THE SADDLE-POINTS OF THE CONFLUENT HYPERGEOMETRIC FUNCTION $M(a, b; x)$

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ABSTRACT. This paper demonstrates the existence of an infinity of saddle-points of the function and gives a method of evaluating them suitable for automatic computation. A table of about 100 points is given to 5 decimal places.

1. *Introduction.* The confluent hypergeometric function is defined by

$$M(a, b; x) = {}_1F_1(a; b; x) = \sum_{r=0}^{\infty} \frac{(a)_r x^r}{(b)_r r!}, \tag{1}$$

where $(a)_r = a(a+1)\dots(a+r-1)$ and $(a)_0 = 1$. Accounts of its properties and its relation to other functions may be found in Erdélyi (1), Slater (5) and Tricomi (8). Slater ((3), (4)) gives methods of evaluating the function and of computing its real zeros. She also gives the value of one saddle-point to 4 decimal places and an indication of the position of a second saddle-point.

2. *The existence of the saddle-points.* Only the properties of the function for $x > 0$ will be considered here. The corresponding properties for $x < 0$ may be obtained from Kummer's transformation

$$M(a, b; x) = e^x M(b-a, b; -x). \tag{2}$$

Kienest (2) shows by means of suitably defined Sturmian chains that, given a and b, $M(a, b; x)$ has only a finite number N of positive zeros; that is, positive values of x satisfying $M(a, b; x) = 0$. The distribution of N is shown in Fig. 1, where each region includes the line on the a-negative side. We are interested in this paper only in the sector $b < a$, $a < 0$. In this sector the zero-surface exists only in alternate squares.

Note that on the planes $b = -n$ ($n = 0, 1, 2, \dots$) the function is not defined. Each plane contains the singular lines $x = 0$ and $a = -m$ ($m = 0, 1, \dots, n$). Arbitrarily near these lines $M(a, b; x)$ can take all values including zero. Hence these lines lie asymptotically in the zero-surface.

Consider the intersection of the zero surface with the plane $x = k$. If $k = 0$, $M(a, b; x) = 1$ for all a and b, so that the zero surface can only be intersected by the plane $x = 0$ in the lines $b = -n$. When k is very large and positive, the function is positive in regions where there are an even number of zeros, shown in Fig. 1, and negative where there are an odd number. Across the lines $a = -n$ the function is continuous and changes from positive to negative or vice versa. Hence the zero surface becomes approximately the planes $a = -n$ ($n = 0, 1, 2, \dots$) for positive values of x large compared with a or b.

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Consider the squares

$$\left.\begin{aligned} -2m-1 \leqslant a < -2m \\ -2m'-2 < b < -2m'-1 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} -2m-2 \leqslant a < -2m-1, \\ -2m'-1 < b < -2m', \end{aligned}\right\} \tag{3}$$

where $m, m' = 0, 1, 2, \ldots$ and $a > b$. In each of these squares the zero surface exists everywhere, is continuous, single-valued and infinitely differentiable. Its cross-section by planes parallel to $x = 0$ changes continuously from the lines $b = -n$ when $x = 0$ to the lines $a = -n$ when x becomes infinite, and always includes the corners of each square. Hence we deduce the existence of a saddle-point in each of these squares.

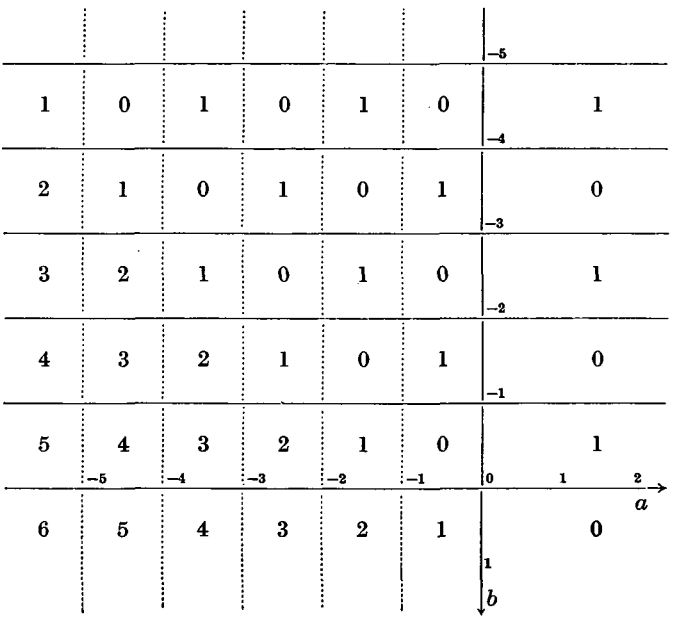


Fig. 1

3. *Nesting processes for a zero, and a saddle-point.* Given a, b and an approximation k to a zero in x, a better approximation is

$$x_1 = k - \frac{M(a, b; k)}{M'(a, b; k)}, \tag{4}$$

where $M' = \partial M/\partial x$. This is the Newtonian approximation. It can be developed into the cyclic process

$$x_{n+1} = x_n - \frac{M(a, b; x_n)}{M'(a, b; x_n)}. \tag{5}$$

Given a sufficiently good first approximation, x_n converges to the required zero with second-order convergence. As was pointed out by Slater (4), this is a very suitable process for automatic computation. Starting values for k when $b > 0$ have been given by Tricomi. Other methods of computation exist but are not discussed here.

A saddle-point of the zero surface is characterized by the three equations

$$\frac{\partial M}{\partial a} = 0, \quad \frac{\partial M}{\partial b} = 0, \quad M = 0, \tag{6}$$

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where the arguments a, b and x are omitted. Denote $\partial M/\partial a$ by A, $\partial M/\partial b$ by B, $\partial^2 M/\partial a^2$ by A' and $\partial^2 M/\partial b^2$ by B'. These are all functions of a, b and x. Infinite series expansions for these functions are given below.

A three-dimensional analogue of the nesting process for a zero seems practicable. It is given by the three cyclic equations

$$x_{n+1} = x_n - M/M', \tag{7}$$

$$a_{n+1} = a_n - A/A', \tag{8}$$

$$b_{n+1} = b_n - B/B'. \tag{9}$$

The order of iteration is to a large extent arbitrary, but the arguments on the right-hand sides of the equations will be the latest available in the course of the computation. The process requires a suitable initial approximation (a_0, b_0, x_0) for each saddle-point. It may then be expected to converge with second-order convergence.

4. *Series expansions, and approximations to the saddle-points.* The following relations are easily derived from the definition of $M(a, b; x)$:

$$M' = \frac{a}{b} M(a+1, b+1; x), \tag{10}$$

$$A = \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\sum_{n=1}^{r} \frac{1}{a+n-1} \right], \tag{11}$$

$$A' = \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\left(\sum_{n=1}^{r} \frac{1}{a+n-1} \right)^2 - \sum_{n=1}^{r} \frac{1}{(a+n-1)^2} \right], \tag{12}$$

$$B = -\sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\sum_{n=1}^{r} \frac{1}{b+n-1} \right], \tag{13}$$

$$B' = \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\left(\sum_{n=1}^{r} \frac{1}{b+n-1} \right)^2 + \sum_{n=1}^{r} \frac{1}{(b+n-1)^2} \right]. \tag{14}$$

A suitable approximation (a_0, b_0, x_0) was derived as follows. $M(a, b; x)$ satisfies the recurrence relation

$$M(a, b; x) = b^{-1}\{bM(a-1, b; x) + xM(a, b+1; x)\}. \tag{15}$$

On the plane $a = b$, $M(b, b; x) = e^x$. Using the recurrence relation, we may obtain by an inductive argument

$$M(b+r, b; x) = \frac{e^x}{(b)_r} \left[x^r + \binom{r}{1} (b+r-1) x^{r-1} + \ldots + \binom{r}{s} (b+r-s) + \ldots + (b+r-1) x^{r-s} + \ldots + (b)_r \right]. \tag{16}$$

Now $M(b+r, b; x) = 0$ if the polynomial in square brackets is zero. When

$$a - b = r = 2n+1,$$

an approximation to a root of this polynomial is

$$x = -b - \tfrac{4}{3}n = \tfrac{1}{3}(2 - 2a - b). \tag{17}$$

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But this is an approximation to the zero of $M(a, b; x)$ on the planes $a = b + 2n + 1$. The centre of each square containing a saddle-point lies on one of these planes. So, assuming as a first approximation that the saddle-points lie at the centre of each square, we obtain

$$\left.\begin{aligned} a_0 &= -2m - \tfrac{1}{2} \\ b_0 &= -(2m' + 1) - \tfrac{1}{2} \\ x_0 &= \tfrac{1}{3}(2 - 2a_0 - b_0) \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} a_0 &= -(2m + 1) - \tfrac{1}{2}, \\ b_0 &= -2m' - \tfrac{1}{2}, \\ x_0 &= \tfrac{1}{3}(2 - 2a_0 - 2b_0), \end{aligned}\right\} \tag{18}$$

where $m, m' = 0, 1, 2, \ldots$, and $a_0 > b_0$.

5. *The computations.* The computations were performed on the computer EDSAC II. Given the starting values, successive corrections to a, b and x were calculated from (7), (8) and (9) by summing the series given for A, B, M and their derivatives. The series were curtailed when the terms became smaller than 10^{-10}. The program first computed the zero x_1 for (a_0, b_0) by repeated iteration of (7), then used one iteration of (8) to give a_1. The zero x_1' of (a_1, b_0) was then computed, followed by one iteration of (9) to give b_1. The complete cycle was then repeated.

The iterations for x were terminated when corrections became smaller than $10^{-9}h$, and those for a and b when corrections became smaller than $5 \times 10^{-9}h$, where h is the maximum modulus of the partial sums of the series for $M(a, b; x)$. (Note $h \geqslant 1$.) When single-length floating-point numbers are used on EDSAC II, the accuracy with which $M(a, b; x)$ can be calculated is limited to about $10^{-10}h$. Unfortunately, near the plane $a = b$, h is of the same order as e^{-b}, which severely limits the accuracy of calculations near this plane. In order to give the results to the same accuracy throughout the table, a double-length fixed-point program was used to compute the saddle-points for which both $a < -10$ and $b < -10$.

Though the above arrangement of iterations is not the best possible one, the calculation of each point took only about 15 sec. using the single-length program, and about 2 min. using the double-length program. Normally convergence to the required accuracy was achieved in 4 or 5 cycles.

To calculate saddle-points for greater negative values of b, particularly near the plane $a = b$, it will be necessary to use either multi-length programs or asymptotic expansions. Such expansions exist but have not yet been programmed for automatic computation.

6. *The results.* Table 1 gives the saddle-points as far as $b = -20$, to 5 decimal places. By Kummer's transform (2), if $(a, b; x)$ is a saddle-point, $(b - a, b; -x)$ is also a saddle-point. Hence the saddle-points for $x < 0$ may also be obtained from the table.

It may be noted that, though the saddle-points all lie close to the plane

$$3x + 2a + b = 2$$

they are not accurately coplanar.

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Table 1. *Saddle-points of the confluent hypergeometric function*

a	b	x	a	b	x
− 0·34235	− 1·57064	1·45640	− 0·22846	−15·60947	5·51739
− 1·38312	− 2·55446	2·46906	− 2·29615	−15·59057	7·14759
			− 4·33403	−15·57670	8·62076
− 0·29986	− 3·58752	2·05230	− 6·36252	−15·56501	10·03592
− 2·40304	− 3·54593	3·47480	− 8·38662	−15·55445	11·42072
			−10·40873	−15·54430	12·78720
− 1·34341	− 4.57183	3·08794	−12·43077	−15·53385	14·14184
− 3·41539	− 4·54045	4·47822	−14·45579	−15·52168	15·48888
− 0·27670	− 5·59565	2·63947			
− 2·36660	− 5·56249	4·10470	− 1·26610	−16·60002	6·65922
− 4·42401	− 5·53655	5·48056	− 3·31269	−16·58481	8·19732
			− 5·34425	−16·57265	9·64365
− 1·31978	− 6·58135	3·69677	− 7·36950	−16·56204	11·04725
− 3·38173	− 6·55608	5·11469	− 9·39157	−16·55223	12·42635
− 5·43045	− 6·53360	6·48228	−11·41225	−16·54268	13·78986
			−13·43317	−16·53271	15·14296
− 0·26135	− 7·60053	3·22101	−15·45719	−16·52101	16·48924
− 2·34382	− 7·57224	4·72540			
− 4·39262	− 7·55132	6·12143	− 0·22344	−17·61063	6·08711
− 6·43550	− 7·53126	7·48362	− 2·28892	−17·59306	7·74310
			− 4·32544	−17·58010	9·23362
− 1·30334	− 8·58755	4·29819	− 6·35272	−17·56920	10·66171
− 3·36000	− 8·56569	5·74301	− 8·37550	−17·55944	12·05659
− 5·40097	− 8·54760	7·12631	−10·39592	−17·55027	13·43113
− 7·43960	− 8·52935	8·48470	−12·41539	−17·54122	14·79218
			−14·43534	−17·53168	16·14395
− 0·25013	− 9·60383	3·79869	−16·45847	−17·52040	17·48956
− 2·32746	− 9·57887	5·33870			
− 4·37191	− 9·56068	6·75506	− 1·26021	−18·60177	7·24229
− 6·40763	− 9·54459	8·13004	− 3·30530	−18·58753	8·80055
− 8·44301	− 9·52775	9·48559	− 5·33574	−18·57612	10·26148
			− 7·35990	−18·56621	11·67634
− 1·29092	−10·59196	4·89407	− 9·38073	−18·55715	13·06443
− 3·34404	−10·57244	6·36433	−11·39979	−18·54852	14·43524
− 5·38119	−10·55667	7·76389	−13·41822	−18·53990	15·79421
− 7·41310	−10·54208	9·13299	−15·43731	−18·53075	17·14484
− 9·44591	−10·52638	10·48634	−17·45964	−18·51983	18·48986
− 0·24140	−11·60621	4·37357	− 0·21909	−19·61160	6·65572
− 2·31483	−11·58376	5·94612	− 2·28265	−19·59515	8·33585
− 4·35645	−11·56740	7·38222	− 4·31802	−19·58295	9·84296
− 6·38868	−11·55337	8·77067	− 6·34432	−19·57271	11·28362
− 8·41771	−11·53996	10·13540	− 8·36610	−19·56358	12·68848
−10·44841	−11·52520	11·48698	−10·38534	−19·55511	14·07111
			−12·40325	−19·54693	15·43882
− 1·28105	−12·59529	5·48566	−14·42078	−19·53870	16·79602
− 3·33152	−12·57753	6·97989	−16·43911	−19·52989	18·14565
− 5·36623	−12·56331	8·39549	−18·46071	−19·51932	19·49014
− 7·39491	−12·55059	9·77606			
− 9·42165	−12·53813	11·13740	− 1·25507	−20·60324	7·82341
−11·45059	−12·52416	12·48755	− 3·29886	−20·58984	9·40078
			− 5·32834	−20·57907	10·87580
− 0·23434	−13·60803	4·94632	− 7·35162	−20·56973	12·30170
− 2·30463	−13·58754	6·54880	− 9·37152	−20·56125	13·69871
− 4·34417	−13·57257	8·00385	−11·38945	−20·55327	15·07688
− 6·37422	−13·55989	9·40577	−13·40637	−20·54550	16·44198
− 8·40019	−13·54821	10·78045	−15·42311	−20·53760	17·79763
−10·42508	−13·53653	12·13910	−17·44077	−20·52910	19·14638
−12·45252	−13·52324	13·48804	−19·46171	−20·51884	20·49039
− 1·27294	−14·59790	6·07383			
− 3·32129	−14·58155	7·59063			
− 5·35423	−14·56847	9·02185			
− 7·38090	−14·55697	10·41398			
− 9·40475	−14·54613	11·78410			
−11·42809	−14·53511	13·14056			
−13·45424	−14·52242	14·48848			

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