Canad. Math. Bull. Vol. **69** (3), 2026, pp. 1054–1067
http://dx.doi.org/10.4153/S0008439526101623
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

Restriction problem for mod p representations of GL_2 over a finite field

Eknath Ghate and Shubhanshi Gupta

Abstract. Let $\mathbb{F}_q$ be the finite field with $q = p^f$ elements. We study the restriction of two classes of mod p representations of $G_q = \mathrm{GL}_2(\mathbb{F}_q)$ to $G_p = \mathrm{GL}_2(\mathbb{F}_p)$. We first study the restrictions of principal series which are obtained by induction from a Borel subgroup B_q. We then analyze the restrictions of inductions from an anisotropic torus T_q which are related to cuspidal representations. Complete decompositions are given in both cases according to the parity of f. The proofs depend on writing down explicit orbit decompositions of $G_p \backslash G_q / H$, where $H = B_q$ or T_q using the fact that G_q/H is an explicit orbit in a certain projective line, along with Mackey theory.

1 Introduction

The *restriction problem*, which examines how representations behave when restricted from a group to a subgroup, is a fundamental topic in representation theory. In general, the restriction of an irreducible representation of a group to a subgroup is neither irreducible nor completely reducible. The general theory by which the restriction of an irreducible representation of a group decomposes into irreducible representations of its subgroup is explained by *branching rules*. One is interested in computing the multiplicity with which an irreducible representation of the subgroup appears (as a subquotient) in the restriction of an irreducible representation of the given group. More generally, one may ask how a member of a class of representations of a group decomposes under restriction into member of this class for the subgroup. In Kobayashi [9], the author gives a survey of branching problems for complex representations of real reductive groups. See also, for instance, [4, 5, 10, 12] for other aspects of the restriction problem.

The decomposition of a finite tensor product of representations of a group G into its irreducible representations can be considered as part of a restriction problem as it can be seen as the restriction of the representation of the group $G \times G \times \cdots \times G$ (finitely many times) to the diagonal subgroup which is isomorphic to G. In this case, the branching is explained by what are called Clebsch–Gordan formulas. This situation occurs naturally in the study of the restriction of irreducible mod p representations of finite groups of Lie type, and in particular in the study of the restriction of irreducible mod p representations (weights) of $\mathrm{GL}_2(\mathbb{F}_q)$ to $\mathrm{GL}_2(\mathbb{F}_p)$, where $\mathbb{F}_q = \mathbb{F}_{p^f}$ is a finite

Received by the editors June 16, 2025; revised December 20, 2025; accepted December 30, 2025.
Published online on Cambridge Core January 7, 2026.
AMS subject classification: **20C20**, **20C33**, 12E20, 14L30.
Keywords: Modular representations, restriction problem, projective space, Mackey's formula.

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

extension of degree $f \geq 1$ of the finite field $\mathbb{F}_p$ with p elements. See, for instance, [8] for an explicit decomposition in the case that $\mathbb{F}_q$ is a degree $f = 3$ extension of $\mathbb{F}_p$ using the Clebsch–Gordan formula from [7] for $\mathrm{GL}_2(\mathbb{F}_p)$. See also [2] for Clebsch–Gordan formulas for $\mathrm{GL}_2(\mathbb{F}_q)$ (which would be useful for solving the restriction problem when the group that one is restricting to is $\mathrm{GL}_2(\mathbb{F}_q)$, cf. Remark 3.11).

In this context, it would be interesting to study the restriction problem for a slightly more complicated but natural class of representations of these groups. Let $G_p \coloneqq \mathrm{GL}_2\left(\mathbb{F}_p\right)$ and $G_q \coloneqq \mathrm{GL}_2\left(\mathbb{F}_q\right)$. Let B_p and B_q denote the Borel subgroups of all upper triangular matrices of G_p and G_q, respectively. A natural problem would be to study the restriction to G_p of principal series representations of G_q obtained by inducing a mod p character of B_q. Similarly, let T_p and T_q denote the anisotropic torus of G_p and G_q, respectively. One could also study the restrictions from G_q to G_p of representations induced from a mod p character of T_q. The latter representations are closely connected to the mod p reductions of complex cuspidal representations of G_q.

In this article, we solve the restriction problem for both these classes of representations of G_q completely, by decomposing their restrictions to G_p into a direct sum of analogous representations of G_p (possibly twisted by the mod p Steinberg representation of G_p). The explicit decompositions are contained in Theorems 2.18 and 3.9 which are the main results of this article. The key tool is Mackey's restriction formula. However, to use it, one needs to make the orbit decomposition of $G_p \backslash G_q / H$ reasonably explicit where $H = B_q$ or T_q. This is done using the projective line as a crutch, noting that $G_q/B_q \simeq \mathbb{P}^1(\mathbb{F}_q)$ and G_q/T_q is the orbit of an element in $\mathbb{P}^1(\mathbb{F}_{q^2})$ outside $\mathbb{P}^1(\mathbb{F}_q)$. These orbit decompositions may be of independent interest.

2 Restriction of principal series representations

Recall that for a mod p character χ of B_q, the mod p principal series representation of G_q is the induced representation $\mathrm{ind}_{B_q}^{G_q}\chi$. Its Jordan–Hölder factors are described in [1, 3]. While a complete decomposition of the restriction of such a principal series representation of G_q to G_p is not immediately evident, one might expect that the principal series $\mathrm{ind}_{B_p}^{G_p}(\chi|_{B_p})$ of G_p (which has two Jordan–Hölder factors) be contained in the restriction to G_p.

Lemma 2.1 *We have* $\mathrm{ind}_{B_p}^{G_p}\left(\chi|_{B_p}\right) \hookrightarrow \mathrm{ind}_{B_q}^{G_q}\chi\Big|_{G_p}$.

Proof By Frobenius reciprocity, this is equivalent to checking whether $\chi|_{B_p} \hookrightarrow \mathrm{ind}_{B_q}^{G_q}\chi\Big|_{B_p}$, which is true as $\chi \hookrightarrow \mathrm{ind}_{B_q}^{G_q}\chi\Big|_{B_q}$ (again, by Frobenius reciprocity). ∎

The real question then is what the other representations appearing in the decomposition of the restriction are. In this section, we shall give a complete answer in terms of principal series representations, the Steinberg representation, and cuspidal representations of G_p.

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

2.1 The action of G_p on $\mathbb{P}^1(\mathbb{F}_q)$

There is a natural action of G_q on the projective space $\mathbb{P}^1(\mathbb{F}_q)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy],$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$ and $[x : y] \in \mathbb{P}^1(\mathbb{F}_q)$. This action is transitive with $\mathrm{Stab}_{G_q}([1:0]) = B_q$. We identify $\mathbb{P}^1(\mathbb{F}_q)$ with $\{[1 : x] : x \in \mathbb{F}_q\} \cup \{[0 : 1]\}$ throughout the article.

Notation 2.2 *Denote $\widehat{x} := [1 : x]$ and $\widehat{\infty} := [0 : 1]$.*

Notation 2.3 *Fix $\varepsilon \in \mathbb{F}_{q^2} \smallsetminus \mathbb{F}_q$ such that $\varepsilon^2 \in \mathbb{F}_q$. Denote the anisotropic torus of G_q by*

$$T_q := \left\{ \begin{pmatrix} a & b \\ b\varepsilon^2 & a \end{pmatrix} \in G_q : a, b \in \mathbb{F}_q \text{ not both zero} \right\}.$$

Analogously, for fixed $\eta \in \mathbb{F}_{p^2} \smallsetminus \mathbb{F}_p$ such that $\eta^2 \in \mathbb{F}_p$, let

$$T_p := \left\{ \begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix} \in G_p : a, b \in \mathbb{F}_p \text{ not both zero} \right\}$$

denote the anisotropic torus of G_p.

Lemma 2.4 *Under the action of G_q, we have the orbit decomposition*

$$\mathbb{P}^1(\mathbb{F}_{q^2}) = O_{G_q}(\widehat{0}) \sqcup O_{G_q}(\widehat{\varepsilon}).$$

Moreover, the stabilizers of $\widehat{0}$ and $\widehat{\varepsilon}$ are equal to B_q and T_q, respectively.

Proof The transitivity of the G_q action on $\mathbb{P}^1(\mathbb{F}_q)$ gives the orbit $O_{G_q}(\widehat{0}) = \mathbb{P}^1(\mathbb{F}_q) = \{\widehat{x} : x \in \mathbb{F}_q\} \sqcup \{\widehat{\infty}\}$. That $\mathrm{Stab}_{G_q}(\widehat{0}) = B_q$ can easily be verified. Note that $\widehat{\varepsilon} \notin O_{G_q}(\widehat{0})$ since $\varepsilon \in \mathbb{F}_{q^2} \smallsetminus \mathbb{F}_q$. If we show that $\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| = \left|O_{G_q}(\widehat{0}) \sqcup O_{G_q}(\widehat{\varepsilon})\right|$, then we are done. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}_{G_q}(\widehat{\varepsilon})$, that is, $g \cdot \widehat{\varepsilon} = \widehat{\varepsilon}$. Then, for some $0 \neq \lambda \in \mathbb{F}_{q^2}$, $(a + b\varepsilon, c + d\varepsilon) = (\lambda, \lambda\varepsilon)$ which gives $(a + b\varepsilon)\varepsilon = c + d\varepsilon$. As $1, \varepsilon$ are linearly independent over $\mathbb{F}_q$, we get $a = d$ and $c = b\varepsilon^2$. Thus, $\mathrm{Stab}_{G_q}(\widehat{\varepsilon}) = T_q$ and so the number of elements in $O_{G_q}(\widehat{\varepsilon})$ is $\frac{(q^2-1)(q^2-q)}{q^2-1} = q^2 - q$ and $\left|O_{G_q}(\widehat{0})\right| + \left|O_{G_q}(\widehat{\varepsilon})\right| = q^2 + 1 = \left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|$. ∎

Notation 2.5 *For $f > 2$, define the set* $X = \begin{cases} \mathbb{F}_q \smallsetminus \mathbb{F}_p, & \text{if } 2 \nmid f, \\ \mathbb{F}_q \smallsetminus \mathbb{F}_{p^2}, & \text{if } 2 \mid f. \end{cases}$

Lemma 2.6 *For all $x \in X$, the set $\{1, x, x^2\}$ is linearly independent over $\mathbb{F}_p$.*

Proof Let $x \in \mathbb{F}_q$ and let $\{1, x, x^2\}$ be linearly dependent over $\mathbb{F}_p$. Then, there exist $a_0, a_1, a_2 \in \mathbb{F}_p$, not all zero, such that $a_0 + a_1x + a_2x^2 = 0$. That is, x satisfies a

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

polynomial of degree at most 2 over $\mathbb{F}_p$. If either $a_2 = 0$ or the polynomial is reducible, then $x \in \mathbb{F}_p$. If $a_2 \neq 0$ and the polynomial is irreducible, then $\mathbb{F}_{p^2} \simeq \mathbb{F}_p(x) \hookrightarrow \mathbb{F}_q$. But $\mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_q$ iff $2 \mid f$. Thus, $x \in \mathbb{F}_p$ when $2 \nmid f$ and $x \in \mathbb{F}_{p^2}$ when $2 \mid f$. ∎

Lemma 2.7 *With respect to the G_p action on $\mathbb{P}^1(\mathbb{F}_q)$,*

$$\mathrm{Stab}_{G_p}(\widehat{x}) = Z_p, \text{ for all } x \in X,$$

where Z_p is the center of G_p.

Proof Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}_{G_p}(\widehat{x})$, then $(a+bx, c+dx) = (\lambda, \lambda x)$, for some $0 \neq \lambda \in \mathbb{F}_q$. So, we get $c + (d-a)x - bx^2 = 0$ and by Lemma 2.6, $a = d$ and $b = c = 0$. ∎

Proposition 2.8 *The decomposition of $\mathbb{P}^1(\mathbb{F}_q)$ into its G_p-orbits is given as*

$$\mathbb{P}^1(\mathbb{F}_q) = \begin{cases} O_{G_p}(\widehat{0}) \sqcup \bigsqcup_{\widehat{x} \in \mathcal{I}_1} O_{G_p}(\widehat{x}), & \text{if } 2 \nmid f, \quad (2.1) \\ O_{G_p}(\widehat{0}) \sqcup O_{G_p}(\widehat{\eta}) \sqcup \bigsqcup_{\widehat{x} \in \mathcal{I}_2} O_{G_p}(\widehat{x}), & \text{if } 2 \mid f, \quad (2.2) \end{cases}$$

where $\eta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ with $\eta^2 \in \mathbb{F}_p$ and where $\mathcal{I}_1$ and $\mathcal{I}_2$ form a complete set of representatives of the G_p-orbits in $\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)$ and $\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_{p^2})$, respectively. Also,

$$|\mathcal{I}_1| = \frac{p^{f-1}-1}{p^2-1} \text{ and } |\mathcal{I}_2| = \frac{p\left(p^{f-2}-1\right)}{p^2-1}.$$

Proof Suppose $2 \nmid f$. Since $\mathbb{P}^1(\mathbb{F}_p) \hookrightarrow \mathbb{P}^1(\mathbb{F}_q)$ is G_p-stable, we have

$$G_p \backslash \mathbb{P}^1(\mathbb{F}_q) = G_p \backslash \mathbb{P}^1(\mathbb{F}_p) \sqcup G_p \backslash \left(\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)\right).$$

Note that $G_p \backslash \mathbb{P}^1(\mathbb{F}_p) = \{O_{G_p}(\widehat{0})\}$ as G_p acts transitively on $\mathbb{P}^1(\mathbb{F}_p)$. Moreover, for any $\widehat{x} \in \mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)$, $\mathrm{Stab}_{G_p}(\widehat{x}) = Z_p$ by Lemma 2.7, and so $\left|O_{G_p}(\widehat{x})\right| = p\left(p^2-1\right)$. Since the orbit of each element in $\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)$ has the same cardinality, the number of G_p-orbits in $\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)$ is

$$|\mathcal{I}_1| = \frac{\left|\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_p)\right|}{\left|O_{G_p}(\widehat{x})\right|} = \frac{\left(p^f+1\right)-(p+1)}{p\left(p^2-1\right)} = \frac{p^{f-1}-1}{p^2-1}.$$

Suppose $2 \mid f$. Since $\mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_q$ and $\mathbb{P}^1(\mathbb{F}_{p^2}) \hookrightarrow \mathbb{P}^1(\mathbb{F}_q)$ is G_p-stable, we have

$$G_p \backslash \mathbb{P}^1(\mathbb{F}_q) = G_p \backslash \mathbb{P}^1(\mathbb{F}_{p^2}) \sqcup G_p \backslash \left(\mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_{p^2})\right).$$

Note that $G_p \backslash \mathbb{P}^1(\mathbb{F}_{p^2}) = \{O_{G_p}(\widehat{0}), O_{G_p}(\widehat{\eta})\}$ by taking $f = 1$ in Lemma 2.4. Again, for any $\widehat{x} \in \mathbb{P}^1(\mathbb{F}_q) \setminus \mathbb{P}^1(\mathbb{F}_{p^2})$, $\mathrm{Stab}_{G_p}(\widehat{x}) \simeq Z_p$ by Lemma 2.7, and so, $\left|O_{G_p}(\widehat{x})\right| = p\left(p^2-1\right)$. So, by the same argument as in the previous case, the number of G_p-orbits

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

in $\mathbb{P}^1(\mathbb{F}_q) \smallsetminus \mathbb{P}^1(\mathbb{F}_{p^2})$ is

$$|\mathcal{I}_2| = \frac{\left|\mathbb{P}^1(\mathbb{F}_q) \smallsetminus \mathbb{P}^1(\mathbb{F}_{p^2})\right|}{\left|O_{G_p}(\widehat{x})\right|} = \frac{\left(p^f+1\right) - \left(p^2+1\right)}{p\left(p^2-1\right)} = \frac{p\left(p^{f-2}-1\right)}{p^2-1}. \qquad \blacksquare$$

2.2 Decomposition of the restriction

For a mod p character χ of B_q, let $\operatorname{ind}_{B_q}^{G_q}\chi$ be a principal series representation of G_q. Then, by Mackey's restriction formula, we have

$$\operatorname{ind}_{B_q}^{G_q}\chi\Big|_{G_p} \simeq \bigoplus_{\gamma\in\Gamma} \operatorname{ind}_{B_q^\gamma}^{G_p}\chi^\gamma, \tag{2.3}$$

where $B_q^\gamma \coloneqq \gamma B_q \gamma^{-1} \cap G_p$ and $\chi^\gamma(b) = \chi\left(\gamma^{-1}b\gamma\right)$ for all $b \in B_q^\gamma$ and where Γ is a set of double coset representatives of $G_p\backslash G_q/B_q$.

Notation 2.9 *For $x \in \mathbb{F}_q$, let $g_x \coloneqq \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ and $w \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in G_q.*

Remark 2.10 A complete set of coset representatives of G_q/B_q is given by $\{g_x : x \in \mathbb{F}_q\} \cup \{w\}$, and under the identification $G_q/B_q \simeq \mathbb{P}^1(\mathbb{F}_q)$, we see that $g_x B_q \mapsto \widehat{x}$ and $wB_q \mapsto \widehat{\infty}$.

Under the action of G_p on G_q/B_q, we have $\operatorname{Stab}_{G_p}\left(gB_q\right) = gB_qg^{-1} \cap G_p = B_q^g$. Hence, taking $g = g_x$ and using Remark 2.10, we have

$$B_q^{g_x} = \operatorname{Stab}_{G_p}(\widehat{x}), \text{ for all } x \in \mathbb{F}_q. \tag{2.4}$$

Now, the double cosets in $G_p\backslash G_q/B_q$ are in one-to-one correspondence with the orbits in $G_p\backslash\left(G_q/B_q\right)$ and hence with the orbits in $G_p\backslash\mathbb{P}^1(\mathbb{F}_q)$ by Remark 2.10. Also,

$$G_p g_x B_q \text{ is the union of left cosets in the } G_p\text{-orbit of } g_x B_q.$$

Moreover, $G_p w B_q = G_p g_0 B_q$. Thus, if Γ is a set of double coset representatives of $G_p\backslash G_q/B_q$, then we may assume $\Gamma \subseteq \{g_x : x \in \mathbb{F}_q\}$. Again, by Remark 2.10,

$$\Gamma \xleftrightarrow{1\text{–}1} G_p\backslash\mathbb{P}^1(\mathbb{F}_q), \text{ where } g_x \mapsto O_{G_p}(\widehat{x}). \tag{2.5}$$

Thus, substituting (2.4) and (2.5) in (2.3), we get

$$\operatorname{ind}_{B_q}^{G_q}\chi\Big|_{G_p} \simeq \bigoplus_{O_{G_p}(\widehat{x})\in G_p\backslash\mathbb{P}^1(\mathbb{F}_q)} \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{x})}^{G_p}\chi^{g_x}.$$

Now, Proposition 2.8 gives the description of all the G_p-orbits in $\mathbb{P}^1(\mathbb{F}_q)$ and therefore,

$$\operatorname{ind}_{B_q}^{G_q}\chi\Big|_{G_p} \simeq \begin{cases} \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{0})}^{G_p}\chi^{g_0} \oplus \bigoplus\limits_{\widehat{x}\in\mathcal{I}_1} \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{x})}^{G_p}\chi^{g_x}, & \text{if } 2 \nmid f, \\ \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{0})}^{G_p}\chi^{g_0} \oplus \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{\eta})}^{G_p}\chi^{g_\eta} \oplus \bigoplus\limits_{\widehat{x}\in\mathcal{I}_2} \operatorname{ind}_{\operatorname{Stab}_{G_p}(\widehat{x})}^{G_p}\chi^{g_x}, & \text{if } 2 \mid f. \end{cases}$$

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

Note that, by Lemma 2.4, the stabilizers of $\widehat{0}$ and $\widehat{\eta}$ in G_p are B_p and T_p, respectively, and $\chi^{g_0} = \chi|_{B_p}$. Also, for all $\widehat{x}$ in $\mathcal{I}_1$ or $\mathcal{I}_2$, the corresponding field element x is in X (see the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$) and so, $\mathrm{Stab}_{G_p}(\widehat{x}) = Z_p$, for all $\widehat{x}$ in $\mathcal{I}_1$ or $\mathcal{I}_2$ by Lemma 2.7. Lastly, we note that $\chi^{g_x}|_{Z_p} = \chi|_{Z_p}$, for all $g_x \in G_q$ and so $\mathrm{ind}_{\mathrm{Stab}_{G_p}(\widehat{x})}^{G_p} \chi^{g_x} = \mathrm{ind}_{Z_p}^{G_p}\left(\chi|_{Z_p}\right)$, for all $\widehat{x} \in \mathcal{I}_1$ or $\widehat{x} \in \mathcal{I}_2$. Combining this information, we get

$$\mathrm{ind}_{B_q}^{G_q}\chi\Big|_{G_p} \simeq \begin{cases} \mathrm{ind}_{B_p}^{G_p}\left(\chi|_{B_p}\right) \oplus |\mathcal{I}_1|\,\mathrm{ind}_{Z_p}^{G_p}\left(\chi|_{Z_p}\right), & \text{if } 2 \nmid f, \\ \mathrm{ind}_{B_p}^{G_p}\left(\chi|_{B_p}\right) \oplus \mathrm{ind}_{T_p}^{G_p}\chi^{g_\eta} \oplus |\mathcal{I}_2|\,\mathrm{ind}_{Z_p}^{G_p}\left(\chi|_{Z_p}\right), & \text{if } 2 \mid f, \end{cases} \tag{2.6}$$

where nV denotes the direct sum of n copies of the representation V.

Notation 2.11 *For all positive integers f, any character of B_q is of the form $\chi_{r,s}$ for $0 \le r, s < q-1$, where*

$$\chi_{r,s}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a^r d^s = a^{r-s}(ad)^s, \text{ for all } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_q.$$

Denote $\chi_r := \chi_{r,0}$.

Since $\mathrm{ind}_{B_q}^{G_q}(\chi_{r,s}) \simeq \mathrm{ind}_{B_q}^{G_q}(\chi_{r-s} \otimes \det^s) \simeq \mathrm{ind}_{B_q}^{G_q}\chi_{r-s} \otimes \det^s$, for all $r, s \in \{0, 1, \ldots, q-2\}$, it is sufficient to analyze the restriction $\mathrm{ind}_{B_q}^{G_q}\chi_r\Big|_{G_p}$.

Notation 2.12 *The split torus of G_p is denoted by $S_p := \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}_p^*\right\}$.*

Notation 2.13 *Let $f \ge 1$. The character $\omega_{2f} : T_q \to \mathbb{F}_{q^2}^*$ is defined by*

$$\omega_{2f}\left(\begin{pmatrix} a & b \\ b\varepsilon^2 & a \end{pmatrix}\right) = a + b\varepsilon, \text{ for all } \begin{pmatrix} a & b \\ b\varepsilon^2 & a \end{pmatrix} \in T_q.$$

All the characters of T_q are then precisely of the form ω_{2f}^r, where $0 \le r < q^2 - 1$.

In particular, the character ω_2 of T_p is given as

$$\omega_2\left(\begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix}\right) = a + b\eta, \text{ for all } \begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix} \in T_p.$$

Lemma 2.14 *If f is even, then as characters of T_p, we have $\chi_r^{g_\eta} = \omega_2^r$.*

Proof Indeed, for all $\begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix} \in T_p$, we have

$$\begin{aligned} \chi_r^{g_\eta}\left(\begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix}\right) &= \chi_r\left(\begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix}\right) \\ &= \chi_r\left(\begin{pmatrix} a + b\eta & b \\ 0 & a - b\eta \end{pmatrix}\right) \end{aligned}$$

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

$$= (a + b\eta)^r \quad (\text{since the matrix lies in } B_q \text{ because } \mathbb{F}_{p^2} \subseteq \mathbb{F}_q)$$
$$= \omega_2^r \left(\begin{pmatrix} a & b \\ b\eta^2 & a \end{pmatrix} \right). \qquad \blacksquare$$

Notation 2.15 *For a complex representation V of a finite group, let $\overline{V}$ denote the mod p reduction of an integral model of V. This is only well defined up to semisimplification.*

Remark 2.16 It is well known that the mod p reduction $\overline{\mathrm{St}}$ of the (complex) Steinberg representation St of G_p is irreducible. So, it is independent of the choice of an integral model.

Lemma 2.17 *Let χ be a mod p character of B_p. Then,*

$$\mathrm{ind}_{S_p}^{G_p} \chi|_{S_p} \simeq \mathrm{ind}_{B_p}^{G_p} \chi \otimes \overline{\mathrm{St}}.$$

Proof See Remark 1 of [6]. $\blacksquare$

Theorem 2.18 *Let $0 \leq r < q-1$. Then, for the character χ_r of B_q, we have*

(1)

$$\mathrm{ind}_{B_q}^{G_q} \chi_r \Big|_{G_p} \simeq \begin{cases} \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus \frac{p^{f-1}-1}{p^2-1} \left(\bigoplus_{i=1}^{p-1} \mathrm{ind}_{B_p}^{G_p} \chi_{i,(r-i)} \otimes \overline{\mathrm{St}} \right), & \text{if } 2 \nmid f, \\ \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus \mathrm{ind}_{T_p}^{G_p} \omega_2^r \oplus \frac{p(p^{f-2}-1)}{p^2-1} \left(\bigoplus_{i=1}^{p-1} \mathrm{ind}_{B_p}^{G_p} \chi_{i,(r-i)} \otimes \overline{\mathrm{St}} \right), & \text{if } 2 \mid f. \end{cases}$$

(2)

$$\mathrm{ind}_{B_q}^{G_q} \chi_r \Big|_{G_p} \simeq \begin{cases} \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus \frac{p^{f-1}-1}{p^2-1} \left(\bigoplus_{i=0}^{p} \mathrm{ind}_{T_p}^{G_p} \omega_2^{r+i(p-1)} \right), & \text{if } 2 \nmid f, \\ \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus \mathrm{ind}_{T_p}^{G_p} \omega_2^r \oplus \frac{p(p^{f-2}-1)}{p^2-1} \left(\bigoplus_{i=0}^{p} \mathrm{ind}_{T_p}^{G_p} \omega_2^{r+i(p-1)} \right), & \text{if } 2 \mid f. \end{cases}$$

Remark 2.19 Of the two decompositions of the restriction to G_p of the principal series representation of G_q given in the theorem, part (1), respectively, part (2) maximizes the number of principal series representations of G_p, respectively, induced representations from an anisotropic torus of G_p. As the reader will see, these decompositions are arrived at by a bifurcation in the proof of the theorem, rather than by deducing one from the other. A similar remark applies to the two decompositions in Theorem 3.9 below.

Proof Replace χ by χ_r in the isomorphism in (2.6). Write $\chi_r|_{B_p} = \chi_r$. Using Lemma 2.14, when f is even, we get

$$\mathrm{ind}_{B_q}^{G_q} \chi_r \Big|_{G_p} \simeq \begin{cases} \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus |\mathcal{I}_1| \, \mathrm{ind}_{Z_p}^{G_p} \left(\chi|_{Z_p} \right), & \text{if } 2 \nmid f, \\ \mathrm{ind}_{B_p}^{G_p} \chi_r \oplus \mathrm{ind}_{T_p}^{G_p} \omega_2^r \oplus |\mathcal{I}_2| \, \mathrm{ind}_{Z_p}^{G_p} \left(\chi|_{Z_p} \right), & \text{if } 2 \mid f. \end{cases}$$

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

Note that $|\mathcal{I}_1| = \frac{p^{f-1}-1}{p^2-1}$ and $|\mathcal{I}_2| = \frac{p(p^{f-2}-1)}{p^2-1}$ by Proposition 2.8.

By the transitivity of induction, we have

$$\operatorname{ind}_{Z_p}^{G_p}\left(\chi_r|_{Z_p}\right) \simeq \operatorname{ind}_{S_p}^{G_p}\operatorname{ind}_{Z_p}^{S_p}\left(\chi_r|_{Z_p}\right) \simeq \operatorname{ind}_{T_p}^{G_p}\operatorname{ind}_{Z_p}^{T_p}\left(\chi_r|_{Z_p}\right).$$

We investigate each of these isomorphisms separately.

(1) Note that $\operatorname{ind}_{Z_p}^{S_p}\left(\chi_r|_{Z_p}\right)$ is a sum of $p-1$ characters as S_p is abelian of order prime to p. For a character ψ of S_p, by Frobenius reciprocity, $\psi \hookrightarrow \operatorname{ind}_{Z_p}^{S_p}\left(\chi_r|_{Z_p}\right)$ if and only if $\psi|_{Z_p} = \chi_r|_{Z_p}$. Now, for all $i \in \{1, 2, \ldots, p-1\}$, one can easily verify that $\chi_r|_{Z_p} = \chi_{i,(r-i)}\big|_{Z_p}$, and $\chi_{i,(r-i)}$ are (distinct) characters of S_p. So we get $\operatorname{ind}_{Z_p}^{S_p}\left(\chi_r|_{Z_p}\right) \simeq \bigoplus_{i=1}^{p-1}\chi_{i,(r-i)}$. So

$$\begin{aligned}\operatorname{ind}_{Z_p}^{G_p}\left(\chi_r|_{Z_p}\right) &\simeq \operatorname{ind}_{S_p}^{G_p}\left(\bigoplus_{i=1}^{p-1}\chi_{i,(r-i)}\right)\\ &\simeq \bigoplus_{i=1}^{p-1}\operatorname{ind}_{S_p}^{G_p}\chi_{i,(r-i)}\\ &\simeq \bigoplus_{i=1}^{p-1}\left(\operatorname{ind}_{B_p}^{G_p}\chi_{i,(r-i)} \otimes \overline{\mathrm{St}}\right) \quad \text{(by Lemma 2.17)}.\end{aligned}$$

(2) Similarly, $\operatorname{ind}_{Z_p}^{T_p}\left(\chi_r|_{Z_p}\right)$ is a sum of $p+1$ characters. Again, for any character ψ of T_p, we have $\psi \hookrightarrow \operatorname{ind}_{Z_p}^{T_p}\left(\chi_r|_{Z_p}\right)$ if and only if $\psi|_{Z_p} = \chi_r|_{Z_p}$. Here, one can easily verify that $\chi_r|_{Z_p} = \omega_2^{r+i(p-1)}\big|_{Z_p}$, for all $i \in \{0, 1, \ldots, p\}$, and $\omega_2^{r+i(p-1)}$ are (distinct) characters of T_p. So we get $\operatorname{ind}_{Z_p}^{T_p}\left(\chi_r|_{Z_p}\right) \simeq \bigoplus_{i=0}^{p}\omega_2^{r+i(p-1)}$. So

$$\begin{aligned}\operatorname{ind}_{Z_p}^{G_p}\left(\chi_r|_{Z_p}\right) &\simeq \operatorname{ind}_{T_p}^{G_p}\left(\bigoplus_{i=0}^{p}\omega_2^{r+i(p-1)}\right)\\ &\simeq \bigoplus_{i=0}^{p}\operatorname{ind}_{T_p}^{G_p}\omega_2^{r+i(p-1)}.\end{aligned}$$

■

3 Restriction in the cuspidal case

We know that for a non self-conjugate complex character χ of T_q, one has

$$\operatorname{ind}_{T_q}^{G_q}\chi \simeq \Theta(\chi) \otimes \mathrm{St}, \tag{3.1}$$

where $\Theta(\chi)$ is the complex cuspidal representation of G_q associated with χ. Fixing an integral structure on each of the terms on the right induces an integral structure on the left. Taking the mod p reduction of both sides of (3.1), we see that the mod p reduction $\overline{\Theta(\chi)}$ appears as a factor in the mod p reduction $\overline{\operatorname{ind}_{T_q}^{G_q}\chi}$. The former representation depends on the choice of the integral model of $\Theta(\chi)$. This is not easily written down

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

(one approach involves crystalline cohomology [11]). However, since χ takes integral values, the reduction in the latter representation can be taken with respect to the natural induced integral structure, and can be identified with $\mathrm{ind}_{T_q}^{G_q}\overline{\chi}$. It therefore seems more difficult to study the restriction of the former representation to G_p, so instead we study the restriction of the latter representation to G_p. In any case, this is consistent with our theme of studying the restrictions of induced representations.

As in the principal series case (cf. Lemma 2.1), one might expect that an obvious induction from the anisotropic torus T_p of G_p is contained in the restriction to G_p of the induction of a mod p character χ from the anisotropic torus T_q of G_q:

$$\mathrm{ind}_{T_p}^{G_p}\left(\chi|_{T_p}\right) \hookrightarrow \mathrm{ind}_{T_q}^{G_q}\chi\Big|_{G_p}.$$

By Frobenius reciprocity, this is equivalent to checking if $\chi|_{T_p} \hookrightarrow \mathrm{ind}_{T_q}^{G_q}\chi\Big|_{T_p}$. This follows whenever $T_p \subseteq T_q$ since $\chi \hookrightarrow \mathrm{ind}_{T_q}^{G_q}\chi\Big|_{T_q}$ as representations of T_q (again by Frobenius reciprocity) and restriction to T_p here makes sense. However, the containment $T_p \subseteq T_q$ is only possible if f is odd (and ε is taken to be η, see Lemma 3.1). Indeed, if f is even and $\left(\begin{smallmatrix} 0 & 1 \\ \eta^2 & 0 \end{smallmatrix}\right)$ lies in T_q, then $\varepsilon^2 = \eta^2$ so that $\varepsilon = \pm\eta$ lies in $\mathbb{F}_{p^2} \subseteq \mathbb{F}_q$, which is a contradiction. See also Remark 3.6. Nevertheless, we shall prove in the second part of Theorem 3.9 the somewhat surprising fact that irrespective of the parity of f, one always has

$$\mathrm{ind}_{T_p}^{G_p}\omega_2^r \hookrightarrow \mathrm{ind}_{T_q}^{G_q}\omega_{2f}^r\Big|_{G_p}$$

even though $\omega_{2f}^r\Big|_{T_p} = \omega_2^r$ (only) when $T_p \subseteq T_q$.

In any case, it is of interest to find all the terms in the decomposition of the restriction which we proceed to do now.

3.1 The action of G_p on $\mathbb{P}^1(\mathbb{F}_{q^2})$

The action of G_q on $\mathbb{P}^1(\mathbb{F}_{q^2})$ has two orbits $O_{G_q}(\widehat{0})$ and $O_{G_q}(\widehat{\varepsilon})$ (cf. Lemma 2.4). Also, the orbits of the G_p action on $\mathbb{P}^1(\mathbb{F}_{q^2}) = \mathbb{P}^1(\mathbb{F}_{p^{2f}})$ are given by (2.2). Hence,

$$\mathbb{P}^1(\mathbb{F}_{q^2}) \overset{G_q \text{ action}}{=} \underbrace{O_{G_q}(\widehat{0})}_{\mathbb{P}^1(\mathbb{F}_q)} \sqcup O_{G_q}(\widehat{\varepsilon}) \tag{3.2}$$

$$\| \tag{3.3}$$

$$\mathbb{P}^1(\mathbb{F}_{p^{2f}}) \overset{G_p \text{ action}}{=} \underbrace{O_{G_p}(\widehat{0}) \sqcup O_{G_p}(\widehat{\eta})}_{\mathbb{P}^1(\mathbb{F}_{p^2})} \sqcup \bigsqcup_{\widehat{x}\in\mathcal{I}_2} O_{G_p}(\widehat{x}),$$

where $\mathcal{I}_2$ forms a complete set of representatives of the G_p-orbits in $\mathbb{P}^1(\mathbb{F}_{p^{2f}}) \smallsetminus \mathbb{P}^1(\mathbb{F}_{p^2})$ and $|\mathcal{I}_2| = \frac{p\left(p^{2f-2}-1\right)}{p^2-1}$.

Lemma 3.1 *Let $q = p^f$ with f odd. Then, $x \in \mathbb{F}_{p^2} \smallsetminus \mathbb{F}_p$ with $x^2 \in \mathbb{F}_p$ implies that $x \in \mathbb{F}_{q^2} \smallsetminus \mathbb{F}_q$ with $x^2 \in \mathbb{F}_q$.*

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

Proof Let f be odd and $x \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ with $x^2 \in \mathbb{F}_p$. Then, $x \in \mathbb{F}_{q^2}$ and $x^2 \in \mathbb{F}_q$. It remains to show that $x \notin \mathbb{F}_q$. If not, then $x \in \mathbb{F}_{p^2} \cap \mathbb{F}_q = \mathbb{F}_p$, a contradiction. ∎

Notation 3.2 *In view of Lemma 3.1, we can and do assume that $\varepsilon = \eta$ when f is odd.*

Lemma 3.3 *With respect to the above notation,*

$$O_{G_p}(\widehat{\eta}) \subseteq \begin{cases} O_{G_q}(\widehat{0}), & \text{if } 2 \mid f, \\ O_{G_q}(\widehat{\varepsilon}) \text{ with } \varepsilon = \eta, & \text{if } 2 \nmid f. \end{cases}$$

Proof Suppose $2 \mid f$. Then, $\mathbb{P}^1(\mathbb{F}_{p^2}) \subseteq \mathbb{P}^1(\mathbb{F}_q)$ and so by comparing (3.2) and (3.3), $O_{G_p}(\widehat{\eta}) \subseteq O_{G_q}(\widehat{0})$. Suppose $2 \nmid f$. Then, taking $\varepsilon = \eta$, clearly $O_{G_p}(\widehat{\eta}) \subseteq O_{G_q}(\widehat{\varepsilon})$. ∎

Proposition 3.4 *Let $\varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\varepsilon^2 \in \mathbb{F}_q$. Then, the decomposition of $O_{G_q}(\widehat{\varepsilon})$ into its G_p-orbits is given by*

$$O_{G_q}(\widehat{\varepsilon}) = \begin{cases} \bigsqcup_{\widehat{x} \in \mathcal{J}} O_{G_p}(\widehat{x}), & \text{if } 2 \mid f, \\ O_{G_p}(\widehat{\varepsilon}) \sqcup \bigsqcup_{\widehat{x} \in \mathcal{J}'} O_{G_p}(\widehat{x}), & \text{if } 2 \nmid f, \end{cases}$$

where $\mathcal{J}$ and $\mathcal{J}' \sqcup \{\widehat{\varepsilon}\}$ form a complete set of representatives of G_p-orbits in $O_{G_q}(\widehat{\varepsilon}) = \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ when f is even and odd, respectively. Also, $|\mathcal{J}| = \frac{p^{f-1}(p^f-1)}{p^2-1}$ and $|\mathcal{J}'| = \frac{(p^{f-1}-1)(p^f+p-1)}{p^2-1}$.

Proof It is obvious that $O_{G_p}(\widehat{0}) \subseteq O_{G_q}(\widehat{0})$. By Equations (3.2) and (3.3), $O_{G_q}(\widehat{\varepsilon})$ is a (disjoint) union of some G_p-orbits of $\widehat{x} \in \mathcal{I}_2$ and possibly $O_{G_p}(\widehat{\eta})$. In fact, by Lemma 3.3,

$$O_{G_q}(\widehat{\varepsilon}) = \begin{cases} \bigsqcup_{\widehat{x} \in \mathcal{J}} O_{G_p}(\widehat{x}), & \text{if } 2 \mid f, \\ O_{G_p}(\widehat{\varepsilon}) \sqcup \bigsqcup_{\widehat{x} \in \mathcal{J}'} O_{G_p}(\widehat{x}), & \text{if } 2 \nmid f, \end{cases}$$

where we recall that $\eta = \varepsilon$ when f is odd and, $\mathcal{J}$ and $\mathcal{J}'$ are as described in the hypothesis. Their cardinalities can be computed as follows. Note that $\left|O_{G_q}(\widehat{\varepsilon})\right| = q^2 - q$, and when f is odd, $\left|O_{G_p}(\widehat{\varepsilon})\right| = p^2 - p$ since $\varepsilon = \eta$. Also, $\left|O_{G_p}(\widehat{x})\right| = p(p^2-1)$ for all $\widehat{x} \in \mathcal{I}_2$, as can be seen in the proof of Proposition 2.8 and hence for all $\widehat{x} \in \mathcal{J}, \mathcal{J}' \subseteq \mathcal{I}_2$. Thus,

$$|\mathcal{J}| = \frac{\left|O_{G_q}(\widehat{\varepsilon})\right|}{\left|O_{G_p}(\widehat{x})\right|} = \frac{q^2-q}{p(p^2-1)} = \frac{p^{f-1}\left(p^f-1\right)}{p^2-1} \quad \text{and}$$

$$\left|\mathcal{J}'\right| = \frac{\left|O_{G_q}(\widehat{\varepsilon}) \setminus O_{G_p}(\widehat{\varepsilon})\right|}{\left|O_{G_p}(\widehat{x})\right|} = \frac{(q^2-q)-(p^2-p)}{p(p^2-1)} = \frac{\left(p^{f-1}-1\right)\left(p^f+p-1\right)}{p^2-1}. \quad ∎$$

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

Remark 3.5 To summarize, the G_p-orbits of $\mathbb{P}^1(\mathbb{F}_{q^2})$ that appear in $O_{G_q}(\widehat{0})$ and $O_{G_q}(\widehat{\varepsilon})$, respectively, are given as follows:

$$\mathbb{P}^1(\mathbb{F}_{q^2}) \overset{\text{Equation (3.2)}}{=} O_{G_q}(\widehat{0}) \sqcup O_{G_q}(\widehat{\varepsilon}) \overset{\text{Equation (3.3)}}{=} O_{G_p}(\widehat{0}) \sqcup O_{G_p}(\widehat{\eta}) \sqcup \bigsqcup_{\widehat{x}\in I_2} O_{G_p}(\widehat{x})$$

$$\overset{\text{Proposition 3.4}}{=} \begin{cases} \underbrace{O_{G_p}(\widehat{0}) \sqcup O_{G_p}(\widehat{\eta}) \sqcup \bigsqcup_{\widehat{x}\in I_2\setminus\mathcal{J}} O_{G_p}(\widehat{x})}_{O_{G_q}(\widehat{0})} \sqcup \underbrace{\bigsqcup_{\widehat{x}\in\mathcal{J}} O_{G_p}(\widehat{x})}_{O_{G_q}(\widehat{\varepsilon})}, & \text{if } 2 \mid f, \\ \underbrace{O_{G_p}(\widehat{0}) \sqcup \bigsqcup_{\widehat{x}\in I_2\setminus\mathcal{J}'} O_{G_p}(\widehat{x})}_{O_{G_q}(\widehat{0})} \sqcup \underbrace{O_{G_p}(\widehat{\varepsilon}) \sqcup \bigsqcup_{\widehat{x}\in\mathcal{J}'} O_{G_p}(\widehat{x})}_{O_{G_q}(\widehat{\varepsilon})}, & \text{if } 2 \nmid f, \end{cases}$$

where $\widehat{\varepsilon} = \widehat{\eta}$ if $2 \nmid f$.

3.2 Decomposition of the restriction

For a character χ of T_q, by Mackey's restriction formula, we have

$$\operatorname{ind}_{T_q}^{G_q}\chi\Big|_{G_p} \simeq \bigoplus_{\gamma\in\Gamma'} \operatorname{ind}_{T_q^\gamma}^{G_p}\chi^\gamma, \tag{3.4}$$

where $T_q^\gamma = \gamma T_q \gamma^{-1} \cap G_p$ and $\chi^\gamma(t) = \chi\left(\gamma^{-1}t\gamma\right)$ for all $t \in T_q^\gamma$ and where Γ' is a set of double coset representatives in $G_p\backslash G_q/T_q$.

Remark 3.6 For $\gamma = \mathrm{Id} \in \Gamma'$, we have $T_q^\gamma = T_q \cap G_p = \begin{cases} Z_p, & \text{if } 2 \mid f, \\ T_p, & \text{if } 2 \nmid f. \end{cases}$ As a consequence, $T_p \subseteq T_q$ if and only if f is odd.

Proof Clearly, $Z_p \subseteq T_q \cap G_p$ always. Suppose f is even then we show that $T_q \cap G_p \subseteq Z_p$. Let $g = \begin{pmatrix} a & b \\ b\varepsilon^2 & a \end{pmatrix} \in T_q \cap G_p$. Then, $a, b, b\varepsilon^2 \in \mathbb{F}_p$ and $\varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Suppose, if possible, that $b \neq 0$. Then, $\varepsilon^2 \in \mathbb{F}_p$. This implies that $\varepsilon^p = \pm\varepsilon$ and consequently $\varepsilon^q = \varepsilon$ since f is even. This is a contradiction as $\varepsilon \notin \mathbb{F}_q$. So, $b = 0$ and $g \in Z_p$, as desired. In particular, if $T_p \subseteq T_q$, then $T_p \subseteq T_q \cap G_p = Z_p$, a contradiction.

Suppose f is odd. If $g = \begin{pmatrix} a & b \\ b\varepsilon^2 & a \end{pmatrix} \in T_q \cap G_p$, then again $a, b, b\varepsilon^2 \in \mathbb{F}_p$, but since f is odd, $\varepsilon = \eta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. So g is in T_p. So $T_q \cap G_p = T_p$. ∎

Notation 3.7 *For $a, b \in \mathbb{F}_q$ with $b \neq 0$, define* $g_{a,b} \coloneqq \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in G_q$.

Remark 3.8 Recall that $\operatorname{Stab}_{G_q}(\widehat{\varepsilon}) = T_q$ and $G_q/T_q \simeq O_{G_q}(\widehat{\varepsilon})$ via $T_q \mapsto \widehat{\varepsilon}$, by Lemma 2.4. Then, a complete set of coset representatives of G_q/T_q is given by $\{g_{a,b} : a, b \in \mathbb{F}_q, b \neq 0\}$. Indeed,

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

$$\begin{aligned} G_q/T_q \simeq O_{G_q}(\widehat{\varepsilon}) &= \{\widehat{x} : x \in \mathbb{F}_{q^2} \smallsetminus \mathbb{F}_q\} \\ &= \{\widehat{a+b\varepsilon} : a, b \in \mathbb{F}_q, b \neq 0\} \\ &= \{g_{a,b} \cdot \widehat{\varepsilon} : a, b \in \mathbb{F}_q, b \neq 0\} \end{aligned}$$

and since $T_q \mapsto \widehat{\varepsilon}$, we get $g_{a,b}T_q \mapsto g_{a,b} \cdot \widehat{\varepsilon}$.

Under the action of G_p on G_q/T_q, we have $\mathrm{Stab}_{G_p}(gT_q) = gT_qg^{-1} \cap G_p = T_q^g$. Hence, by taking $g = g_{a,b}$ and using Remark 3.8, we have

$$T_q^{g_{a,b}} = \mathrm{Stab}_{G_p}(\widehat{a+b\varepsilon}), \text{ for all } a, b \in \mathbb{F}_q, b \neq 0. \tag{3.5}$$

Now, the double cosets in $G_p\backslash G_q/T_q$ are in one-to-one correspondence with the orbits in $G_p\backslash\left(G_q/T_q\right)$ and hence with the orbits in $G_p\backslash O_{G_q}(\widehat{\varepsilon})$ by Remark 3.8. Also,

$$G_p g_{a,b} T_q \text{ is the union of left cosets in the } G_p\text{-orbit of } g_{a,b}T_q.$$

Thus, if Γ' is a set of double coset representatives of $G_p\backslash G_q/T_q$, then we may assume that $\Gamma' \subseteq \{g_{a,b} : a, b \in \mathbb{F}_q, b \neq 0\}$ and again by Remark 3.8,

$$\Gamma' \overset{1-1}{\longleftrightarrow} G_p\backslash(O_{G_q}(\widehat{\varepsilon})) \text{ where } g_{a,b} \mapsto O_{G_p}(\widehat{a+b\varepsilon}). \tag{3.6}$$

Thus, substituting (3.5) and (3.6) in (3.4), we get

$$\mathrm{ind}_{T_q}^{G_q}\chi\Big|_{G_p} \simeq \bigoplus_{O_{G_p}(\widehat{a+b\varepsilon}) \in G_p\backslash(O_{G_q}(\widehat{\varepsilon}))} \mathrm{ind}_{\mathrm{Stab}_{G_p}(\widehat{a+b\varepsilon})}^{G_p} \chi^{g_{a,b}}.$$

Now, Proposition 3.4 gives the description of all the G_p-orbits in $O_{G_q}(\widehat{\varepsilon})$ and therefore,

$$\mathrm{ind}_{T_q}^{G_q}\chi\Big|_{G_p} \simeq \begin{cases} \bigoplus\limits_{\widehat{a+b\varepsilon} \in \mathcal{J}} \mathrm{ind}_{\mathrm{Stab}_{G_p}(\widehat{a+b\varepsilon})}^{G_p} \chi^{g_{a,b}}, & \text{if } 2 \mid f, \\ \mathrm{ind}_{\mathrm{Stab}_{G_p}(\widehat{\varepsilon})}^{G_p} \chi^{g_{0,1}} \oplus \bigoplus\limits_{\widehat{a+b\varepsilon} \in \mathcal{J}'} \mathrm{ind}_{\mathrm{Stab}_{G_p}(\widehat{a+b\varepsilon})}^{G_p} \chi^{g_{a,b}}, & \text{if } 2 \nmid f. \end{cases}$$

Note that for $\widehat{x} \in \mathcal{J}, \mathcal{J}' \subseteq \mathcal{I}_2$, we have $x \in \mathbb{F}_{q^2} \smallsetminus \mathbb{F}_{p^2}$ by the definition of $\mathcal{I}_2$, and for such x, $\mathrm{Stab}_{G_p}(\widehat{x}) = Z_p$ by Lemma 2.7. Thus, $\mathrm{Stab}_{G_p}(\widehat{a+b\varepsilon}) = Z_p$, for all $\widehat{a+b\varepsilon} \in \mathcal{J}$ and $\mathcal{J}'$. In this case, $\chi^{g_{a,b}}|_{Z_p} = \chi|_{Z_p}$. Also, for f odd, $\varepsilon = \eta$ and so $\mathrm{Stab}_{G_p}(\widehat{\varepsilon}) = T_p$. Note that $g_{0,1} = \mathrm{Id}$ and so $\chi^{g_{0,1}} = \chi|_{T_p}$. Combining this information, we get

$$\mathrm{ind}_{T_q}^{G_q}\chi\Big|_{G_p} \simeq \begin{cases} |\mathcal{J}|\,\mathrm{ind}_{Z_p}^{G_p}\left(\chi|_{Z_p}\right), & \text{if } 2 \mid f, \\ \mathrm{ind}_{T_p}^{G_p}\left(\chi|_{T_p}\right) \oplus |\mathcal{J}'|\,\mathrm{ind}_{Z_p}^{G_p}\left(\chi|_{Z_p}\right), & \text{if } 2 \nmid f. \end{cases} \tag{3.7}$$

Theorem 3.9 *Let $0 \leq r < q^2 - 1$. Then, for the character ω_{2f}^r of T_q, we have*

(1)

$$\mathrm{ind}_{T_q}^{G_q}\omega_{2f}^r\Big|_{G_p} \simeq \begin{cases} \frac{p^{f-1}(p^f-1)}{p^2-1}\left(\bigoplus\limits_{i=1}^{p-1} \mathrm{ind}_{B_p}^{G_p}\chi_{i,(r-i)} \otimes \overline{\mathrm{St}}\right), & \text{if } 2 \mid f, \\ \mathrm{ind}_{T_p}^{G_p}\omega_2^r \oplus \frac{(p^{f-1}-1)(p^f+p-1)}{p^2-1}\left(\bigoplus\limits_{i=1}^{p-1} \mathrm{ind}_{B_p}^{G_p}\chi_{i,(r-i)} \otimes \overline{\mathrm{St}}\right), & \text{if } 2 \nmid f. \end{cases}$$

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

(2)

$$\operatorname{ind}_{T_q}^{G_q}\omega_{2f}^r\Big|_{G_p} \simeq \begin{cases} \frac{p^{f-1}(p^f-1)}{p^2-1}\left(\bigoplus_{i=0}^{p}\operatorname{ind}_{T_p}^{G_p}\omega_2^{r+i(p-1)}\right), & \text{if } 2 \mid f, \\ \operatorname{ind}_{T_p}^{G_p}\omega_2^r \ \oplus \ \frac{(p^{f-1}-1)(p^f+p-1)}{p^2-1}\left(\bigoplus_{i=0}^{p}\operatorname{ind}_{T_p}^{G_p}\omega_2^{r+i(p-1)}\right), & \text{if } 2 \nmid f. \end{cases}$$

Proof Replace χ by ω_{2f}^r in the isomorphism in (3.7). Then, $\omega_{2f}^r\Big|_{T_p} = \omega_2^r$ as characters of T_p if f is odd. Also, $|\mathcal{J}| = \frac{p^{f-1}(p^f-1)}{p^2-1}$ and $|\mathcal{J}'| = \frac{(p^{f-1}-1)(p^f+p-1)}{p^2-1}$ by Proposition 3.4. By the transitivity of induction, we have

$$\operatorname{ind}_{Z_p}^{G_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) \simeq \operatorname{ind}_{S_p}^{G_p}\operatorname{ind}_{Z_p}^{S_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) \simeq \operatorname{ind}_{T_p}^{G_p}\operatorname{ind}_{Z_p}^{T_p}\left(\omega_{2f}^r\Big|_{Z_p}\right).$$

We investigate each of these isomorphisms separately.

(1) Note that $\operatorname{ind}_{Z_p}^{S_p}\left(\omega_{2f}^r\Big|_{Z_p}\right)$ is a sum of $p-1$ characters as S_p is abelian of order prime to p. For a character ψ of S_p, by Frobenius reciprocity, $\psi \hookrightarrow \operatorname{ind}_{Z_p}^{S_p}\left(\omega_{2f}^r\Big|_{Z_p}\right)$ if and only if $\psi|_{Z_p} = \omega_{2f}^r\Big|_{Z_p}$. Now, for all $i \in \{1, 2, \ldots, p-1\}$, one can easily verify that $\omega_{2f}^r\Big|_{Z_p} = \chi_{i,(r-i)}\Big|_{Z_p}$, where $\chi_{i,(r-i)}$ are (distinct) characters of S_p. So we get $\operatorname{ind}_{Z_p}^{S_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) \simeq \bigoplus_{i=1}^{p-1}\chi_{i,(r-i)}$. So,

$$\begin{aligned} \operatorname{ind}_{Z_p}^{G_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) &\simeq \operatorname{ind}_{S_p}^{G_p}\left(\bigoplus_{i=1}^{p-1}\chi_{i,(r-i)}\right) \\ &\simeq \bigoplus_{i=1}^{p-1}\left(\operatorname{ind}_{S_p}^{G_p}\chi_{i,(r-i)}\right) \\ &\simeq \bigoplus_{i=1}^{p-1}\left(\operatorname{ind}_{B_p}^{G_p}\chi_{i,(r-i)} \otimes \overline{\mathrm{St}}\right) \quad \text{(by Lemma 2.17)}. \end{aligned}$$

(2) Similarly, $\operatorname{ind}_{Z_p}^{T_p}\left(\omega_{2f}^r\Big|_{Z_p}\right)$ is a sum of $p+1$ characters. Again, for any character ψ of T_p, we have $\psi \hookrightarrow \operatorname{ind}_{Z_p}^{T_p}\left(\omega_{2f}^r\Big|_{Z_p}\right)$ if and only if $\psi|_{Z_p} = \omega_{2f}^r\Big|_{Z_p}$. Here, one can easily verify that $\omega_{2f}^r\Big|_{Z_p} = \omega_2^{r+i(p-1)}\Big|_{Z_p}$, for all $i \in \{0, 1, \ldots, p\}$, where $\omega_2^{r+i(p-1)}$ are (distinct) characters of T_p. So we get $\operatorname{ind}_{Z_p}^{T_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) \simeq \bigoplus_{i=0}^{p}\omega_2^{r+i(p-1)}$. Thus,

$$\begin{aligned} \operatorname{ind}_{Z_p}^{G_p}\left(\omega_{2f}^r\Big|_{Z_p}\right) &\simeq \operatorname{ind}_{T_p}^{G_p}\left(\bigoplus_{i=0}^{p}\omega_2^{r+i(p-1)}\right) \\ &\simeq \bigoplus_{i=0}^{p}\operatorname{ind}_{T_p}^{G_p}\omega_2^{r+i(p-1)}. \end{aligned}$$

■

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press

Remark 3.10 Consider ω_2 both as a character of T_p taking values in a mod p field and in a characteristic zero field by taking its Teichmüller lift. Then, for $0 \le r < p^2 - 1$ with ω_2^r not self-conjugate, by using the mod p reduction of the isomorphism in (3.1) with $\chi = \omega_2^r$, we can further substitute $\overline{\Theta(\omega_2^r)} \otimes \overline{\mathrm{St}}$ for $\mathrm{ind}_{T_p}^{G_p} \omega_2^r$ in the statements of Theorems 2.18 and 3.9. We also remark that the mod p reductions of complex cuspidal representations of G_p have been studied in, for instance, [6, 11, 13].

Remark 3.11 Relative versions of the results in this article studying the restrictions of mod p representations of $\mathrm{GL}_2(\mathbb{F}_{q'})$ of the type considered in this article to $\mathrm{GL}_2(\mathbb{F}_q)$ for q' a power of q could easily be further derived. For brevity, we have decided to treat only the absolute case $q = p$.

Acknowledgements The second author thanks Prof. Eknath Ghate for suggesting the problem, for his invitation to visit TIFR, Mumbai and for the warm hospitality extended during her visit and thanks Prof. Gautam Borisagar for his valuable inputs. She is also grateful to CSIR for the Ph.D. fellowship. Both authors thank the referee for several suggestions that have improved the article.

References

[1] M. Bardoe and P. Sin, *The permutation modules for* $\mathrm{GL}(n+1, \mathbb{F}_q)$ *acting on* $\mathbb{P}^n(\mathbb{F}_q)$ *and* $\mathbb{F}_q^{n-1}$. J. London Math. Soc. (2) **61**(2000), no. 1, 58–80.
[2] S. Bhattacharjee, E. Ghate, S. Pandey, and S. Veerapaneni, *Clebsch–Gordan and the theta filtration for modular representations of* $\mathrm{GL}_2(\mathbb{F}_q)$, Springer proceedings (2026). https://arxiv.org/pdf/2511.02299.
[3] C. Breuil and V. Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2. Mem. Amer. Math. Soc. **216**(2012), no. 1016, vi+114 pp.
[4] C. Cheng, *Mod p representations of* $\mathrm{SL}_2(\mathbb{Q}_p)$. J. Number Theory **133**(2013), no. 4, 1312–1330.
[5] W. T. Gan, B. H. Gross, and D. Prasad, *Restrictions of representations of classical groups: Examples*. Astérisque **346**(2012), 111–170.
[6] E. Ghate and A. Jana, *Modular representations of* $\mathrm{GL}_2(\mathbb{F}_q)$ *using calculus*. Forum Math. **37**(2025), no. 5, 1503–1543.
[7] D. J. Glover, *A study of certain modular representations*. J. Algebra **51**(1978), no. 2, 425–475.
[8] S. Gupta, *On the restriction of some irreducible mod p representations*. Rend. Circ. Mat. Palermo (2) (2026), 9 pp.
[9] T. Kobayashi, *Recent advances in branching problems of representations*. Sugaku Expos. **37**(2024), no. 2, 129–177.
[10] V. Paškūnas, *On the restriction of representations of* $\mathrm{GL}_2(F)$ *to a Borel subgroup*. Compos. Math. **143**(2007), no. 6, 1533–1544.
[11] D. A. Reduzzi, *Reduction mod p of cuspidal representations of* $\mathrm{GL}_2(\mathbb{F}_{p^n})$ *and symmetric powers*. J. Algebra **324**(2010), no. 12, 3507–3531.
[12] M. Reeder, On the restriction of Deligne-Lusztig characters. J. Amer. Math. Soc. **20**(2007), no. 2, 573–602.
[13] M. M. Schein, *Reduction modulo p of cuspidal representations and weights in Serre's conjecture*. Bull. Lond. Math. Soc. **41**(2009), no. 1, 147–154.

Tata Institute of Fundamental Research, India
e-mail: eghate@math.tifr.res.in

Institute of Infrastructure, Technology, Research and Management, India
e-mail: shubhanshi.gupta.20pm@iitram.ac.in

https://doi.org/10.4153/S0008439526101623 Published online by Cambridge University Press