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ARTICLES

OPTIMAL DYNAMIC LABOR TAXATION

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We analyze optimal dynamic taxation when labor supply is indivisible. As in Hansen (1985) and Rogerson (1988), markets are complete, and an employment lottery determines who works. The consumer can buy insurance to diversify this income uncertainty. The optimal wage tax is generally positive except for some special cases when leisure is nonnormal and the government can use debt as a policy instrument in addition to its tax instruments. We derive a HARA class of preferences, for which we characterize the dynamic paths of the wage tax. The optimal paths of the labor tax differ between divisible- and indivisible-labor economies.

Keywords: Optimal Taxation, Dynamic Taxation, Indivisible Labor

1. INTRODUCTION

There are two ways of introducing second-best government policy in a competitive economy. The first is when the government has to raise an exogenously specified amount of revenue without recourse to lump-sum taxation. The second-best tax system then minimizes the distortions.[1] The second alternative is to highlight the redistributive role of the government when individuals are heterogeneous in terms of factor ownership. The government then resorts to distortionary taxation for redistributive reasons. Under both approaches, considerable research has been devoted to finding the optimal capital-income tax. The central result is typically that the optimal capital-income tax is zero in the steady state. This is the well-known Judd-Chamley result [Judd (1985) and Chamley (1986)].[2]

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In this paper, we focus our attention on analyzing the second-best labor income taxation in the Judd-Chamley model. The goal of our paper is to find conditions on preferences that give rise to positive labor taxation.[3] We do this in a fairly general setting, which includes physical capital and public debt, and allows for both capital and labor to be taxed.[4] The issue is important because one may question whether the labor income tax is also zero in the steady state like the capital tax. If the government can accumulate capital, it could raise all necessary revenues by taxing capital and labor at the beginning of the optimization period, and set all taxes zero at the steady state.[5] Renström (1999) investigates this issue in an economy with divisible labor, and finds that the labor tax is positive in the steady state if leisure is normal. In this paper, we allow labor supply to be indivisible, as in Hansen (1985) and Rogerson (1988), and we establish conditions not only for positive labor taxation but also for zero and negative taxation.[6,7]

Why is the issue of optimal taxation in an environment with indivisible labor worth exploring? With indivisible labor, we can unravel a rich set of preferences for which labor is taxable. We accomplish this by establishing a connection between the household's demand for unemployment insurance and the normality of leisure. When leisure is normal, in order to raise the tax base, the fiscal authority would tax labor to induce the household to work harder. When labor supply is indivisible, and the leisure is normal, the individual buys insurance to equate the utility gain from not working to the utility cost of the insurance purchase. The novelty of our approach is that we further derive a HARA class of preferences (i.e., preferences with Hyperbolic Absolute Risk Aversion) with nonseparable leisure for which this insurance is a linear function of after-tax wage income. Because positive insurance demand and normality of leisure are inextricably connected, one can then answer the tax policy question posed in this paper: What class of preferences would warrant positive labor taxation? For those preferences, we can derive the time path of the optimal labor tax. In addition, we can compare the optimal time paths of labor income taxes between divisible and indivisible labor economies, which turn out to be very different between these two scenarios. This happens because labour supply elasticities are different between these two models.

There is also a pedagogical reason for exploring the optimal labor taxation in an indivisible labor scenario. There is a literature on optimal taxation arguing that labor should be taxed when market is incomplete in the sense that there is neither an insurance market nor a set of contingent-claims markets (where the household can diversify income risk). The government then uses corrective taxes for the missing markets.[8] However, the issue still remains whether labor should be taxed in a complete-markets environment. Our exercise with indivisible labor provides insights why labor may be taxed even when there are no missing insurance markets.[9]

We also characterize the short-run dynamics of the labor income tax. We find that when leisure is normal, the optimal labor tax is increasing over time as long as capital is taxed. When leisure is inferior, and labor is subsidized, the subsidy is increasing over time as long as capital is taxed. We also look at two special cases

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of the HARA class, iso-elastic utility and negative exponential utility. We find that for widely used iso-elastic preferences, the optimal labor income tax is positive both in the short and long run. For negative exponential utility, where leisure is neutral, the optimal labor-income tax is zero in the short and long run when the government can use debt as an instrument. Optimal taxation of labor is, therefore, generally positive except for some special types of preferences.

The paper is organized as follows. The following section presents the model. Section 3 derives the optimal-tax implications in a second-best world. Section 4 derives a class of HARA preferences, and presents examples of the time path of the optimal labor tax. Section 5 compares the time paths of optimal taxes between divisible and indivisible labor economies. Section 6 briefly discusses the case when public debt is zero. Section 7 concludes.

2. AN ECONOMY WITH INDIVISIBLE LABOR

2.1. Individual Economic Behavior

Following Hansen (1985) and Rogerson (1988), we consider an economy where labor supply is indivisible. The consumption set is restricted so that that the individuals can work either full time, h_0, or not at all. Households have access to an insurance market where they can buy insurance. In each period, the household member engages in an employment lottery, choosing the probability of working, $\alpha(t)$, and the probability of not working, $(1-\alpha(t))$. This makes her wage income uncertain. She has access to an insurance market where she buys unemployment insurance, $y(t)$.The household's consumption ($c^s(t)$) and asset accumulation ($\dot{a}^s(t)$) are thus potentially contingent on whether the household works ($s=1$) or not ($s=2$). There is no intrinsic uncertainty, which means that preferences and technology are nonstochastic. The household thus solves the following maximization problem:

$$J(a_0) \equiv \max_{c,y,\alpha} \int_0^\infty e^{-\theta t}[\alpha(t)u(c^1(t), 1-h_0) + (1-\alpha(t))u(c^2(t), 1)]\,dt \quad \textbf{(1)}$$

$$\dot{a}^1(t) = \rho(t)a(t) + \omega(t)h_0 - p(\alpha(t))y(t) - c^1(t) \quad \textbf{(2)}$$

$$\dot{a}^2(t) = \rho(t)a(t) + y(t) - p(\alpha(t))y(t) - c^2(t) \quad \textbf{(3)}$$

$$a(0) = a_0, \quad \textbf{(4)}$$

where $a(t)$ equals the sum of outstanding public debt, $b(t)$, and the capital stock, $k(t)$, that earn the after-tax interest at rate $\rho(t) = (1-\tau^K(t))r(t)$, and $\omega(t) = (1-\tau^L(t))w(t)$ is the after tax wage; $r(t)$ and $w(t)$ are the rental- and wage-rates, respectively, $\tau^K(t)$ and $\tau^L(t)$ are the proportional tax rates on capital- and labor-income, respectively, and $p(\alpha(t))$ is the competitive price of insurance.[10] The insurance company behaves competitively and maximizes the expected profit, $p(\alpha(t))y(t) - (1-\alpha(t))y(t)$, which gives rise to the zero-profit condition, $p(\alpha(t)) = 1-\alpha(t)$.[11]

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Substituting the zero-profit condition into (2) and (3), the current value Hamiltonian of the representative household can be written as

$$\begin{aligned} H &= \alpha(t)u(c^1(t), 1-h_0) + (1-\alpha(t))u(c^2(t), 1) \\ &\quad + \alpha(t)q^1(t)[\rho(t)a(t) + \omega(t)h_0 - (1-\alpha(t)y(t) - c^1(t)] \\ &\quad + (1-\alpha(t))q^2(t)[\rho(t)a(t) + \alpha(t)y(t) - c^2(t)]. \end{aligned} \tag{5}$$

The first-order conditions are (subscripts denoting partial derivatives)

$$\frac{\partial H}{\partial c^1(t)} = u_c(c^1(t), 1-h_0) - q^1(t) = 0 \tag{6}$$

$$\frac{\partial H}{\partial c^2(t)} = u_c(c^2(t), 1) - q^2(t) = 0 \tag{7}$$

$$\frac{\partial H}{\partial y(t)} = q^1(t) - q^2(t) = 0. \tag{8}$$

Using (6), (7), and (8) it follows that

$$u_c(c^1(t), 1-h_0) = u_c(c^2(t), 1) = q(t), \tag{9}$$

which means perfect insurance. In other words, by buying insurance, the individual equalizes the marginal utilities across states. Equation (9) gives the optimal time paths of state-contingent consumption, $c^1(q(t))$ and $c^2(q(t))$, as functions of the co-state variable $q(t)$. However, this does not necessarily imply that household will equalize consumption across states. For consumption equalization, one requires an additional restriction on the preferences that the utility function is additively separable between consumption and leisure, meaning $u_{c(1-h)} = 0$. It turns out that without any such restriction on the preferences, the household will not choose to have full consumption insurance as in Hansen (1985).[12] This can be seen from (6), (7), and (8). Because $1 - h_0$ is not equal to unity, c^1 cannot equal c^2 unless $u_{c(1-h)}$ is equal to 0. Next, because $q^1(t) = q^2(t)$ it follows that the optimal asset holding decisions must satisfy

$$\dot{a}^1(t) = \dot{a}^2(t), \tag{10a}$$

$$\dot{q}(t) = (\theta - \rho(t))q(t). \tag{10b}$$

An individual's asset accumulation is thus independent of her employment history. This implies that individuals starting with the same a_0 will have the same $a(t)$ at all t, regardless of their employment history. Substituting (10a) in (2) and (3) gives

$$y(t) = \omega(t)h_0 + c^2(q(t)) - c^1(q(t)). \tag{11}$$

Notice now that the household chooses full insurance if the optimal consumption bundles are such that $c^1(q(t)) = c^2(q(t))$. In the absence of any restriction on $y(t)$,

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the household can choose to have positive, negative or zero insurance.[13] Finally, the optimal choice of $\alpha(t)$ must be such that

$$\frac{\partial H}{\partial \alpha(t)} = u(c^1(q(t)), 1 - h_0) - u(c^2(q(t)), 1)$$
$$- q(t)[c^1(q(t)) - c^2(q(t)) - \omega(t)h_0] = 0, \tag{12}$$

which upon the use of (11) can be rewritten as

$$u(c^2(q(t)), 1) - u(c^1(q(t)), 1 - h_0) = q(t)y(t). \tag{12'}$$

The household chooses to buy a positive insurance, $y(t)$, if the utility gain from not working balances the utility cost of the insurance purchase.

2.2. Production

There is large number of competitive firms in the economy each operating under the following constant returns-to-scale technology:

$$f(k(t), \alpha(t)h_0) = f_1 \cdot k(t) + f_2 \cdot \alpha(t)h_0, \tag{13}$$

where $f_1 = \frac{\partial f}{\partial k}$ and $f_2 = \frac{\partial f}{\partial \alpha h_0}$.

2.3. The Government

The government taxes labor and capital income to finance an exogenously specified sequence of public spending, $g(t)$, the use of which is not explicitly modeled. It adjusts two tax rates, $\tau^L(t)$ and $\tau^K(t)$, continuously. The government is assumed to borrow and lend freely at the market rate of interest, $r(t)$. The government's budget constraint is, therefore, given by

$$\dot{b} = r(t)b(t) - \tau^k(t)r(t)a(t) - \tau^L(t)w(t)\alpha(t)h_0 + g(t), \tag{14}$$

with $b(0) = b_0$.

2.4. Equilibrium

The equilibrium is characterized by the following conditions:

(a) Facing $w(t)$, $r(t)$, $\tau^L(t)$, $\tau^K(t)$, the household chooses optimal sequences of $c(t)$, $a(t)$, $\alpha(t)$,$y(t)$ that solves the problem stated in (1), (2) and (3).[14]
(b) Given an exogenous steam of government spending, $g(t)$, the government precommits to a tax sequence, $\tau^L(t)$ and $\tau^K(t)$, and a debt sequence, $b(t)$, that satisfies the government budget constraint (14).
(c) Goods, labor, rental markets clear meaning

$$\dot{k}(t) = f(k(t), \alpha(t)h_0) - \alpha(t)c^1(t) - [1 - \alpha(t)]c^2(t) - g(t), \tag{15}$$
$$\omega(t) = (1 - \tau^L(t))f_2(k(t), \alpha(t)h_0), \tag{16}$$
$$\rho(t) = (1 - \tau^K(t))f_1(k(t), \alpha(t)h_0). \tag{17}$$

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Notice that the equilibrium level of employment, $h(t)$ ($\equiv \alpha(t)h_0$) is determined by the time path of the probability of work, $\alpha(t)$. The equilibrium time path of $\alpha(t)$ can be determined in two steps. First, using (12) one determines the market clearing after tax wage, $\omega(t)$ as a function of $q(t)$. Define that equilibrium wage function as

$$\omega(t) = \Omega(q(t)). \tag{18}$$

Next, using (16) and (18), one can characterize the path of $\alpha(t)$ as a function of $k(t)$, $q(t)$, and $\tau^L(t)$ as follows:

$$\alpha(t) = \alpha(k(t), \tau^L(t), q(t)) \tag{19}$$

PROPOSITION 1. *In equilibrium*, $y(t) > 0$, *if and only if* $\frac{\partial \alpha(t)}{\partial a_0} < 0$ (*leisure is normal at date t*); $y(t) < 0$, *if and only if* $\frac{\partial \alpha(t)}{\partial a_0} > 0$ (*leisure is inferior at date t*).

Proof. See Appendix A.

Positive insurance demand and normality of leisure are inextricably connected.[15] To see the intuition, start from a scenario where the initial wealth, a_0 is such that $y(t) = 0$. In this case, the individual is indifferent between work and no work (see Equation 12′). Starting from this scenario suppose a_0 increases. A higher wealth makes the consumer value leisure more if leisure is normal. The household cannot choose hours of work in this indivisible labor world. The only choice is to decrease the probability of work and buy positive insurance in response to increase in wealth. This is why a positive insurance demand is associated with a positive utility gain from not working as in (12′).[16]

3. SECOND-BEST OPTIMAL TAXATION

We now solve for the optimal tax problem for the government for this economy with indivisible labor. The government solves a Ramsey problem for precommitted tax sequences, $\tau^L(t)$ and $\tau^K(t)$, that maximize the household's utility functional (1) subject to its own budget constraint (14), the economy wide resource constraint (15), the first-order optimality conditions (9), (10b), and (12), and a no-confiscation constraint on capital income as follows:[17]

$$\rho(t) \geq 0. \tag{20}$$

Using (16) and (17), and the CRS property of the production function, the government's budget constraint, (14), can be rewritten as

$$\dot{b}(t) = \rho(t)b(t) + \rho(t)k(t) + \alpha(t)\omega(t)h_0 - f(k(t), \alpha(t)h_0) + g(t). \tag{21}$$

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We may write the government's current value Hamiltonian as follows (ignoring the time indices from now on):

$$\begin{aligned} H^g &= \alpha u(c^1, 1-h_0) + (1+\alpha)u(c^2, 1) \\ &\quad + \mu\{\rho b + \rho k + \alpha\omega h_0 - f(k, \alpha h_0) + g\} \\ &\quad + \lambda\{f(k, \alpha h_0) - \alpha c^1 - (1-\alpha)c^2 - g\} + \psi(\theta - \rho)q + \nu\rho. \end{aligned} \tag{22}$$

In principle, the government faces the states, k, b, and q, and chooses the controls ρ and τ^L. For algebraic convenience, we pose the government's problem as follows. The government chooses the controls ρ and α. Then using the equilibrium sequence of α as in (19), one can determine the optimal labor tax, τ^L.[18] Denoting $u^1 = u(c^1,1-h_0)$ and $u^2 = u(c^2,1)$, the first-order conditions facing the government are as follows:

$$\frac{\partial H^g}{\partial \alpha} = u^1 - u^2 + \mu[\omega h_0 - f_2 \cdot h_0] + \lambda[f_2 h_0 + c^2 - c^1] = 0 \tag{23}$$

$$\frac{\partial H^g}{\partial q} = -\dot{\psi} + \theta\psi$$

$$\Rightarrow \dot{\psi} = \rho\psi - \alpha u_c^1 c_q^1 - (1-\alpha)u_c^2 c_q^2 - \mu\alpha\Omega'(q)h_0 + \lambda\left[\alpha c_q^1 + (1-\alpha)c_q^2\right] \tag{24}$$

$$\frac{\partial H^g}{\partial \rho} = \mu(b+k) - \psi q + \nu = 0 \tag{25}$$

$$\frac{\partial H^g}{\partial k} = \mu(\rho - f_1) + \lambda f_1 = \theta\lambda - \dot{\lambda} \tag{26}$$

$$\frac{\partial H^g}{\partial b} = \mu\rho = \theta\mu - \dot{\mu}. \tag{27}$$

3.1. The Optimal Capital-Income Tax

One may now establish, as in Judd (1985) and Chamley (1986), that the optimal capital-income tax is zero in the steady state, also in our economy with indivisible labor. This can be verified as follows. At steady state the individual's marginal utility of consumption is constant over time, implying, by (10b), that $\theta = \rho$. This in turn implies, by (27), that μ is constant in the steady state. With μ being constant (and q constant), equation (23) implies that also λ is constant in the steady state. Setting the time derivative of λ to zero in (26), and using $\theta = \rho$ gives $(\rho - f_1)(\lambda - \mu) = 0$, which holds if and only if $f_1 = \rho$ because $(\lambda - \mu) > 0$. We summarize this result in terms of the following proposition.

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PROPOSITION 2. *If the indivisible labor economy converges to a steady state under the second-best tax program, then in the steady state the optimal capital-income tax is zero,* $\tau^K = 0$.

3.2. The Optimal Labor-Income Tax

Our primary interest in this paper is to explore carefully the optimal labor-tax implications. The problem is complicated by the fact that in an economy with indivisible labor and lottery, the consumer has the option to transfer consumption not only across dates but also across states. Whether the government should tax wage income in the second best depends crucially on the household's risk preference, which will determine its demand for insurance. The following proposition states a key result about the relationship between the household's demand for insurance and the second-best wage taxation.

PROPOSITION 3. *Under the second-best tax program, from the date when the no-confiscation constraint* (20) *on capital ceases to bind, the optimal wage-income tax is positive* (*negative*) *if the household's demand for insurance,* y, *is positive* (*negative*).

Proof. See Appendix B.

Thus, there is a direct link between the sign of y and the sign of the labor-income tax. The question arises whether the optimal labor-income tax is zero when $y = 0$, the case when leisure is neutral. This cannot be directly inferred from Proposition 3, because it is based on an equilibrium wage equation subject to the condition that y is nonzero.[19] We next analyze a benchmark case when the optimal labor income tax is zero.

PROPOSITION 4. *If the preferences of the consumer are such that the consumer chooses to buy zero insurance* (*meaning* $y = 0$), *the optimal labor-income tax is zero at all dates.*

Proof. Plugging (12′) into (23) and using (11), one obtains

$$f_2 - \omega = \frac{(q - \lambda)y}{(\lambda - \mu)h_0}. \tag{29}$$

It immediately follows that when $y = 0$, $f_2 = \omega$, which means $\tau^L = 0$. ■

Because Proposition 1 gave us a one-to-one relationship between normality/neutrality/inferiority of leisure and positive/zero/negative insurance, we can conclude from Propositions 3 and 4 that there is a one-to-one relationship between normality/neutrality/inferiority of leisure and the sign of the optimal labor tax. That is, the labor tax should be positive/zero/negative if leisure in normal/neutral/inferior. Notice that this is not just a steady-state result but holds at least from date when the nonconfiscation constraint does not bind.

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The issue arises whether for normal/inferior leisure, the labor-income tax/subsidy rises or falls over time. It is difficult to characterize the exact time path without fully specifying the preferences. In the next section, we derive a broad class of HARA preferences for which it is possible to characterize the exact short run path of the labor tax.

4. DYNAMICS OF THE LABOR TAX: A PARAMETRIC EXAMPLE

Consider a class of preferences for which insurance demand is proportional to the wage income as follows:

$$y = (1 - \pi)\omega h_0, \tag{30}$$

where π is a parameter that can be either positive, negative, or zero. It turns out that the parameter π pins down alternative preference structures for which leisure may be normal, neutral or inferior.

4.1. Full Insurance

The benchmark case of full insurance arises $\pi = 0$. We have the following result.

PROPOSITION 5. *Necessary and sufficient for* $\pi = 0$ *is that the utility function* $u\ (c, 1 - h_0)$ *is additively separable in leisure*: $u_{c(1-h)} = 0$.

Proof. If $\pi = 0$, from (11) and (30) it follows that $c^1 = c^2$. Next, note that when $c^1 = c^2$, the only way marginal utilities can be equalized in (9) is by setting $u_{c(1-h)} = 0$. This proves necessity. Next we prove the sufficiency. If $u_{c(1-h)} = 0$, the = from (9) it follows that $c^1 = c^2$, which on plugging in (11) and (30) gives $\pi = 0$. ■

The additive separable leisure in the utility function is widely used in the literature. When the utility function is additively separable in leisure, it implies that leisure is a normal good. In this case, evidently labor is taxed as per Proposition 2. However, the converse is not true. For normality of leisure, it is not necessary that $u_{c(1-h)v} = 0$. We now derive a wide range of preferences involving nonseparable leisure that belongs to a family of preferences known as the Hyperbolic Absolute Risk Aversion (HARA) class. Such a family of preferences can be derived when $\pi \neq 0$.

4.2. HARA Preferences

We shall establish now that for $\pi \neq 0$, the derived HARA class can encompass various possibilities: (a) $0 < \pi < 1$, partial insurance; (b) $\pi = 1$ no insurance; (c) $\pi > 1$, negative insurance, and (d) $\pi < 0$, over-insurance.

Plugging (30) into (11) to eliminate ω, we get

$$y = \frac{1 - \pi}{\pi}(c^1 - c^2). \tag{31}$$

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Plugging (31) into (12′) gives

$$u^1 - u^2 + q\frac{1-\pi}{\pi}(c^1(q) - c^2(q)) = 0, \tag{32}$$

where $u^1 = u(c^1(q), 1-h_0)$ and $u^2 = u(c^2(q), 1)$. Equation (32) holds for all preferences for which $\pi \neq 0$.[20] We wish to find the class of preferences for which π is constant (i.e., independent of q). We have the following lemma.

LEMMA 1. *It is necessary that the class of preferences for which $\pi \neq 0$, satisfies the following condition*:

$$u_c^1 c_q^1 - u_c^2 c_q^2 + (1-\pi)(c^1 - c^2) = 0, \tag{33}$$

where u_c^s and c_q^s denote the derivatives of u and c in state $s = \{1, 2\}$ w.r.t. c and q, respectively.

Proof. Define (32) as the implicit function $J(q, \pi) = 0$. Using the implicit function theorem,

$$\frac{\partial \pi}{\partial q} = -\frac{J_q}{J_\pi}. \tag{34}$$

For (34) to be zero, it is necessary that either $J_q = 0$ or $J_\pi = \infty$. However, $J_\pi = \infty$ would mean

$$-q(c^1 - c^2)/\pi^2 = \infty, \tag{35}$$

q being the marginal utility of consumption cannot be infinity at the optimum. Thus, for (35) to hold, π must equal zero, which violates the restriction that $\pi \neq 0$. Thus, it is necessary that $Jq = 0$, which yields (33). ∎

LEMMA 2. *For $J_q = 0 \ \forall \ q$, it is necessary that the utility function is of the following Hyperbolic Absolute Risk Aversion (HARA) form*:

$$u(c_s, 1-h^s) = D^s - \frac{B^s}{\pi}(A + (1-\pi)\, c_s)^{\pi/(\pi-1)}, \tag{36}$$

for $s = \{1, 2\}$. B^s and D^s are constants (possibly dependent of h), with $B > 0$, and A is a constant independent of h.

Proof. Taking the derivative of (9) w.r.t. q gives $c_q^s(q) = 1/u_{cc}^s$, $s = \{1, 2\}$. Plugging this into (33), we get

$$\frac{u_c^1}{u_{cc}^1} + (1-\pi)\, c^1 = \frac{u_c^2}{u_{cc}^2} + (1-\pi)\, c^2. \tag{37}$$

Because preferences are state independent (i.e., expected utility), the left- and right-hand sides must equal the same constant, (say $-A$), that is,

$$\frac{u_c^s}{u_{cc}^s} = -A - (1-\pi)\, c^s, \tag{38}$$

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for $s = \{1, 2\}$. This means that A and π cannot be state dependent (i.e., dependent of h). Integrating (38) twice (see, Appendix C for derivation) yields (36). ■

We next show that for the class of HARA preferences (36), the constant D is state independent (i.e., independent of leisure).

LEMMA 3. *For* (32) *and* (36) *to hold, it is necessary that that* $D^1 = D^2$.

Proof. Take the derivative of (36)

$$u_c(c^s, 1 - h^s) = B^s(A + (1 - \pi)\, c^s)^{1/(\pi - 1)}. \tag{39}$$

Then, by using (9), we find c^s as functions of q:

$$c^s = \frac{1}{1 - \pi}\left(\frac{q}{B^s}\right)^{\pi - 1} - \frac{A}{\pi - 1}. \tag{40}$$

Inserting (40) into (36) gives

$$u^s = D^s - (B^s)^{1-\pi}\, q^\pi / \pi. \tag{41}$$

Plugging (40) and (41) into (32), it follows that $D^1 = D^2$. ■

Because in the preference class (36), only B^s and c^s can be state-dependent, it must be the case that B^s is a function of leisure. Using this insight, define $B^s = \phi^h$, where h can take two possible states 0 and h_0. Our next task is to characterize the precise restrictions on π, which generates leisure as a "good" (with positive marginal utility) in the utility function. We are ready to state a key proposition.

PROPOSITION 6. *Necessary and sufficient for insurance to be a constant fraction of the after-tax wage income is that the preferences belong to the following class*:

$$u(c, 1 - h) = D - \frac{\phi^h}{\pi}(A + (1 - \pi)c)^{\pi/(\pi - 1)}, \tag{42}$$

where A and D are constants. ϕ^h *is a function of leisure as follows*:

(a) *If* $\pi > 0$, *then* $\phi^h = \phi(h)$, *with* $\phi(h_0) > \phi(0)$.
(b) *If* $\pi < 0$, *then* $\phi^h = \phi(1 - h)$, *with* $\phi(1) > \phi(1 - h_0)$.

Proof. Necessity of (42) follows from Lemma 2 and 3. Sufficiency can be checked by evaluating equation (32) for the class in (42). Finally, conditions (a) and (b) imply that utility of working is less than of not working, everything else equal. ■

4.3. The Case of Zero Labor-Income Taxation

In the next step we derive a subclass of preferences from the HARA class for which labor should not be taxed at the second-best optimum. This is a special case when leisure is neutral.

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PROPOSITION 7. *If the utility function is of the following exponential class*

$$u(c, 1-h) = D - \phi^h e^{-c}, \tag{43}$$

where D is a constant, and $\phi^h = \phi(h)$, *with* $\phi(h_0) > \phi(0)$, *then the household self-insures meaning* $y = 0$, *and the optimal labor-income tax is zero at all dates.*

Proof. Recall from Proposition 3 that when the individual takes no insurance, the optimal wage tax is zero. In the present context, $y = 0$ if $\pi = 1$ (see equation 30). Take limits of (42) as $\pi \to 1$ (which requires setting A equal to unity) and apply l'Hôpital's rule to obtain (43). ■

Proposition 7 has an important implication. It basically provides a special case where self-insurance could be optimal. Even if the insurance market is present, households choose to self-insure (meaning the demand for insurance, $y = 0$) in this special environment by using saving and labor supply as instruments.[21]

4.4. Why Does Normality of Leisure Mean a Positive Tax on Labor?

We have found that whether labor should be taxed or not depends on whether the individual demands positive insurance or not, which in turn depends on whether leisure is normal or inferior. A relationship thus exists between normality of leisure and the insurance demand via the individual's preferences. To see this connection clearly, differentiate (19) with respect to $\tau^L(t)$, (keeping $k(t)$ constant), to obtain the following useful decomposition of a change in tax rate, $\tau^L(t)$ on work effort, $\alpha(t)$:

$$\left.\frac{d\alpha(t)}{d\tau^L(t)}\right|_{k(t)} = \frac{\partial \alpha(t)}{\partial \tau^L(t)} + \frac{\partial \alpha(t)}{\partial q(t)} \cdot \frac{\partial q(t)}{\partial \tau^L(t)}. \tag{44}$$

The first term in (44) represents the compensated labor-supply response when the tax rate changes (the substitution effect). If $q(t)$ is held constant, it follows from (18) that $\omega(t)$ is also constant and it is straightforward to verify from (16) that the first term is: $f_2/[(1-\tau^L) f_{22} \cdot h_0]$, and is negative. This substitution effect thus captures the distortionary effect of the wage-income tax. As far as this substitution effect is concerned, a higher labor tax lowers labor supply and thus lowers the tax base.

The second term in (44) reflects the income effect of a change in the wage tax rate, which works through the effect of $\tau^L(t)$ on $\alpha(t)$ via its effect on $q(t)$. When $\tau^L(t)$ is higher, it lowers the permanent income of the household, thus lowering consumption in both states for given $\alpha(t)$. The marginal utility of consumption, $q(t)$, thus rises, which means $\partial q(t)/\partial \tau^L(t) > 0$. A loss of permanent income (represented by higher $q(t)$) would boost the labor supply, $\alpha(t)$, if leisure is a

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normal good. Next, verify that (44) can be rewritten as:[22]

$$\left.\frac{d\alpha(t)}{d\tau^L(t)}\right|_{k(t)} = \frac{f_2}{[1-\tau^L(t)]f_{22}h_0} - \frac{(1-\pi)f_2}{f_{22}q(t)h_0}\frac{\partial q(t)}{\partial \tau^L(t)}. \qquad (44')$$

The normality of leisure ($\pi < 1$) thus makes the income effect of a higher wage tax positive. This positive income effect tends to increase the tax base ($w(t)\alpha(t)h_0$) when $\tau^L(t)$ rises, which countervails the distortionary effect of a higher $\tau^L(t)$. The government can thus tax labor more in those economies. By contrast, if $\pi > 1$, leisure is inferior. The income effect is then negative, reinforcing the distortionary effect of wage taxation. In this case, the government should subsidize labor.

4.5. Should Labor Be Taxed When the Utility Function Is Isoelastic in Consumption?

Consider now a special case of the HARA class, equation (42), when $A = 0$. We have the following lemma concerning the elasticity of interstate substitution (call it σ hereafter).

LEMMA 4. *For a specific class of HARA utility functions with* $A = 0$, *the elasticity of interstate substitution* (σ) *is given by* $1 - \pi$.

Proof. By definition,

$$\sigma = \frac{d\ln\left(\frac{c^2}{c^1}\right)}{d\ln\left(\frac{u_c^1}{u_c^2}\right)}. \qquad (45)$$

Using (42) gives $\sigma = 1 - \pi$. ■

The utility function thus becomes isoelastic in consumption, and the elasticity of interstate substitution in consumption is uniquely characterized by π. Furthermore, we must have $\pi < 1$ when $A = 0$, otherwise positive marginal utility, $u_c > 0$, is violated. This immediately implies that leisure is normal when the utility function is isoelastic in consumption, and thus labor must be taxed.

To summarize, the HARA class of preferences (42) embraces a variety of utility functions, and thus covers a wide range of optimal taxation schemes. These cases include: (a) isoelastic utility in consumption ($A = 0$), in which case the optimal labor tax is positive, (b) negative exponential utility function ($\pi = 1$), where the optimal labor tax is zero, and (c) quadratic when $\pi = 2$ (where $A > \max_t c(t)$), in which case the optimal labor tax is negative (i.e., a subsidy).

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4.6. Transitional Dynamics of the Wage-Income Tax in the HARA Case

We shall now analyze the transitional dynamics of labor income tax for the range of preferences discussed earlier. Using (29) one obtains

$$\frac{\tau^L}{1-\tau^L} = (1-\pi)\cdot\frac{q-\lambda}{\lambda-\mu}, \tag{46}$$

then taking the time derivative of (46) using (10b), (26), and (27), one obtains

$$\frac{d}{dt}\frac{\tau^L}{1-\tau^L} = (1-\pi)\frac{d}{dt}\frac{q-\lambda}{\lambda-\mu} = (1-\pi)(f_1-\rho)\frac{q-\mu}{\lambda-\mu}. \tag{47}$$

Appendix C outlines the steps in deriving the second equality. Based on the second equality in (47), and the fact that $q-\mu>0$, and $\lambda-\mu>0$, we have the following proposition.

PROPOSITION 8. *If $\pi<1$ labor is taxed (at least from the date at which the nonconfiscation constraint does not bind), and the wage tax is increasing as long as capital is taxed. If $\pi=1$ labor is always untaxed. If $\pi>1$ labor is subsidized (at least from the date at which the nonconfiscation constraint does not bind), and the subsidy is increasing as long as capital is taxed.*[23]

5. COMPARING THE LABOR TAX IN INDIVISIBLE AND DIVISIBLE LABOR ECONOMIES

In this section, we compare the optimal tax paths of the indivisible and divisible labor economies. We establish here in terms of an example that the optimal labor tax paths could be very different between these two economies because of different labor supply elasticities. The current value Hamiltonian for the household for the divisible labor economy is

$$H = u(c(t), 1-h(t)) + q(t)\{\rho(t)a(t) + \omega(t)h(t) - c(t)\}. \tag{48}$$

The first-order conditions are

$$u_c(c(t), 1-h(t)) - q(t) = 0, \tag{49}$$

$$u_{1-h}(c(t), 1-h(t)) - q(t)\omega(t) = 0, \tag{50}$$

$$q(t)\rho(t) = \theta q(t) - \dot{q}(t). \tag{51}$$

Equations (49) through (50) form a system of equations such that c and h can be solved in terms of q and ω. For our purpose we are interested in the compensated labor supply response with respect to a change in ω holding q constant. We get

$$\frac{\partial h(t)}{\partial \omega(t)} = \frac{-u_{cc}q(t)}{u_{cc}u_{ll} - u_{lc}^2} > 0. \tag{52}$$

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Denoting $\eta(t) = \frac{\partial h(t)}{\partial \omega(t)} \frac{\omega(t)}{h(t)}$ as the compensated labor supply, and assuming an additively separable utility function, $U(c, 1-h) = V(c) + L(1-h)$, Renström (1999) shows that for this divisible labor economy, the time path of labor tax is given by

$$\frac{\tau^L}{1-\tau^L} = \frac{q-\lambda}{\lambda-\mu} - \eta^{-1} \cdot \frac{\mu}{\lambda-\mu}. \tag{53}$$

In order to make a valid comparison between divisible and indivisible labor economies, we need to assume the same preference structures for both. Recall from Proposition 5 that for the indivisible laboreconomy $\pi = 0$ in (30) if and only if the utility function is additively separable. Substituting (30), with $\pi = 0$, into (29) gives

$$\frac{\tau^L}{1-\tau^L} = \frac{q-\lambda}{\lambda-\mu}. \tag{46a}$$

Comparing (46a) and (53), we see that the time paths of labor taxes are very different for these two economies because of the presence of the labor-supply elasticity term η on the right-hand side of (53).

Hansen (1985) shows that the equilibrium allocation of an indivisible labor economy can be replicated in a divisible labor economy with a utility function linear in leisure. From (53), this immediately means that η is infinite for such a divisible labor economy. The optimal labor tax path described in (53) thus reduces to the indivisible labor case (46a) in such a case. Except this special case, the optimal paths for τ_L are generally different between these two economies.[24]

6. NO DEBT SCENARIO

Until now, we assumed that the government intertemporally balances the budget by using public debt as an instrument. How would the optimal tax formula change when public debt is disallowed meaning $b = 0$? In the absence of public debt, the government is left only with two policy instruments, namely, wage tax and capital tax. The zero capital tax still remains socially optimal in steady state, which means that labor has to be always taxed to finance the exogenous stream of government spending in the steady state. In certain cases, we may even have a corner solution for the labor supply where the labor supply hits the upper bound ($\alpha = 1$).

The intuition for the possibility of corner solution goes as follows. To sustain such a zero capital tax regime, the household may have to stretch the labor supply to the upper bound (a corner solution) to finance the government spending. This happens particularly when the household values leisure as an inferior or neutral good. As the economy-wide capital stock grows, the household does not then mind working harder to finance the government spending just to sustain a socially optimal zero capital tax regime. Thus labor supply hits the corner when leisure is nonnormal. This kind of a corner solution does not arise when debt is allowed as an

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instrument because the benevolent government can always float debt to alleviate the strain on laborsupply.[25]

7. CONCLUSION

In this paper, we address the issue of optimal wage taxation in a dynamic complete-markets setting. We find that the second-best labor tax depends crucially on the degree of complementarity between consumption and leisure. If leisure is a not neutral, there is scope for government intervention in the form of labor taxation or subsidy in a complete market environment. We investigated this question in a fairly general setting with preferences allowing nonseparability between consumption and leisure. Our conclusion is that labor should be taxed if leisure is normal. The optimal wage tax could be zero or negative only for very special scenarios where leisure is nonnormal, and the government has access to public debt as a policy instrument. If the government is restricted to balance the budget period by period, the wage subsidy result ceases to hold. Thus, a normative theory of taxation generally implies that labor should be taxed in the long run. We also find that the optimal labor tax paths are generally different between the divisible and indivisible labor economies. The results obtained here provide useful guidance in designing optimal tax policy in a dynamic environment. A useful extension of this paper would be to investigate similar issues in a third-best environment where the government may not necessarily commit to a specific tax design.

NOTES

1. In the second-best, the government designs an optimal dynamic tax formula at date zero, and remains precommitted to it (i.e., the government chooses all future consumer prices at date zero). If lump-sum taxation is allowed, the analysis reduces to the first-best.

2. Although Chamley (1986) takes the first view of second-best taxation where the government is just revenue raising, Judd (1985) views the government as redistributive. The zero capital-income tax is a result robust to several generalizations. Atkeson et al. (1999) provides a survey of the robustness of the zero capital taxation result using the primal approach. See also Renström (1999) for a survey of the dynamic optimal tax literature using the dual approach.

3. Chamley (1985) analyzes second-best wage taxation in the absence of a capital tax and suggests that the exact tax policy depends on the utility function. We explore this issue and show that the functional form of the utility function is important in determining the sign of the labor-income tax.

4. There is a literature looking at this issue from various perspectives. Lucas and Stokey (1983), and Ayagiri et al. (2002) look at optimal labor taxation in the absence of capital. For our purposes, it is important to include physical capital. One may then explore various optimal-tax outcomes, including the possibility of zero tax on both capital and labor. Jones, Manuelli, and Rossi (1993) address the issue of optimal labor taxation including physical and either human capital or productive government spending. However, their analysis is mostly based on simulation with specific functional forms, and does not admit a closed form solution with a fairly general preference structure.

5. Jones, Manuelli, and Rossi (1997) include human capital in addition to physical capital, so that the labor tax has an intertemporal distortion. They show that there are certain cases when the labor tax is zero in steady state. See also Reinhorn (2006) for a clarification of those results, in particular regarding interior solutions of a model with human capital in addition to physical.

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6. Hansen (1985) and Rogerson (1988) establish that indivisible labor explains aggregate fluctuations better than models with divisible labor. A subsequent literature explored further into the business cycle implications of indivisible labor [see Greenwood and Huffman (1988), Hansen and Sargent (1988)]. Mulligan (1999) points out that the tax implications are different for these two classes of models.

7. A related paper, Basu, Marsiliani, and Renström (2004), derives necessary and sufficient conditions on preferences for the *capital tax* to become zero in *finite* time in an indivisible labor setting. Those preferences are special cases of our HARA class derived in Section 4 of this paper.

8. This has been explored in the literature on risk sharing. There, even without a government revenue constraint, a labor tax may be levied. The labor tax then corrects for a market failure (a missing insurance market). In a two-period setting, Hamilton (1987) demonstrates that the optimal wage tax is positive if the second period labor income is uncertain. The issue is indirectly dealt in the macroeconomics literature concerning debt nonneutrality. Chan (1983) and Barsky et al. (1986), as well as Basu (1996), examine the debt nonneutrality hypothesis when future income is uncertain. Although none of these models explicitly deals with the labor-supply decision, there is one common result: taxing future income at a flat rate would be welfare improving if markets were unable to fully insure households from future income risk.

9. As expected, we find that the second-best capital-income tax is zero at the steady state.

10. The optimization problem laid out in our paper resembles Hansen (1985). We make it explicit that the agent starts with the same asset but potentially can transit to two possible states: work or no-work. In equilibrium, they will, however, have the same asset equation via insuring themselves against income risk. Alternatively, we can transform the problem into a single state variable by defining $a = \alpha a^1(t) + (1-\alpha)a^2(t)$, and get exactly the same first-order conditions.

11. The household randomizes the labor-supply decision in this setting by choosing a probability of work $\alpha(t)$. A realistic description of this arrangement is that the representative household consists of a family of N members. In each period, the household decides the proportion, $\alpha(t)$, of members working. The labor supply is then $\alpha(t)h_0N$. The household can buy insurance on a competitive market to diversify the income uncertainty arising from $(1-\alpha(t))N$ of its members not working. After choosing the probability of work, $\alpha(t)$ the household is precommitted to it, and cannot renege. The insurance company then charges the actuarially fair premium, $1-\alpha$. This rules out adverse selection in the model. The household then realizes that, when choosing work probability, the insurance premium is a linear function of the probability.

12. Note that if the utility function is nonseparable, then the household optimally chooses not to have full consumption insurance. One can easily set up a social planning problem and demonstrate that no other consumption allocation dominates the allocation governed by (12).

13. One needs to be careful about the non-negativity constraint on consumption while thinking about negative unemployment benefit. $y(t)$ can be negative as long as $c^2(t)$ is nonnegative. We assume interior solutions, meaning $c^2(t) > 0$.

14. We assume no-Ponzi games.

15. In a divisible labor economy, Renström (1999) finds that noninferiority of leisure is sufficient for the wage-income tax to be positive. In the indivisible-labor setting, normality is both necessary and sufficient for a positive labor-income tax.

16. The insurance market is added to obtain insights about the relationship between normality of leisure and the optimal tax, as well as deriving preference structures for which leisure is normal. One can alternatively construct an environment with contingent-claims markets as in Shell and Wright (1993), and derive the same equilibrium allocation as a sunspot equilibrium. A proof of this equivalence is available from the authors on request.

17. No such confiscation constraint is relevant for labor income taxation because if labor income is confiscated by the government it is optimal for the household to set $\alpha(t) = 0$, which means no production. By contrast, the capital income can be confiscated and the government can eventually own all the capital to run production.

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18. It is straightforward to verify that for given k and q, $\alpha'(\tau^L) < 0$ and hence $\alpha(\cdot)$ can be inverted with respect to τ^L.

19. To see this clearly note that the proof of Proposition 3 rests on the fact that $\Omega'(q) = -y/qh_0$, which holds when y is nonzero.

20. Note that $\pi = 0$ cannot be nested under this case, which means additively separable utility function does not belong to our derived class of preferences.

21. In general, savings alone would not equalize marginal utilities across states. To see this clearly, note that in the absence of an insurance market, there is no equation (8), and hence (9) will not hold generally. Thus, savings alone cannot equalize marginal utilities across states. If the insurance market is dropped, the model reduces to an incomplete market scenario where the intertemporal marginal rates of substitution in consumption (which is governed by the two co-state variables q_1 and q_2 in our model) will not be equalized across states because of the emergence of uninsurable income risk. This uninsurable risk arises because a fraction of the households do not work.

22. To see this, insert (30) into (A.5) and use (16) to obtain $\partial\alpha(t)/\partial q(t) = -(1-\pi) f_2/f_{22} q(t) h_0$.

23. When $\pi = 1$, it turns out that the nonconfiscation constraint is binding for a period $t \in [0, T^*)$, and capital income is taxed at 100%. From T^* and onward, capital is untaxed, and the economy is at its steady state level. T^* is a function of the present value of the stream of g_t, $t \in [0, \infty)$ discounted at the rate θ.

24. One suspects that in general the optimal labor tax is lower in an indivisible labor economy because of a higher labor supply elasticity. In order to make such a quantitative comparison, we need to solve the differential equations in λ, q and μ in both economies. Although differential equations look the same for both economies, the solution will be different. The reason is that the initial values of these co-state variables are different for these two economies (except the special case when η is infinite), and also the time from which the nonconfiscation constraint does not bind also will differ. The details of these derivations are available from the authors on request.

25. The details of the derivation of the zero debt case are available from the authors on request.

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APPENDIX A

Proof of Proposition 1. First note that leisure in an indivisible labor economy is normal if $\frac{\partial \alpha(t)}{\partial a_0} < 0$, which means the household chooses a lower probability of work when its wealth is higher. Next observe that

$$\frac{\partial \alpha(t)}{\partial a_0} = \frac{\partial \alpha(t)}{\partial q(t)} \cdot \frac{\partial q(t)}{\partial a_0}. \tag{A.1}$$

Using (10a), one obtains

$$\frac{\partial q(t)}{\partial a_0} = \exp\left(\int_0^\infty (\theta - \rho(s))\, ds\right) \cdot \frac{\partial q_0}{\partial a_0}. \tag{A.2}$$

Next note by the application of Envelope property of the value function $J(a_0)$ in (1) that $J'(a_0) = q_0$. By strict concavity of the value function ($J''(a_0) < 0$), it follows that $\frac{\partial q_0}{\partial a_0} < 0$. Thus from (A.2), it follows that $\frac{\partial q(t)}{\partial a_0} < 0$. From (A.1), it means that

$$\operatorname{sign}\left(\frac{\partial \alpha(t)}{\partial a_0}\right) = -\operatorname{sign}\left(\frac{\partial \alpha(t)}{\partial q(t)}\right). \tag{A.3}$$

Using (12) and (16), define the following implicit function (time indices suppressed)

$$G(q, \alpha) = U(c^1(q), 1-h_0) - U(c^2(q), 1) + q \cdot \lfloor (1-\tau^L) \cdot f_2(k, \alpha h_0) h_0 + c^2(q) - c^1(q) \rfloor. \tag{A.4}$$

Using the implicit function theorem and (9) one obtains

$$\frac{\partial \alpha(t)}{\partial q(t)} = -\frac{G_q}{G_\alpha} = -\frac{y(t)}{q(t)(1-\tau^L(t))\, f_{22} \cdot h_0^2}. \tag{A.5}$$

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From (A.3) and (A.5), it follows that $\frac{\partial \alpha(t)}{\partial a_0}$ is negative, zero, or positive if and only $y(t)$ is positive, negative, or zero. ■

APPENDIX B

Proof of Proposition 2. Plugging (12) into (23) and using (11), one obtains

$$f_2 - \omega = \frac{(q-\lambda)y}{(\lambda-\mu)h_0}. \tag{B.1}$$

Premultiply (24) by q and exploiting the fact that $\Omega'(q) = [c^1 - c^2 - \omega h_0]/qh_0$, one obtains

$$q\dot{\psi} = q\rho\psi + (\lambda - q)\left[\alpha\, c_q^1\, q + (1-\alpha)\, c_q^2\, q\right] + \mu\alpha[\omega h_0 - c^1 + c^2]. \tag{B.2}$$

Next note that

$$\frac{d}{dt}(\psi q) = \dot{\psi} q + \psi \dot{q} = \dot{\psi} q + \psi(\theta - \rho) q. \tag{B.3}$$

Plugging (B.3) into (B.2) gives

$$\frac{d}{dt}(\psi q) = \theta\psi q + (\lambda - q)\left[\alpha\, c_q^1\, q + (1-\alpha)\, c_q^2\, q\right] + \mu\alpha\left[\omega h_0 - c^1 + c^2\right]. \tag{B.4}$$

Next note that

$$\frac{d}{dt}(\mu a) = \dot{\mu} a + \mu \dot{a}. \tag{B.5}$$

Plugging (27) into (B.5)

$$\frac{d}{dt}(\mu a) = \mu(\theta - \rho) a + \mu \dot{a}. \tag{B.6}$$

Using (2), (3), and (10a), the household's budget constraint can be rewritten as

$$\dot{a} = \rho a + \alpha\omega h_0 - \alpha c^1 - (1-\alpha)c^2, \tag{B.7}$$

which after plugging into (B.6) gives

$$\frac{d}{dt}(\mu a) = \theta\mu a + \mu[\alpha\omega h_0 - \alpha c^1 - (1-\alpha)c^2]. \tag{B.8}$$

Next noting that $a = b + k$, rewrite (25) as

$$\nu = \psi q - \mu a. \tag{B.9}$$

Taking the time derivative of (B.9), one obtains

$$\dot{\nu} = \frac{d}{dt}(\psi q) - \frac{d}{dt}(\mu a). \tag{B.10}$$

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Using (B.4) and (B.8) in (B.10) one obtains

$$\dot{\nu} = \theta\nu + (\lambda - q)\left[\alpha\, c_q^1\, q + (1-\alpha)\, c_q^2\, q\right] + \mu c^2. \tag{B.11}$$

Chamley (1986) shows that the confiscation constraint, (20) cannot be binding forever. In our case, if it is binding forever, consumption falls to zero in both states. Suppose that it ceases to bind at date t_1. Because ν is the multiplier associated with the confiscation constraint, (20), this implies that

$$\nu(t) = \dot{\nu}(t) = 0, \tag{B.12}$$

for $t \geq t_1$. Plugging (B.12) into (B.11) and simplifying terms, we get

$$\frac{q-\lambda}{\mu} = \frac{c^2}{\left[\alpha c_q^1 q + (1-\alpha) c_q^2 q\right]}, \tag{B.13}$$

for $t \geq t_1$.

Plugging (B.13) into (B.1), one obtains

$$\frac{\lambda - \mu}{-\mu}(f_2 - \omega) = \frac{-c^2 y}{\left[\alpha\, c_q^1\, q + (1-\alpha)\, c_q^2\, q\right] h_0}. \tag{B.14}$$

Because $\lambda > 0$ and $\mu < 0$ and $c_q^1 < 0$, and $c_q^2 < 0$ (by concavity of u), it follows that the sign of $(f_2 - \omega)$ is the same as the sign of y. Hence, the labor income tax is positive (negative) when $y > (<)\, 0$. ■

APPENDIX C

Derivation of equation (36): Inverting (38) gives

$$u_{cc}^s \big/ u_c^s = -1/[A + (1-\pi)c^s], \tag{C.1}$$

for $s = \{1, 2\}$. Or equivalently

$$\frac{d \ln\left(u_c^s\right)}{dc^s} = -\frac{1}{1-\pi}\frac{d \ln(A + (1-\pi)c^s)}{dc^s}. \tag{C.2}$$

Integrating both sides with respect to c^s gives

$$\ln\left(u_c^s\right) = M^s - \frac{1}{1-\pi}\ln(A + (1-\pi)\, c^s), \tag{C.3}$$

where M^s is any constant, possibly dependent on s.

Taking exponents of both sides gives

$$u_c^s = B^s (A + (1-\pi)\, c^s)^{-1/(1-\pi)}, \tag{C.4}$$

where $B^s = \exp(M^s)$, and consequently is positive.

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Integrating both sides with respect to c^s finally gives

$$u^s = D^s - \frac{B^s}{\pi}(A + (1-\pi)\, c^s\,)^{-\pi/(1-\pi)}, \tag{C.5}$$

where D^s is any constant, possibly dependent on s. ■

Derivation of equation (46). Note that

$$\frac{d}{dt}\left[\frac{q-\lambda}{\lambda-\mu}\right] = \left[\frac{(dq/dt)-(d\lambda/dt)}{\lambda-\mu}\right] - \left[\frac{q-\lambda}{\lambda-\mu}\right].\left[\frac{(d\lambda/dt)-(d\mu/dt)}{\lambda-\mu}\right]. \tag{C.6}$$

Next plug in (10b), (26), and (27) into the right-hand side of (C.6) to obtain (47). ■

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