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A CELLULAR ABSOLUTE MOTIVIC RING SPECTRUM REPRESENTING HERMITIAN K-THEORY

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Abstract In the Morel-Voevodsky motivic stable homotopy category of a quasi-compact quasi-separated scheme S, several candidates exist for a motivic spectrum representing hermitian K-theory. This note shows that the cellular absolute motivic spectrum constructed in [9] via the geometry of orthogonal and hyperbolic Grassmannians over S coincides with the motivic ring spectrum constructed in [4].

1. Introduction

With his creation of higher algebraic K-theory, Daniel Quillen constructed an interesting cohomology theory of schemes. The Morel-Voevodsky $\mathbf{P}^1$-stable $\mathbf{A}^1$-homotopy theory provides a natural home for this cohomology theory over regular schemes, similar to the classical stable homotopy category hosting complex topological K-theory. More precisely, higher algebraic K-groups of any smooth S-scheme X are represented by a motivic ring spectrum KGL_S in the motivic stable homotopy category $\mathbf{SH}(S)$ for any Noetherian regular scheme S; see [26, 14, 19]. Questions around algebraic K-theory, such as the existence of the motivic spectral sequence analogous to the Atiyah-Hirzebruch spectral sequence for complex topological K-theory, were a major driving force for motivic stable homotopy theory.

The motivic spectrum KGL_S is constructed via the infinite Grassmannian, the union of the finite Grassmannians of d-dimensional linear subspaces of a $c+d$-dimensional affine

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space $\mathbf{A}^{c+d}_S$ over S. Since these are built in a suitable sense out of algebraic spheres over S, the motivic spectrum KGL_S is cellular [5]. To explain the notion 'absolute' appearing in the title, the collection $\{\mathsf{KGL}_S\}$ of motivic spectra is absolute in the sense that it pulls back along any morphism, that is, $f^*\mathsf{KGL}_S \simeq \mathsf{KGL}_R$ for any morphism $f\colon R \to S$.

Hermitian K-theory, as conceived by Karoubi, adds the complexity of suitable bilinear forms to the vector bundles underlying algebraic K-theory. From the topological perspective, it is related to algebraic K-theory in the same way as real topological K-theory is related to complex topological K-theory. Under the additional restriction that 2 be invertible in the regular affine Noetherian base scheme S, a motivic spectrum in $\mathbf{SH}(S)$ representing Karoubi's hermitian K-theory (also known as higher Grothendieck-Witt theory) was provided in [6]. However, the restriction on the invertibility of 2 prevents its absoluteness, and several authors have worked on extending the theory, in particular to the terminal scheme $\mathrm{Spec}(\mathbb{Z})$ [21], [24].

The first author, in a 2020 PhD thesis [9] supervised by the second author, obtained a cellular absolute motivic spectrum KQ'_S over any scheme via a geometric description. The essential parts of the thesis have been published in [10]. If 2 is invertible in the Noetherian regular scheme S, this geometric description coincides with the one given in [23], and in particular represents Karoubi's hermitian K-theory. However, a ring structure, obtained previously in [11], as well as several structural properties, were missing, as well as an identification of what this motivic spectrum represents over regular Noetherian bases in which 2 is not invertible.

In 2024, [4] provided an absolute motivic ring spectrum KQ_S over any quasi-compact quasi-separated scheme S, with further desirable properties, such as absolute purity and twisted Thom isomorphisms. Their work is based on further deep results on hermitian K-theory in great generality, similar to [24], which is partly in progress. A geometric description, known for regular bases where 2 is invertible at least since Girja Tripathi's thesis, is missing from this impressive list of properties. Theorem 2.3 below proves that the two motivic spectra are equivalent over any quasi-compact quasi-separated scheme. As one consequence, work by the second author with Kolderup and Østvær extends slice and coefficient computations to hermitian K-theory over Dedekind domains, including $\mathbb{Z}$ [8]. Applications to motivic stable homotopy groups of spheres will be pursued elsewhere.

To prove the equivalence we use two main results. As a first result, the collaboration of Panin and Walter [16, 17] provides a description of $\mathsf{KQ}^{*+(\star)}(\mathbf{HGr}_{\mathbb{Z}})$. Using this description allows us to construct a map

$$\phi\colon \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}$$

of motivic spaces. Its source features $\mathbf{HGr}_{\mathbb{Z}}$, the quaternionic Grassmannian defined over $\mathbb{Z}$. The target of ϕ is the infinite $\mathbf{P}^1$-loop space of the suitably shifted hermitian K-theory spectrum over $\mathbb{Z}$. We then show this is compatible with the structure maps of $\mathsf{KQ}'_{\mathbb{Z}}$ and $\mathsf{KQ}_{\mathbb{Z}}$. The resulting map $\phi\colon \mathsf{KQ}'_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$ of motivic spectra is then an equivalence. To prove this we use as a second main result, the Grothendieck-Witt groups defined in [1] agree with the classical hermitian K-groups of Karoubi in positive degrees for polynomial rings over fields. This fact is proved in a recent preprint of Schlichting [22], but it follows rather directly from the results of [1] as described below.

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Notation

Let S be a qcqs (quasi-compact and quasi-separated) base scheme. We denote by $S^{t,w} = S^{t-w} \wedge \mathbb{G}_m^w = S^{t-w+(w)}$ the motivic sphere of topological degree t and weight w. The corresponding suspension functor on the stable motivic homotopy category $\mathbf{SH}(S)$ is denoted by $\Sigma^{t,w} = \Sigma^{t-w+(w)}$. For any pointed motivic space X with associated motivic suspension spectrum $\Sigma^{\infty+(\infty)}X$ and any motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$, set

$$\mathsf{E}^{t+(w)}(X) = [\Sigma^{\infty+(\infty)}X, \Sigma^{t+(w)}\mathsf{E}]_{\mathbf{SH}(S)}$$

as the corresponding group of homomorphisms in $\mathbf{SH}(S)$. The notation '$\Sigma^{\infty+(\infty)}$' may be left out for simplicity. In the particular case of motivic spheres, one associates to any motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ the following bigraded motivic homotopy group

$$\pi_{t,w}\mathsf{E} = \pi_{t-w+(w)}\mathsf{E} = [S^{t,w}, \mathsf{E}]_{\mathbf{SH}(S)}$$

for $t,w \in \mathbb{Z}$. For example, the hermitian K-groups of S can be defined as $\mathsf{KQ}_{t,w}(S) = \pi_{t,w}\mathsf{KQ}_S$. In the literature, the notation GW_t^w is also used to denote hermitian K-groups; these are related to $\mathsf{KQ}_{t,w}$ via the indexing conventions visible in the identifications

$$\mathsf{GW}_t^w(S) \cong [S^{t,0}, \Sigma^{(w)}\mathsf{KQ}_S] = \pi_{t-2w,-w}\mathsf{KQ}_S = \pi_{t-w-(w)}\mathsf{KQ}_S.$$

For an affine scheme $X = \mathrm{Spec}(R)$, we use both $\mathrm{KSp}(X)$ and $\mathrm{KSp}(R)$ to denote the K-theory spectrum of the category of non-degenerate alternating (or, in the language of [2], genuine (-1)-even) forms on X. As a consequence, KSp denotes the corresponding spectrum valued presheaf $X \mapsto \mathrm{KSp}(X)$, with $\mathrm{KSp}_n(X)$ and $\mathrm{KSp}_n(R)$ short for the homotopy groups $\pi_n\mathrm{KSp}(X)$.

2. The two candidates

Recent work achieved a satisfying status for Karoubi's hermitian K-theory (also known as higher Grothendieck-Witt theory), which until recently was available only over base schemes in which 2 is invertible.

Theorem 2.1 [4]. *Let S be a qcqs scheme. There exists a motivic E_∞ ring spectrum $\mathsf{KQ} \in \mathbf{SH}(S)$ with the following properties.*

1. *There exists an element $\alpha\colon \Sigma^{4+(4)}\mathbf{1}_S \to \mathsf{KQ}_S$ such that multiplication with α is an equivalence $\mathsf{KQ}_S \simeq \Sigma^{4+(4)}\mathsf{KQ}_S$.*
2. *There exists a ring map $\mathsf{Forg}\colon \mathsf{KQ}_S \to \mathsf{KGL}_S$ and a KQ_S-module map $\mathsf{Hyp}\colon \mathsf{KGL}_S \to \mathsf{KQ}_S$ such that*

$$\Sigma^{(1)}\mathsf{KQ}_S \xrightarrow{\eta\mathsf{KQ}_S} \mathsf{KQ}_S \xrightarrow{\mathsf{Forg}} \mathsf{KGL}_S \xrightarrow{\Sigma^{1+(1)}\mathsf{Hyp}\circ\beta} \Sigma^{1+(1)}\mathsf{KQ}_S$$

   is a cofiber sequence, where β denotes the Bott periodicity equivalence induced by multiplication with the Bott element $\beta\colon \Sigma^{1+(1)}\mathbf{1}_S \to \mathsf{KGL}_S$.
3. *For every morphism $f\colon R \to S$ there exists a canonical equivalence $f^*\mathsf{KQ}_S \simeq \mathsf{KQ}_R$.*
4. *For every closed embedding $i\colon R \to S$ of regular schemes of codimension c and normal bundle Ni, the purity transformation $i^*\mathsf{KQ}_S \to \mathsf{Th}(Ni)i^!\mathsf{KQ}_S$ is an equivalence.*

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5. *For every vector bundle $V \to S$ of rank r, there exists a natural Thom equivalence $\mathsf{Th}(\det V)\Sigma^{r+(r)}\mathsf{KQ}_S \simeq \mathsf{Th}(V)\Sigma^{1+(1)}\mathsf{KQ}_S$, where $\det V \to S$ is the determinant line bundle of $V \to S$.*
6. *If S is regular and 2 is invertible on S, KQ_S represents Karoubi's hermitian K-groups over S.*

Proof. The first statement is [4, Remark 8.1.2]; note that $\mathsf{Forg}(\alpha) = \beta^4$. The second statement follows from [4, Corollary 8.1.7] and [4, Remark 8.1.8]. The absoluteness statement 3 is [4, Proposition 8.2.1]. Statement 4 follows from [4, Theorem 8.4.2] and statement 5 can be deduced from [4, Proposition 8.3.1]. Statement 6 follows from [3, Proposition B.2.2] and [4, Corollary 8.1.5]. □

Theorem 2.1 relies on joint work with six further colleagues, which is partly in progress, and partly available [2, 3, 1]. The long list of properties is still not sufficient to transfer the known arguments for slice computations of hermitian K-theory over fields of characteristic zero to the more general setting. Some geometry for an effectivity statement is required. To state the geometric description, let $\mathbf{HGr}_S$ denote the infinite quaternionic Grassmannian over S constructed in [16].

Theorem 2.2 [9]. *Let S be a qcqs scheme. There exists a motivic spectrum $\mathsf{KQ}'_S \in \mathbf{SH}(S)$ with the following properties.*

6′ *If S is regular and 2 is invertible on S, KQ'_S represents Karoubi's hermitian K-groups over S.*
7′ *There exists a structure map $\Sigma^{4+(4)}\mathbb{Z} \times \mathbf{HGr}_S \to \mathbb{Z} \times \mathbf{HGr}_S$ of pointed motivic spaces over S such that $\Sigma^{2+(2)}\mathsf{KQ}'_S$ is the motivic spectrum associated to the Bousfield-Friedlander type $\Sigma^{4+(4)}$-spectrum $(\mathbb{Z} \times \mathbf{HGr}_S, \mathbb{Z} \times \mathbf{HGr}_S, \dots)$ obtained from this structure map.*

In particular, KQ'_S is cellular.

The cellularity basically follows as in [20]. It is worth remarking that other work on hermitian K-theory over schemes in which 2 is not necessarily invertible existed [21, 24]. Also the ring structure has been addressed already in [11], but is not easily transferable to the geometric version KQ'.

Theorem 2.3. *Let S be a qcqs scheme. There exists an equivalence $\mathsf{KQ}'_S \to \mathsf{KQ}_S$.*

Proof. Since both motivic spectra are stable under pullback by Theorem 2.1 and Theorem 2.2, it suffices to construct an equivalence $f\colon \mathsf{KQ}'_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$. Its main ingredient, a map

$$\phi\colon \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}$$

of motivic spaces over $\mathbb{Z}$, is given in Lemma 2.4 below. Via the explicit periodicity equivalence $\Sigma^{4+(4)}\mathsf{KQ}_{\mathbb{Z}} \simeq \mathsf{KQ}_{\mathbb{Z}}$, one deduces a map

$$\phi_n\colon \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{4n+2+(4n+2)}\mathsf{KQ}_{\mathbb{Z}}$$

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of motivic spaces over $\mathbb{Z}$ for every natural number n. That these maps define a map of motivic spectra requires, for every $n \in \mathbb{N}$, the commutativity of the following diagram

$$\begin{array}{ccc} \Sigma^{4+(4)}\mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} & \xrightarrow{\Sigma^{4+(4)}\phi_n} & \Sigma^{4+(4)}\Omega^{\infty+(\infty)}\Sigma^{4n+2+(4n+2)}\mathsf{KQ}_{\mathbb{Z}} \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} & \xrightarrow{\phi_{n+1}} & \Omega^{\infty+(\infty)}\Sigma^{4n+6+(4n+6)}\mathsf{KQ}_{\mathbb{Z}} \end{array} \tag{1}$$

where the vertical arrows depict the respective structure maps of the motivic spectra. Using the adjunction $(\Sigma^{4+(4)},\Omega^{4+(4)})$, each of the two paths in diagram (1) defines a homotopy class $\mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{4n+2+(4n+2)}\mathsf{KQ}_{\mathbb{Z}}$. To conclude that these coincide, observe that the Thom isomorphism [4, Proposition 8.3.1] for $\mathsf{KQ}_{\mathbb{Z}}$ provides, for every special linear vector bundle $V \to X$ of rank r with X smooth over $\mathrm{Spec}(\mathbb{Z})$, a class

$$\mathrm{th}(V \to X) \in \mathsf{KQ}^{r+(r)}(\Sigma^{\infty+(\infty)}\mathsf{Th}(V)) \cong \mathsf{KQ}^{0+(0)}(X)$$

corresponding to the unit in the latter ring. It is straightforward to check that these classes constitute a normalized SL-orientation on $\mathsf{KQ}_{\mathbb{Z}}$ in the sense of [17, Section 5]. Hence the motivic spectrum $\mathsf{KQ}_{\mathbb{Z}}$ obtains a symplectic orientation, that is, a map $\mathsf{MSp}_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$ which is a homomorphism of commutative monoids in the homotopy category; see [17, Theorem 1.1].

The theory of Borel classes associated with the symplectic orientation on KQ_S supplies an isomorphism

$$\mathsf{KQ}_S^{s+(w)}(\mathbf{HGr}) \cong \left(\mathsf{KQ}_S^{*+(\star)}(S)[\![b_1,b_2,\ldots]\!]\right)^{s+(w)}$$

where $b_j \in \mathsf{KQ}_S^{2j+(2j)}(\mathbf{HGr}_S)$, see [16, Theorem 11.4]. This holds in particular for $S = \mathrm{Spec}(\mathbb{Z})$. Thus the two homotopy classes under consideration can be identified with elements in the group

$$\mathsf{KQ}_{\mathbb{Z}}^{4n+2+(4n+2)}(\mathbf{HGr}_{\mathbb{Z}}) \cong \left(\mathsf{KQ}^{*+(\star)}(\mathbb{Z})[\![b_1,b_2,\ldots]\!]\right)^{4n+2+(4n+2)}$$

where $b_j \in \mathsf{KQ}_{\mathbb{Z}}^{2j+(2j)}(\mathbf{HGr}_{\mathbb{Z}})$. It follows that the respective power series coefficients for both of the homotopy classes reside in the groups $\mathsf{KQ}^{2m+(2m)}(\mathbb{Z})$, which by periodicity reduce to the groups $\mathsf{KQ}^{0+(0)}(\mathbb{Z})$ and $\mathsf{KQ}^{2+(2)}(\mathbb{Z})$. Since the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces an injection (even an isomorphism) on these groups, the commutativity of diagram (1) up to homotopy follows from the respective commutativity over $\mathbb{R}$. The latter follows from the fact that $\mathsf{KQ}_{\mathbb{R}}$ and $\mathsf{KQ}'_{\mathbb{R}}$ are equivalent to the motivic spectrum representing Karoubi's hermitian K-groups of smooth $\mathbb{R}$-varieties, as listed in Theorem 2.1 and Theorem 2.2. Since 2 is invertible in $\mathbb{R}$, ϕ_n is mapped to τ_{4n+2} in Section 12 of [15] under the equivalence $S^{2+(2)} \simeq \mathbf{HP}^1$ with the quaternionic projective line. As a consequence, diagram (1) commutes up to homotopy for every n.

To provide an actual map of motivic spectra, commuting up to homotopy does not suffice. However, the argument for the commutativity up to homotopy shows that the sequence (ϕ_n) is an element in the group $\lim_{n\to\infty} \mathsf{KQ}^{4n+2+(4n+2)}(\mathbb{Z} \times \mathbf{HGr})$. This group sits

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in a short exact sequence

$$0 \to \lim_n{}^1 \mathsf{KQ}^{4n+1+(4n+2)}(\mathbb{Z}\times\mathbf{HGr}) \to [\mathsf{KQ}',\mathsf{KQ}]_{\mathbf{SH}(\mathbb{Z})} \to \lim_{n\to\infty} \mathsf{KQ}^{4n+2+(4n+2)}(\mathbb{Z}\times\mathbf{HGr}) \to 0 \tag{2}$$

by [14, Lemma 1.3.3], where the middle group represents the set of maps from KQ' to KQ in the motivic stable homotopy category $\mathbf{SH}(\mathbb{Z})$. Hence there exists a lift $f\colon \mathsf{KQ}'_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$ of the sequence (ϕ_n), that is, a map f of motivic spectra whose level $f_n\colon \mathbb{Z}\times\mathbf{HGr} \to \mathsf{KQ}_{4n+2}$ coincides with ϕ_n up to homotopy. To prove that this map is an equivalence, consider the diagram

$$\begin{array}{ccccccc}
\mathbb{Z}\times\mathbf{HGr}_{\mathbb{Z}} & \longrightarrow & \Omega^{4+(4)}\mathbb{Z}\times\mathbf{HGr}_{\mathbb{Z}} & \longrightarrow & \dots & \longrightarrow & \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}'_{\mathbb{Z}} \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \Omega^{4+(4)}f_1} & & & & \downarrow \\
\Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}} & \longrightarrow & \Omega^{4+\infty+(4+\infty)}\Sigma^{6+(6)}\mathsf{KQ}_{\mathbb{Z}} & \longrightarrow & \dots & \longrightarrow & \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}
\end{array} \tag{3}$$

whose horizontal maps are induced by the structure maps of the motivic spectra $\mathsf{KQ}'_{\mathbb{Z}}$ and $\mathsf{KQ}_{\mathbb{Z}}$, respectively, and which ends with the respective colimit on the right-hand side. Diagram (3) commutes, since so does diagram (1). The bottom horizontal maps in diagram (3) are all equivalences of motivic spaces by Theorem 2.1. Lemma 2.4 implies that the vertical map $\Omega^{4n+(4n)}(f_n)$ is an equivalence of motivic spaces for $n>0$. Hence the map induced on the colimit is an equivalence of motivic spaces. It follows that the map $f\colon \mathsf{KQ}'_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$ is an equivalence. □

In the Milnor short exact sequence (2), the $\lim^1$-term vanishes, as one can show by an argument similar to the one used in the proof of [15, Theorem 13.1]. However, this vanishing is not relevant for the existence of the constructed map $\mathsf{KQ}'_{\mathbb{Z}} \to \mathsf{KQ}_{\mathbb{Z}}$.

Lemma 2.4. *Let $\mathbf{HGr}_{\mathbb{Z}}$ be the infinite quaternionic Grassmannian over $\mathbb{Z}$. There exists a map*

$$\phi\colon \mathbb{Z}\times\mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}$$

of motivic spaces inducing an injection on π_0 and isomorphisms on π_n for every $0<n\in\mathbb{N}$.

Proof. To construct ϕ, the theory of Borel classes associated with the symplectic orientation on KQ_S, as explained in the proof of Theorem 2.3, will be used. This results in an isomorphism

$$\mathsf{KQ}_S^{s+(w)}(\mathbf{HGr}) \cong \left(\mathsf{KQ}_S^{*+(\star)}(S)[\![b_1,b_2,\dots]\!]\right)^{s+(w)}$$

where $b_j \in \mathsf{KQ}_S^{2j+(2j)}(\mathbf{HGr}_S)$, see [16, Theorem 11.4]. This holds in particular for $S=\operatorname{Spec}(\mathbb{Z})$. By [1, Section 3.2] and [4, Corollary 8.1.5] there is an inclusion of groups

$$\mathbb{Z}\oplus\mathbb{Z} \cong \mathrm{KSp}_0(\mathbb{Z})\oplus\mathbb{Z}\{b_1\} \to \mathsf{KQ}_{\mathbb{Z}}^{2+(2)}(\mathbf{HGr})$$

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where $\mathsf{KQ}^{2+(2)}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathrm{KSp}_0(\mathbb{Z}) \cong \mathbb{Z}$ is generated by the hyperbolic form of rank 2 over $\mathbb{Z}$. Let $\phi\colon \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}$ be the morphism given by the sequence of elements $(i+b_1)_{i\in\mathbb{Z}}$ in $\mathrm{KSp}_0(\mathbb{Z}) \oplus \mathbb{Z}\{b_1\}$. This provides a unique element in $[\mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}}, \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}]$, the set of maps in the unstable $\mathbf{A}^1$-homotopy category $\mathbf{Ho}(\mathbb{Z})$ of $\mathbb{Z}$.

Having produced $\phi\colon \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}} \to \Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}$, it remains to prove the assertion on homotopy sheaves. For this purpose, consider the map of homotopy presheaves $X \mapsto [\Sigma^n X_+, \phi]$ inducing the map on homotopy sheaves. The domain $[\Sigma^n X_+, \mathbb{Z} \times \mathbf{HGr}_{\mathbb{Z}}]$ of this map is naturally identified with $\pi_n \mathrm{Sing}^{\mathbf{A}^1}\Omega^\infty \mathrm{KSp}(X)$ in [10, Theorem 3.5]. The map ϕ is then induced by the canonical map

$$\phi(X)\colon \mathrm{Sing}^{\mathbf{A}^1}\Omega^\infty \mathrm{KSp}(X) \to \mathrm{Sing}^{\mathbf{A}^1}\Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}(X) \simeq \Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{Z}}(X)$$

of spectra comparing genuine (-1)-even to (homotopy) (-1)-symmetric forms, the latter being $\mathbf{A}^1$-invariant [4, Theorem 6.3.1]. Hence ϕ arises as the evaluation of a map of presheaves with values in spectra, or rather with values in their associated spaces obtained by applying Ω^∞. Let $i\colon \mathrm{Spec}(\mathbb{F}_2) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ denote the closed embedding with open complement $j\colon \mathrm{Spec}(\mathbb{Z}[2^{-1}]) \hookrightarrow \mathrm{Spec}(Z)$. The S^1-stabilization

$$j_\sharp j^* \to \mathrm{Id}_{\mathbf{Ho}(\mathbb{Z})} \to i_* i^* \to \Sigma j_\sharp j^*$$

of the unstable localization cofiber sequence [13, Theorem 3.2.21] applies to give a natural transformation of long exact sequences

$$\cdots \to [\Sigma^n X_+, j_\sharp j^* \phi] \to [\Sigma^n X_+, \phi] \to [\Sigma^n X_+, i_* i^* \phi] \to [\Sigma^{n-1} X_+, j_\sharp j^* \phi] \to \cdots.$$

By parts 3 and 5 of Theorem 2.1, for any $f\colon R \to S$ and any vector bundle $V \to S$ of rank r with trivial determinant line bundle the square

$$\begin{array}{ccc} f^*\Sigma^{r+(r)}\mathsf{KQ}_S & \xrightarrow{\simeq} & \Sigma^{r+(r)}\mathsf{KQ}_R \\ \downarrow\simeq & & \downarrow\simeq \\ f^*\mathsf{Th}(V)f^*\mathsf{KQ}_S & \xrightarrow{\simeq} & \mathsf{Th}(f^*V)\mathsf{KQ}_R \end{array}$$

commutes. This implies that the corresponding SL-orientation is stable under base change. In particular, the Borel classes are compatible with pullbacks. Our definition of ϕ is the same as Panin and Walter's identification $\mathbb{Z} \times \mathbf{HGr} \simeq \Omega^\infty \mathrm{KSp}$ when 2 is invertible; see [15, Section 11]. The fact that ϕ is compatible with pullbacks, as obtained from the preceding argument, implies that $j^*\phi$ is an equivalence. Hence so is $j_\sharp j^* \phi$. It follows that the required connectivity for ϕ holds if it holds for $i^*\phi$. Having placed ourselves in the S^1-stable $\mathbf{A}^1$-homotopy category over the perfect field $\mathbb{F}_2$, these $\mathbf{A}^1$-homotopy sheaves are strictly $\mathbf{A}^1$-invariant by a theorem of Morel [12, Corollary 6.2.9]. For these, it suffices to evaluate on finitely generated fields F over $\mathbb{F}_2$. The map

$$i^*\phi(F)\colon \mathrm{Sing}^{\mathbf{A}^1}\mathrm{KSp}(F) \to \mathrm{Sing}^{\mathbf{A}^1}\Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{F}_2}(F)$$

is induced by the map

$$i^*\phi(\Delta^d_F)\colon \mathrm{KSp}(\Delta^d_F) \to \Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{F}_2}(\Delta^d_F)$$

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where $\Delta^d_F = \mathrm{Spec} F[t_0,\dots,t_d]/(\sum t_j = 1)$. Its target is equivalent to $\Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{F}_2}(F)$ by $\mathbf{A}^1$-invariance for KQ. Since Δ^d_F has Krull dimension d, [1, Corollary 1.3.9] for $r=1$ implies that $\pi_n i^*\phi(\Delta^d_F)$ is injective for $n \geq d$ and bijective for $n > d$. However, the arguments of [1, Section 1.3] give more in this special situation. The relevant base ring R is a polynomial ring in d variables over the field F, with trivial involution and canonical duality. Every finitely generated projective R-module is free, by the solution of Serre's conjecture [18, 25]. Hence the duality on the resulting derived category of finitely generated projective R-modules interacts with the t-structure in the same way as in the case of the field F; see [1, p. 27, Remark 1.18]. Thus $\pi_n i^*\phi(\Delta^d_F)$ is injective for $n \geq 0$ and bijective for $n > 0$, for every d. The same property then holds after realization, which shows that

$$i^*\phi(F)\colon \mathrm{Sing}^{\mathbf{A}^1}\mathrm{KSp}(F) \to \mathrm{Sing}^{\mathbf{A}^1}\Omega^{(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{F}_2}(F)$$

has the desired connectivity property. This concludes the proof. □

Remark 2.5. The map $\pi_0 i^*\phi$ above fails to be surjective. For F a perfect field of characteristic 2, genuine (1)-symmetric and (-1)-symmetric forms are equivalent, and every unit is a square. Thus

$$\pi_0\Omega^{\infty+(\infty)}\Sigma^{2+(2)}\mathsf{KQ}_{\mathbb{F}_2}(F) \cong \mathsf{GW}^0_0(F) \cong \mathbb{Z}$$

and the morphism $\pi_0 i^*\phi(F)\colon \mathrm{KSp}_0(F) \to \mathsf{GW}^0_0(F)$ is the map $2\colon \mathbb{Z} \to \mathbb{Z}$; see [7, Remark 5.8].

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