Variations on a *sangaku* problem involving kissing spheres

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In their delightful book on Japanese temple geometry [1], Fukagawa and Rothman give a detailed account of the practice of *sangaku* (the Japanese custom of inscribing a geometrical problem on a tablet of wood and hanging it up as an offering in a Buddhist temple or Shinto shrine) as it flourished in Japan during the Edo period (1603-1867). They also present a large number of *sangaku* problems for the delectation of the reader, with solutions to many of them provided. Two of the 'still harder' problems in Chapter 6 are concerned with packings of identical spheres around a central sphere. Problem 14, due to Ishikawa Nagamasa of the Baba Seito school, asks for the radius of 30 identical spheres that can be packed around a central sphere of given radius so that each of the outer spheres touches four others of its kind (in addition to the central sphere). Problem 15, due to Kimura Sadamasa, is similar except that the 30 spheres are replaced by 20 and each is now required to touch three others of its kind. Both problems are accompanied by figures showing the needed arrangement of the spheres, to give the reader a hint; in the former case, the identical spheres are at the vertices of an icosidodecahedron and in the latter case at the vertices of a dodecahedron.

The purpose of this note is to generalise the above problems by asking for what values of N and M it is possible to arrange N identical spheres around a central one so that each of the identical spheres touches M others of its kind, in addition to the central sphere. It is also required to calculate the radius, r, of the central sphere if the radius of the N identical spheres is taken to be unity. The *sangaku* problems of the previous paragraph are the special cases of this problem for $N = 30, M = 4$ and $N = 20, M = 3$.

One class of solutions to this problem becomes obvious if one realises that the regular and semi-regular polyhedra have vertices that lie on the surface of a sphere, with the vertices being identical to each other. A solution is then obtained by placing N identical spheres at the vertices of a regular or semi-regular polyhedron of edge length twice the radius of the spheres; M is given by the number of edges at each vertex of the polyhedron and it remains only to calculate the radius, r, of the central sphere. This can be done with the aid of Figure 1, which shows the central sphere (at the top) and two outer spheres (at the bottom). The line joining the centres of the outer spheres is an edge of the polyhedron and subtends an angle of 2θ at the centre of the upper (or central) sphere. It is easily seen that the radius of the central sphere is given by

$$r = \operatorname{cosec}\theta - 1. \tag{1}$$

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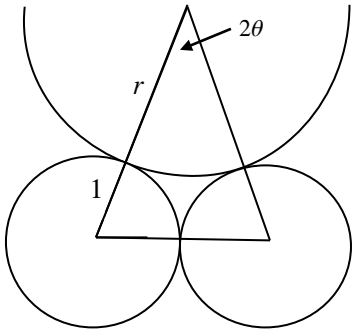


FIGURE 1: Calculation of r in terms of θ

The problem now is to determine θ. For a regular polyhedron whose faces are p-gons, q of which meet at a vertex, it is known [2] that $\cos\theta = \cos(\pi/p)\,\mathrm{cosec}\,(\pi/q)$. However the situation for the semi-regular polyhedra is more complicated because each has two or three different types of polygonal faces. Following Cundy and Rollett [3], one can describe each such polyhedron by a symbol that indicates the types of faces it has in it and the order in which they occur around a vertex. For example, a cuboctahedron is described by the symbol $(3 \cdot 4)^2$, which indicates that a triangle, square, triangle and square occur, in that order, around a vertex. It would be nice to have a formula for θ in terms of the numbers occurring in the symbol of the polyhedron, but such a formula would not be particularly simple, which is doubtless the reason I failed to find one in any of the books I looked at. I recently derived such a formula [4] as a byproduct of a broader study whose purpose was to calculate the isoperimetric quotients of the Archimedean and Catalan solids. Here I use that formula, in conjunction with (1), to calculate the values of r for all the Archimedean solids and list them, together with the corresponding values for all the Platonic solids, in Table 1.

I have a number of comments on the table:

1. For the two snub polyhedra, expressions for r cannot be given in terms of square roots. Closed form expressions can be given in terms of suitably defined trigonometric quantities, but they are unwieldy and so I have simply listed the numerical results here.

2. The results for the prism and antiprism are interesting because, as $n \to \infty$, the central sphere tends to a plane and the spheres lie along an infinite strip of squares (for the prism) or equilateral triangles (for the anti-prism). These two strip patterns complement the well-known regular and semi-regular tessellations of the plane, of which there are three and eight, respectively.

3. For some values of N, such as 12, 24 or 60, solutions exist for several different values of M. One would expect in such cases that the largest

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value of M leads to the packing with the smallest value of r. An inspection of the table shows that this expectation is met in every case. Two (N, M) values that are particularly noteworthy are (24,3) and (60,3), because they each permit two different arrangements with rather different sizes of the central sphere.

Polyhedron	N	M	r(formula)	r(value)
Tetrahedron 3^3	4	3	$\sqrt{\frac{3}{2}} - 1$	0.224745
Octahedron 3^4	6	4	$\sqrt{2} - 1$	0.414214
Cube 4^3	8	3	$\sqrt{3} - 1$	0.732051
Icosahedron 3^5	12	5	$\sqrt{\frac{5+\sqrt{5}}{2}} - 1$	0.902113
Dodecahedron 5^3	20	3	$\frac{\sqrt{3}}{2}(1+\sqrt{5}) - 1$	1.802517
n-gonal Prism $4^2 \cdot n$	$2n$	3	$\sqrt{1+\operatorname{cosec}^2\frac{\pi}{n}} - 1$	
n-gonal Anti-prism $3^3 \cdot n$	$2n$	4	$\sqrt{1+\frac{1}{4}\operatorname{cosec}^2\frac{\pi}{2n}} - 1$	
Truncated Tetrahedron $3 \cdot 6^2$	12	3	$\sqrt{\frac{11}{2}} - 1$	1.345208
Truncated Octahedron $4 \cdot 6^2$	24	3	$\sqrt{10} - 1$	2.162278
Truncated Cube $3 \cdot 8^2$	24	3	$\sqrt{7+4\sqrt{2}} - 1$	2.557647
Truncated Icosahedron $5 \cdot 6^2$	60	3	$\sqrt{\frac{1}{2}(29+9\sqrt{5})} - 1$	3.956037
Truncated Dodecahedron $3 \cdot 10^2$	60	3	$\sqrt{\frac{1}{2}(37+15\sqrt{5})} - 1$	4.938898
Cuboctahedron $(3 \cdot 4)^2$	12	4	1	1
Icosidodecahedron $(3 \cdot 5)^2$	30	4	$\sqrt{5}$	2.236068
Snub Cube $3^4 \cdot 4$	24	5	—	1.687428
Snub Dodecahedron $3^4 \cdot 5$	60	5	—	3.311675
Small Rhombicuboctahedron $3 \cdot 4^3$	24	4	$\sqrt{5+2\sqrt{2}} - 1$	1.797933
Small Rhombicosidodecahedron 3·4·5·4	60	4	$\sqrt{11+4\sqrt{5}} - 1$	3.465901
Great Rhombicuboctahedron 4·6·8	48	3	$\sqrt{13+6\sqrt{2}} - 1$	3.635222
Great Rhombicosidodecahedron 4·6·10	120	3	$\sqrt{31+12\sqrt{5}} - 1$	6.604789

TABLE 1: Solutions to the kissing spheres problem based on regular and semiregular polyhedra

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4. The problem I have solved might be termed the 'exterior' problem because the identical spheres are required to kiss the central sphere externally; the 'interior' problem would require the kissing to be internal instead. The regular and semiregular polyhedra also provide solutions to the interior problem, provided that the radius of the central sphere is taken to be

$$r' = \operatorname{cosec}\theta + 1, \tag{2}$$

with θ being the same angle for each polyhedron as before. This result can be derived from a figure similar to Figure 1, but with the smaller spheres tangent to the larger one from the inside. Alternatively, it can also be obtained by performing an inversion in the larger sphere and using the formulae of inversive geometry.

5. The solutions to the exterior and interior problems are particularly interesting for $N = 4$ and $M = 3$, when the centres of the identical spheres lie at the vertices of a tetrahedron. Since $\theta = \cos^{-1}\left(1/\sqrt{3}\right)$ in this case, (1) and (2) yield the radius of the central sphere as $\sqrt{3/2} - 1$ or $\sqrt{3/2} + 1$ for the exterior or interior problem, respectively. An alternative way of getting these results is to use the prescription given by Frederick Soddy in the third stanza of his poem *The Kiss Precise*:

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

By the 'bend' of a sphere Soddy means the inverse of its radius, with a positive or negative sign attached if the sphere is externally or internally tangent to the other spheres. If $\varepsilon_i \left(i = 1, \ldots, 5\right)$ are the bends of the five kissing spheres, the last two lines of the stanza tell us that

$$\left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5\right)^2 = 3\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2\right). \tag{3}$$

The solution to the exterior problem can be obtained by putting $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = x$ in (3), solving the resulting quadratic for x and inverting the positive root to obtain r. The solution to the interior problem can be obtained similarly but by putting $\varepsilon_5 = -x$ instead. Soddy's method has the great advantage over mine that it applies to five kissing spheres of arbitrary radii. Moreover, Soddy's

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formula also generalises straightforwardly to $N + 1$ mutually kissing spheres in N dimensions, as Thorold Gosset elegantly observed in a fourth stanza he added to Soddy's poem [5].

6. The solutions I have presented were obtained by using spherical trigonometry to calculate θ for the semi-regular polyhedra (see [4] for details), followed by the use of (1). However spherical trigonometry seems to have been unknown to traditional Japanese mathematicians and they would surely have attacked these problems by other methods. Perhaps the best indication of their approach is to be found in the solution to the $N = 20$, $M = 3$ case presented by Fukagawa and Rothman, where a judicious combination of plane sections is used to calculate all the lengths and angles involved without ever leaving the confines of Euclidean geometry and trigonometry. It would be interesting if such 'Japanese' solutions could be found for more of the cases I have listed in Table 1. And with that challenge (my own little *sangaku* offering?), I bid the reader *sayonara* (but not before saying that I have provided my own answers to my challenge in the Appendix).

Appendix

The results for the n-gonal prism and antiprism in Table 1 are easily derived using elementary geometry. Using these results as a springboard, a number of other results in the table are easily derived. The results for the tetrahedron, octahedron and icosahedron follow on putting $n = 2$, 3 or 5 in the formula for the n-gonal antiprism (since these solids can be regarded as a digonal, triangular or pentagonal antiprism, respectively, with two pyramidal endcaps added in the case of the last), while the results for the cube and the small rhombicuboctahedron follow on putting $n = 4$ or 8 in the formula for the n-gonal prism (since the former solid is a square prism and the latter an octahedral one with two endcaps added). The only Platonic solid not netted by this method is the dodecahedron. Fukagawa and Rothman [1] give an elementary derivation of the result for it, but I give an alternative derivation here based on Euclid's construction of the dodecahedron because it allows me to derive a couple of results that I will need later. (Incidentally, I should add that in getting many of the results of this Appendix and Table 1, it is necessary to use the value $\cos\frac{\pi}{5} = \frac{\tau}{2}$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio, and the recursion relation $\tau^2 = \tau + 1$ and simplify all the radicals and/or powers of τ involved in all the expressions of interest).

The basic idea underlying Euclid's construction of the dodecahedron is to use a pair of pentagons to erect a roof over a face of a cube, and to repeat this for all the faces until the cube is surrounded completely by a dodecahedron. The pentagons used in this construction must have diagonals equal in length to the edge length of the cube. One begins by taking two pentagons and placing them with their diagonals along opposite sides of a cube face, and then tilts the pentagons until their edges meet above the

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centre of the cube face to form the ridge of this portion of the roof (the pentagons must be oriented with their edges facing each other for this to happen). If one repeats this process for all the other cube faces, the gaps in the roof get filled and one gets a dodecahedron completely enclosing the cube. A nice dynamic view of Euclid's construction can be seen at [6].

Eight of the vertices of the dodecahedron coincide with those of the cube, while its other twelve are the ends of the ridges above the cube faces. The cube and the dodecahedron share the same centre. If the ridges above the cube faces are moved parallel to themselves until they lie in the cube faces, their ends define the twelve vertices of an icosahedron. Many properties of the dodecahedron and icosahedron can be inferred simply from this construction. For example, it shows that the cube and the derived dodecahedron have the same circumsphere, whose radius is half the body diagonal of the cube, and that the vertices of an icosahedron can be separated into those of three mutually orthogonal golden rectangles. I will derive three results that are directly useful for us here, but before I do so I introduce some important lengths connected with a regular pentagon.

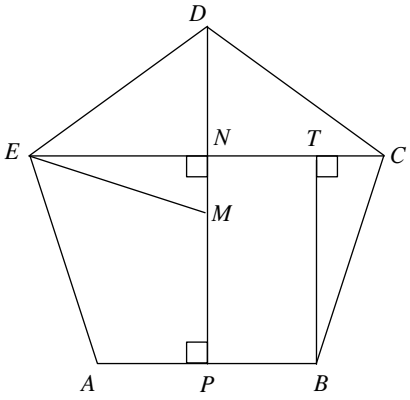


FIGURE 2(a)

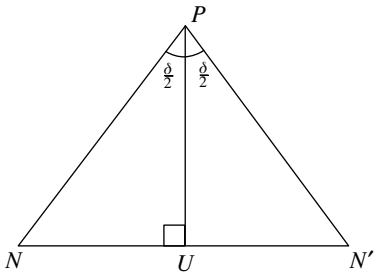


FIGURE 2(b)

Figure 2(a) shows a regular pentagon whose side I will take to have length s. Then the length of a diagonal, such as EC, is $s\tau$. M is the centre of the pentagon and $ME = \frac{1}{2}s \operatorname{cosec} \frac{\pi}{5}$ is its circumradius. Point N is the intersection of EC with the perpendicular from D to the opposite side. I now calculate the lengths of the segments PN and MN, both of which are related to important properties of the dodecahedron. Since the vertex angle of the pentagon is $3\pi/5$ and $\angle BCE = 2\pi/5$, we have

$$PN = BT = BC \sin \frac{2\pi}{5} = s \sin \frac{2\pi}{5}.$$

Also $EN = \frac{1}{2}s\tau$, so $MN = \sqrt{ME^2 - EN^2} = \frac{1}{2}s\sqrt{\operatorname{cosec}^2 \frac{\pi}{5} - \tau^2}$. With these results in hand, I look at three triangles in the dodecahedron and extract some useful information from each of them.

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Consider two segments like PN lying on adjacent sections of a roof and having their common point P on its ridge (see Figure 2(b)). These segments make an angle δ with each other, where δ is the dihedral angle of the dodecahedron, and Euclid's construction shows that their far ends, N and N', are separated by an edge length of the cube; thus $NN' = s\tau$ and we can use triangle PNU to calculate the dihedral angle of the dodecahedron as

$$\sin\frac{\delta}{2} = \frac{NU}{PN} = \frac{\frac{1}{2}s\tau}{s\,\sin\frac{2\pi}{5}} = \frac{\tau}{2\,\sin\frac{2\pi}{5}}. \tag{4}$$

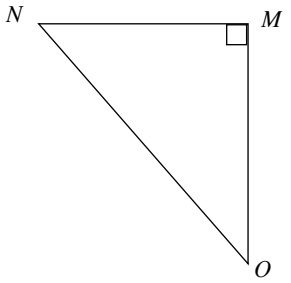


FIGURE 2(c)

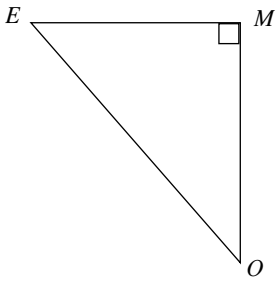


FIGURE 2(d)

Next consider the triangle NMO, where N and M are the points in Figure 2(a) and O is the common centre of the cube and the dodecahedron (see Figure 2(c)). OM is the distance from the centre of the dodecahedron to a face, and so is the inradius R_i of the dodecahedron. Also, Euclid's construction shows that NO has a length equal to half the face diagonal of the cube; thus $NO = s\tau/\sqrt{2}$, and applying the Pythagorean theorem to triangle NMO shows that

$$R_i = OM = \sqrt{ON^2 - NM^2} = \frac{s}{2}\sqrt{3\tau^2 - \operatorname{cosec}^2\tfrac{\pi}{5}}. \tag{5}$$

A bit of manipulation allows the expressions for δ and R_i in (4) and (5) to be put into the neater form given by Coxeter [7]

$$\delta = 2\,\sin^{-1}\left(\tau^{1/2}5^{-1/4}\right) \qquad \text{and} \qquad R_i = 5^{-1/4}\tau^{5/2}\left(\frac{s}{2}\right). \tag{6}$$

I will need these results later, along with the corresponding results for a cube of edge length s (namely $\delta = \frac{\pi}{2}$ and $R_i = \frac{s}{2}$) in analysing some of the more complicated solids.

Finally, I come to the main point of this little interlude, which is the solution of the kissing problem for the dodecahedron. Consider the right-angled triangle OME (see Figure 2(d)), in which O is the centre of the dodecahedron, M the centre of one of its faces and E one of its vertices. We know that $OM = R_i$ and $ME = \frac{s}{2}\operatorname{cosec}\frac{\pi}{5}$, so applying the Pythagorean theorem to triangle OME gives

$$OE = \sqrt{OM^2 + ME^2} = \frac{s}{2}\sqrt{\frac{\tau^5}{\sqrt{5}} + \operatorname{cosec}^2\frac{\pi}{5}},$$

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on making use of (6). The kissing problem can be solved by placing spheres of radius $\frac{1}{2}s$ at the vertices of a dodecahedron of edge length s and choosing the central sphere so that the sum of its radius and that of one of the outer spheres is OE. Thus the radius of the central sphere can be calculated as $r = OE - \frac{1}{2}s$. Finally, on finally putting $s = 2$ (to make the outer spheres have unit radius), we find that the radius of the central sphere is given by

$$r = \sqrt{\frac{\tau^5}{\sqrt{5}} + \operatorname{cosec}^2 \frac{\pi}{5}} - 1, \tag{7}$$

which, upon simplification, matches the result in Table 1.

Having disposed of the Platonics, I turn to the Archimedeans and first take up the cuboctahedron and the icosidodecahedron, the only two 'quasi-regular' polyhedra on the list. As Coxeter points out [7], each has a plane of vertices midway between a pair of opposite faces that defines a regular hexagon (in the case of the cuboctahedron) or decagon (in the case of the icosidodecahedron). Since there is a ring of identical spheres centred at these vertices, it follows that θ is $\frac{\pi}{6}$ or $\frac{\pi}{10}$ in these two cases and r is then readily calculated from (1). This is precisely the solution for the icosidodecahedron given by Fukagawa and Rothman [1].

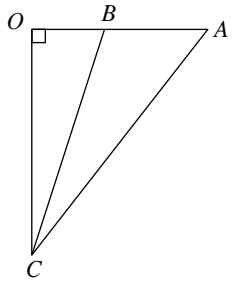


FIGURE 3

I turn next to the Archimedeans with names of the form Truncated X, where X is one of the Platonic solids. These solids can be obtained by dividing the edges of a Platonic into three parts, discarding the two outer parts, and finally adding edges that complete the faces of the new figure. Pugh [8] terms these Archimedeans 'three-frequency figures' and gives nice diagrams showing how the triangular, square or pentagonal faces of a Platonic get converted into the hexagonal, octagonal or decagonal faces of the truncated figure (which also develops new triangular, square or pentagonal faces in the process). An examination of the truncation process shows that a solution of the kissing problem for a Platonic can be converted into one for the derived Archimedean if the unit spheres at the vertices of the former are replaced by spheres of radius l at the vertices of the latter, where $l = \dfrac{1}{3}, \dfrac{1}{1+\sqrt{2}}$ or $\dfrac{\tau}{\tau+2}$ if the parent Platonic has triangular, square or pentagonal faces, respectively. To calculate the radius of the central sphere for the derived Archimedean, refer to Figure 3, in which A, O and C

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are a vertex, midpoint of an edge and centre of the Platonic and B is a vertex of the Archimedean, and note that $OA = 1$, $OB = l$ and $AC = r' + 1$, where r' is the radius of the central sphere for the Platonic with unit spheres at its vertices. From the Pythagorean theorem, $OC = \sqrt{AC^2 - OA^2} = \sqrt{r'^2 + 2r'}$ and $BC = \sqrt{OC^2 + OB^2} = \sqrt{r'^2 + 2r' + l^2}$. But BC is the sum of the radii of the central sphere for the Archimedean and one the spheres touching it, so one can determine the radius of the central sphere by subtracting l from BC. Finally, on rescaling all lengths to make the spheres at the vertices of the Archimedean unit spheres (which amounts simply to a division by l) , we can determine the radius of the central sphere for the Archimedean as

$$r = \frac{1}{l}(BC - l) = \frac{1}{l}\sqrt{r'^2 + 2r' + l^2} - 1, \tag{8}$$

where r' is the radius of the central sphere for the parent Platonic with unit spheres at its vertices. On putting the values of r' for the Platonics from Table 1, along with the appropriate values of l, into (8), one finds that one gets back the results for the five Truncated Platonics in Table 1.

I finally take up the last four solids in Table 1. Each of these solids can be inscribed in a cube or dodecahedron in such a way that its vertices lie in the faces of the cube (in the case of the rhombicuboctahedra) or dodecahedron (in the case of the rhombicosidodecahedra). I will refer to the embedding cube or dodecahedron as the parent and the inscribed Archimedean as the son or daughter [9] if its name bears the prefix great or small, respectively*. The sons have octagonal or decagonal faces lying in the faces of their parents, while the daughters have square or pentagonal ones. I will term the faces of a child lying in the faces of its parent 'prime' faces. Prime faces have the same centres as the parental faces they lie in, and are aligned symmetrically with them. The son and daughter of a parent also have the same centre as it. In the type of solution we want, spheres of unit radius placed at the vertices of a son or daughter touch all their nearest neighbours, implying that the son or daughter has an edge length of 2 units. To calculate the radius of the central sphere for the son or daughter, I introduce the following quantities:

c = circumradius of a prime face of the son or daughter = $\operatorname{cosec}\frac{\pi}{n}$, if n is the number of sides of the prime face;

R_i = inradius of the parent;

h = distance from a vertex of a prime face to the centre of the son or daughter.

* The reader may wonder what the sex of the parent is and whether only single parenting is involved. Actually, I am not the first to introduce sex into this business. The credit for that should go to J. H. Conway, H. Bruegel and C. Goodman-Strauss who, in [9], give a well-reasoned (if politically incorrect) argument for assigning certain family roles to the Platonics and some of the solids derived from them. The sex roles I have introduced here are a little different from theirs, but suit my narrative better. Incidentally, the book by Conway et al contains an encyclopedic treatment of symmetry along with many lavish and well thought out colour illustrations, and should be a veritable delight to all lovers of the subject.

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These three quantities are the sides of a right-angled triangle of which the last is the hypotenuse, so $h = \sqrt{R_i^2 + c^2}$. But h is the sum of the radii of the central sphere and one of the unit spheres touching it, so the radius of the central sphere is

$$r = h - 1 = \sqrt{R_i^2 + c^2} - 1 = \sqrt{R_i^2 + \operatorname{cosec}^2 \tfrac{\pi}{n}} - 1. \tag{9}$$

Now, the inradius of the parent is given in terms of its edge length, s, by the expression

$$R_i = \frac{s}{2} \qquad \text{or} \qquad R_i = 5^{-1/4}\tau^{5/2}\left(\frac{s}{2}\right) \tag{10}$$

if the parent is a cube or dodecahedron, respectively (cf. (6)). Thus, if we can fix the edge length s of the parent, we can use (9) and (10) to calculate r.

The edge length of the parent can be fixed by using the fact that its son or daughter has square faces linking its prime faces. However the edge length dictated by the son is different from that dictated by the daughter, so there are four cases to be considered in all since the parent can be either a cube or a dodecahedron and the child either a son or a daughter (which is just another way of characterising the last four cases in Table 1). I will deal with the two parent-daughter cases first and then take up the parent-son ones.

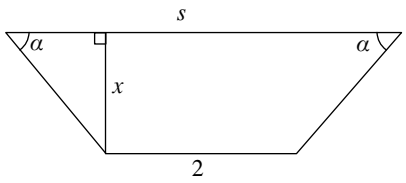


FIGURE 4(a)

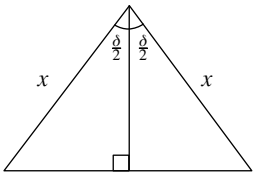


FIGURE 4(b)

Figure 4(a) shows a trapezium whose parallel sides are an edge of a prime face of a daughter (of length 2) and an edge of the parental face it lies in (of length s). The angle α is half the vertex angle of a face of the parent and is $\pi/4$ or $3\pi/10$ if the parent is a cube or a dodecahedron. The distance between the parallel sides of the trapezium is $x = \left(\frac{1}{2}s - 1\right)\tan\alpha$. Figure 4(b) shows two segments like x lying on adjacent faces of the parent and making an angle δ with each other, where δ is the dihedral angle of the parent (note the similarity to Figure 2(b)). But the far ends of these segments are the vertices of a square face of the daughter and so the distance between them must be 2; thus $2x \sin\frac{1}{2}\delta = 2$, and putting this in the earlier expression for x allows us to solve for the edge length of the parent as $s = 2\left[1 + \cot\alpha \operatorname{cosec}\frac{1}{2}\delta\right]$.

I next take up the parent-son case. If one joins the vertices of a prime face of a son to the vertices of the parental face in which it lies, one gets an alternating ring of trapezia and isosceles triangles around the prime face. Figure 5 shows one such trapezium and the two isosceles triangles flanking

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it. The parallel sides of the trapezium have lengths 2 and s and are separated by a distance x, while the other two sides have length d. The isosceles triangles have sides of length 2 and d. The acute angle of the trapezium is β and the vertical angle of the isosceles triangle is γ. Inspection of Figure 5 shows that these lengths and angles obey the equations

$$2\beta + \gamma = 2\alpha, d = \operatorname{cosec}\tfrac{1}{2}\gamma, \tfrac{1}{2}(s-2) = d\cos\beta, x = d\sin\beta \text{ and } 2x\sin\tfrac{1}{2}\delta = 2, \quad (11)$$

where α was defined in the previous paragraph and the last equation arises for the same reason it did in the case of the daughters. These equations can be solved for the edge length of the parent as

$$s = 2\left[1 + \cot\alpha \operatorname{cosec}\tfrac{1}{2}\delta + \operatorname{cosec}\alpha\right]. \quad (12)$$

Note that this is practically the same as the expression we got for the daughters, the only difference being that the last term within the parentheses was missing there. Thus one can use the single expression (12) to calculate the edge length of the parent in all four cases if one uses the appropriate values of α and δ for the parent and keeps or omits the last term within the parentheses depending on whether one is dealing with a son or a daughter. Table 2 shows the values of s in all four cases calculated using this procedure. Using these values of s in conjunction with the appropriate choices in (9) and (10) allows us to recover the results for these four solids given in Table 1. The result for the small rhombicuboctahedron of course agrees with the one we got earlier using the prism formula.

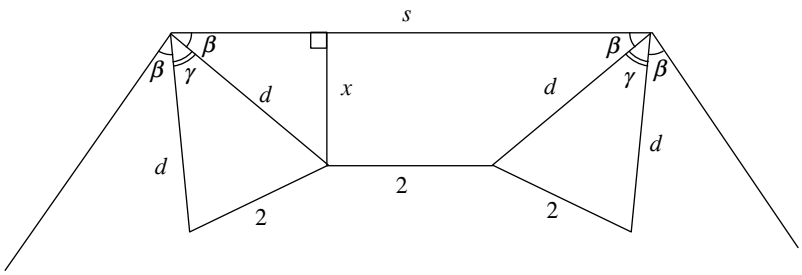


FIGURE 5

Parent	α	δ	Son/Daughter	s
Cube	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3{\cdot}4^3$ (D) $4{\cdot}6{\cdot}8$ (S)	$2\left(1+\sqrt{2}\right)$ $2\left(1+2\sqrt{2}\right)$
Dodecahedron	$\frac{3\pi}{10}$	$2\sin^{-1}\left(\tau^{1/2}5^{-1/4}\right)$	$3{\cdot}4{\cdot}5{\cdot}4$ (D) $4{\cdot}6{\cdot}10$ (S)	$6/\tau$ $10/\tau$

TABLE 2: Calculation of the edge length, s, of the parent for the four different parent-child cases.

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The only solids I have not considered in this account are the two snubs. They can also be inscribed in a cube or a dodecahedron and (9) and (10) can be used for them provided one is able to fix the edge lengths of the parents. However the geometry needed to do this is rather involved, and I will not discuss these cases here.

Endnote: I thank an anonymous referee for suggesting several improvements to an earlier draft of this paper and for pointing out that still further solutions can be obtained for some values of N and M by distorting some of the polyhedra involved. I dedicate this paper to the memory of my high school mathematics teacher, N. C. Sharma of Mayo College, Ajmer, India, a geometer and a gentleman, who revelled in challenging himself along with the young lads around him.

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