMA 3.2. *For the function* $f(u) := \sum_{i=1}^{u} M^{v(\alpha_i)}$, *we have* $f(u) \leqslant u \cdot \lceil \log_M(u) \rceil$. *In particular, for* $u \leqslant M^k/k = |R|/k$, *we have* $f(u) \leqslant |R|$.

Proof. There are $\lfloor u/M^w \rfloor$ terms for which $v(\alpha_i)$ is equal to w. Thus, we may write

$$\begin{aligned} f(u) &= \sum_{w=0}^{\infty} (M^w - M^{w-1}) \cdot \left\lfloor \frac{u}{M^w} \right\rfloor \\ &\leqslant \sum_{w=0}^{\lceil \log_M(u) \rceil} (M^w) \cdot \frac{u}{M^w} \\ &= u \cdot \lceil \log_M(u) \rceil, \end{aligned}$$

which gives the first statement. The second statement follows immediately. □

With this bound, we can now prove Proposition 3.1.

Proof of Proposition 3.1. Inclusion-exclusion gives a lower bound on the size of E:

$$|E| \geqslant \left| \bigcup_{j=1}^{l+1} L_j \right| \geqslant \sum_{j=1}^{l+1} \left(|L_j| - \sum_{i=1}^{j-1} |L_i \cap L_j| \right), \tag{3.8}$$

where $l = \lfloor |R|/k \rfloor$ is as given in Lemma 3.2. We can explicitly write the terms in (3.8) as

$$|L_j| - \sum_{i=1}^{j-1} |L_i \cap L_j| \geqslant |R| - \sum_{i=1}^{j-1} M^{v(\alpha_j - \alpha_i)} = |R| - f(j-1).$$

Applying the upper bound on $f(u)$ from Lemma 3.2 yields

$$\begin{aligned} |E| &\geqslant \sum_{j=1}^{l+1} (|R| - f(j-1)) \\ &= (l+1)|R| - \sum_{j=2}^{l+1} (j-1)\lceil \log_M(j-1) \rceil \\ &\geqslant (l+1)|R| - (\lceil \log_M(l) \rceil) \frac{l(l+1)}{2} \\ &\geqslant |R| \frac{l+1}{2} \\ &> \frac{|R|^2}{2k}, \end{aligned}$$

and this is precisely the desired bound. □

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Now we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. From Proposition 3.1, if E is a Kakeya set in R^2 with either $R = \mathbb{Z}/p^k\mathbb{Z}$ or $R = \mathbb{F}_q[t]/t^k$, then the Minkowski dimension of E is at least $2 - (\log 2k)/(k \log M)$. In particular, if we fix p or q and take $k \to \infty$ then the lower bound goes to 2. The desired result follows, since the Minkowski dimension of any set in R^2 is at most two. □

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