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Lorentz and Gale–Ryser theorems on general measure spaces

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Based on the Gale–Ryser theorem [**2**, **6**], for the existence of suitable $(0,1)$-matrices for different partitions of a natural number, we revisit the classical result of Lorentz [**4**] regarding the characterization of a plane measurable set, in terms of its cross-sections, and extend it to general measure spaces.

Keywords: Cross-sections; Nonincreasing rearrangement; Hardy-Littlewood-Pólya relation

2020 *Mathematics subject classification*: 26D15; 28A35; 46E30

1. Introduction

In [**4**], Lorentz fully characterized the existence of a plane set in terms of its cross-section. The main result reads as follows:

THEOREM 1.1. *Suppose that $P(x)$, $Q(y)$ are two non-negative integrable functions defined for $-\infty < x < +\infty$, $-\infty < y < +\infty$. In order that a measurable set A with cross functions $P(x)$, $Q(y)$ exists, it is necessary and sufficient that the non-increasing rearrangements $p(x)$, $q(y)$ of these functions satisfy the conditions:*

$$\int_0^x p(u)\,du \leqslant \int_0^x q^{-1}(u)\,du, \qquad x > 0, \tag{1.1}$$

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$$\int_0^x q(u)\,du \leqslant \int_0^x p^{-1}(u)\,du, \qquad x > 0. \tag{1.2}$$

In modern terminology, the nonincreasing rearrangement of a function f on a measurable space (X, μ) is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\},$$

where

$$\lambda_f(s) = \mu(\{x \in X : |f(y)| > s\})$$

is the distribution function of f (see [1] for standard definitions and classical properties in this setting). It is worth to mention that, according to Lorentz's notation, we have that $p^{-1}(u) = \lambda_p(u)$. It is also proved in [4] that (1.1) and (1.2) are equivalent to (1.1) and the condition $\|P\|_1 = \|Q\|_1$.

A few years later, Gale and Ryser, studying some graph theoretical conditions for degree sequences on simple graphs [7], proved in [2, 6] a discrete version, namely, they characterized the existence of a $(0,1)$-matrix A, with predetermined $r(A)$, the sums of its rows, and $c(A)$, the sums of its columns (which corresponds to fixing 2 partitions of a given $n \in \mathbb{N}$). For example, if $n = 5 = 3 + 2 = 2 + 2 + 1$, then the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.3}$$

satisfies that $r(A) = \{3, 2\}$ and for the columns we obtain $c(A) = \{2, 2, 1\}$, as desired. Similarly, it is easy to see that for the partitions $n = 5 = 4 + 1 = 2 + 2 + 1$ there is not such a matrix. The aforementioned characterization is given as follows:

THEOREM 1.2 Gale–Ryser, [2, 6]. *Let $p = \{p_1, \dots, p_j\}$ and $q = \{q_1, \dots, q_k\}$ be two nonincreasing partitions of a positive integer (i.e., $p, q \subset \mathbb{N}$, $p_1 \geqslant \cdots \geqslant p_j$, $q_1 \geqslant \cdots \geqslant q_k$ and $p_1 + \cdots + p_j = q_1 + \cdots + q_k$). Then, there exists a $(0,1)$-matrix $A \in \mathcal{M}_{j\times k}$ such that $r(A) = p$ and $c(A) = q$ if and only if for all positive integers m,*

$$\sum_{i=1}^{m} q_i \leqslant \sum_{i=1}^{m} \widehat{p_i},$$

where $\widehat{p_i} = \operatorname{card}\{1 \leqslant l \leqslant j : p_l \geqslant i\}$, if $1 \leqslant i \leqslant p_1$, $\widehat{p_i} = 0$, if $i > p_1$ and $q_i = 0$, if $i > k$.

As we can readily see, these conditions are in exact analogy with those in theorem 1.1. Influenced by this matricial case, we find a new approach (using the geometrical definition of horizontal swappable squares), allowing us to extend and unify both theorems 1.1 and 1.2 by considering products of general resonant measure spaces (see theorem 2.10):

PROBLEM 1.3. Let (X, μ) and (Y, ν) be two σ-finite measure spaces and let $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ be measurable functions. Does there exist a measurable subset

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$E \subset X \times Y$ such that f and g are the corresponding cross-sections of E:

$$f(x) = \int_Y \chi_E(x,y)\,\mathrm{d}\nu(y) \quad \text{and} \quad g(y) = \int_X \chi_E(x,y)\,\mathrm{d}\mu(x), \tag{1.4}$$

μ-a.e. $x \in X$ and ν-a.e. $y \in Y$?

It is clear that a necessary condition for (1.4) to hold is that $\|f\|_{L^1(\mathrm{d}\mu)} = \|g\|_{L^1(\mathrm{d}\nu)}$. However, as the previous matricial example shows for counting finite measures, this equality is not enough.

The following result gives the key estimate to solve problem 1.3 (which will be considered in § 2). For the rest of this section we will assume the following *resonant* condition on the measure space X, which is equivalent to saying that X is either nonatomic or completely atomic, with all atoms having equal measure [**1**, theorem II.2.7].

Proposition 1.4. *Let (X,μ) and (Y,ν) be two σ-finite measure spaces, assume X to be resonant and let $E \subset X \times Y$ be a measurable subset. Let f and g be the cross-sections of E on X and Y, respectively, as in (1.4). Then, for every $t > 0$ we have that*

$$\int_0^t f^*(s)\,\mathrm{d}s \leqslant \int_0^t \lambda_g(s)\,\mathrm{d}s. \tag{1.5}$$

Proof. We first assume that $t > 0$ is in the range of μ, and choose any set $A_t \subset X$, with $\mu(A_t) = t$. Then, setting

$$E_x = \{y \in Y : (x,y) \in E\} \quad \text{and} \quad E^y = \{x \in X : (x,y) \in E\},$$

and using [**1**, lemma II.2.1], we have that

$$\begin{aligned}
\int_{A_t} f(x)\,\mathrm{d}\mu(x) &= \int_{A_t}\int_Y \chi_E(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_{A_t} \chi_E(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\\
&= \int_Y \int_{A_t} \chi_{E^y}(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \leqslant \int_Y \int_0^t \chi_{(0,\mu(E^y))}(s)\,\mathrm{d}s\,\mathrm{d}\nu(y)\\
&= \int_Y \int_0^t \chi_{(0,g(y))}(s)\,\mathrm{d}s\,\mathrm{d}\nu(y) = \int_0^t \int_Y \chi_{(0,g(y))}(s)\,\mathrm{d}\nu(y)\,\mathrm{d}s\\
&= \int_0^t \int_Y \chi_{\{y\in Y : g(y) > s\}}\,\mathrm{d}\nu(y)\,ds = \int_0^t \lambda_g(s)\,\mathrm{d}s.
\end{aligned}$$

Finally, since X is resonant and using [**1**, proposition II.3.3]

$$\int_0^t f^*(s)\,\mathrm{d}s = \sup_{\{A_t \subset X : \mu(A_t) = t\}} \int_{A_t} f(x)\,\mathrm{d}\mu(x) \leqslant \int_0^t \lambda_g(s)\,\mathrm{d}s.$$

Once we have proved (1.5) for $t > 0$ in the range of μ, we observe that the inequality trivially holds for any $t \geqslant \mu(X)$, since then both sides of (1.5) are equal to the measure of E. Now, if $0 < t < \mu(X)$ and X is nonatomic, then we can find a measurable subset $A \subset X$ such that $\mu(A) = t$ (see [**8**]) and we are done. To finish, if

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$0 < t < \mu(X)$ and X is a discrete (totally atomic) measure space, then $\int_0^t f^*(s)\,ds$ is a piecewise linear concave function and $\int_0^t \lambda_g(s)\,ds$ is a concave function greater than the previous integral at the nodes (the μ-measure of a finite collection of atoms). Hence, by the concavity property, the inequality is also true for the intermediate values of $t > 0$. □

Remarks 1.5. A simple remark, when we work with arbitrary general measures, is that (1.4) implies that the cross-sections must take values in the image of the measure. For example, if $X = \{1, 2\}$ and $Y = \{1\}$, both with the cardinality measures, then $f : X \to \mathbb{R}^+$, $f(1) = f(2) = 1/2$ and $g : Y \to \mathbb{R}^+$, $g(1) = 1$ satisfy (1.5), but f is not the cross-section of any set $E \subset X \times Y$, since the cardinality measure only takes nonnegative integer values.

We observe that (1.5) is not a homogeneous inequality. For example, if $f \equiv g \equiv 1$ on $[0, 1]$, corresponding to the case $E = [0, 1] \times [0, 1]$, then (1.5) trivially holds, but it is false for $2f$ and $2g$.

Condition (1.5) is not, a priori, symmetric on f and g. However, we are going to prove, in proposition 1.7, that we can reverse the role of f and g in (1.5), as long as they have the same L^1-norms, which, as we already know, is a necessary condition to solve problem 1.3 (to simplify the proof, continuity of the nonincreasing rearrangements will be also assumed). We start by recalling some well-known equalities:

Lemma 1.6. *Let (X, μ) be a σ-finite measure spaces and let $f : X \to \mathbb{R}^+$ be a measurable function, with $\|f\|_{L^1(\mathrm{d}\mu)} < +\infty$. Then, for every $t > 0$,*

$$\int_t^\infty \lambda_f(s)\,\mathrm{d}s + t\lambda_f(t) = \int_0^{\lambda_f(t)} f^*(s)\,\mathrm{d}s \tag{1.6}$$

and

$$\int_t^\infty f^*(s)\,\mathrm{d}s + tf^*(t) = \int_0^{f^*(t)} \lambda_f(s)\,\mathrm{d}s. \tag{1.7}$$

Proposition 1.7. *Let (X, μ) and (Y, ν) be two σ-finite measure spaces and suppose that $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ are measurable functions such that f^* and g^* are continuous and $\|f\|_{L^1(d\mu)} = \|g\|_{L^1(d\nu)} < \infty$. If (1.5) holds, for $t > 0$, then for every $r > 0$,*

$$\int_0^r g^*(s)\,\mathrm{d}s \leqslant \int_0^r \lambda_f(s)\,\mathrm{d}s. \tag{1.8}$$

Proof. Let us assume that (1.5) holds, for $t > 0$. Since

$$\|f\|_{L^1(\mathrm{d}\mu)} = \int_0^\infty f^*(s)\,\mathrm{d}s = \int_0^\infty \lambda_g(s)\,\mathrm{d}s = \|g\|_{L^1(\mathrm{d}\nu)},$$

it is easy to see that (1.5) is equivalent to the inequality

$$\int_t^\infty \lambda_g(s)\,\mathrm{d}s \leqslant \int_t^\infty f^*(s)\,\mathrm{d}s. \tag{1.9}$$

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Let us now prove (1.8). Using the hypothesis on f and g, and lemma 1.6, condition (1.9) is equivalent to the following inequality:

$$\int_0^{\lambda_g(t)} g^*(s)\,\mathrm{d}s - t\lambda_g(t) \leqslant \int_0^{f^*(t)} \lambda_f(s)\,\mathrm{d}s - tf^*(t). \tag{1.10}$$

To prove (1.8), we observe that if $r \geqslant \|f\|_\infty$, the result is trivial since the right-hand side is equal to $\|f\|_1$. Now, if $0 < r < \|f\|_\infty = f^*(0)$, then by the continuity of f^* and the fact that $\lim_{s\to\infty} f^*(s) = 0$, there exists a $t > 0$ such that $r = f^*(t)$. Let us distinguish the following two possibilities:

If $f^*(t) \leqslant \lambda_g(t)$, then $\lambda_g(t) > 0$ and hence $0 < t < \|g\|_\infty$. Now, since g^* is nonincreasing, then (1.10) implies

$$\begin{aligned}\int_0^{f^*(t)} g^*(s)\,\mathrm{d}s - \int_0^{f^*(t)} \lambda_f(s)\,\mathrm{d}s &\leqslant t(\lambda_g(t) - f^*(t)) - \int_{f^*(t)}^{\lambda_g(t)} g^*(s)\,\mathrm{d}s \\ &= \int_{f^*(t)}^{\lambda_g(t)} (t - g^*(s))\,\mathrm{d}s \\ &\leqslant \int_{f^*(t)}^{\lambda_g(t)} (t - g^*(\lambda_g(t)))\,\mathrm{d}s \\ &= \int_{f^*(t)}^{\lambda_g(t)} (t - g^*(\lambda_{g^*}(t)))\,\mathrm{d}s = 0,\end{aligned}$$

which implies (1.8). The last equality follows because by the hypotheses on g^*, we have that for every $0 < s < t$, $|\{y > 0 : s < g^*(y) \leqslant t\}| > 0$, which is equivalent to the equality $g^*(\lambda_{g^*}(t)) = t$. In fact, $g^*(\lambda_{g^*}(t)) \leqslant t$ is always true [**1**, proposition II.1.7] and $g^*(\lambda_{g^*}(t)) \geqslant t$ means that

$$t \leqslant \inf\{s > 0 : \lambda_{g^*}(s) \leqslant \lambda_{g^*}(t)\};$$

that is, if $s < t$, then $\lambda_{g^*}(s) > \lambda_{g^*}(t)$, which is the hypothesis.

Similarly, if $f^*(t) > \lambda_g(t)$, (1.10) and the monotonicity of g^* imply

$$\begin{aligned}\int_0^{f^*(t)} g^*(s)\,\mathrm{d}s - \int_0^{f^*(t)} \lambda_f(s)\,\mathrm{d}s &\leqslant t(\lambda_g(t) - f^*(t)) + \int_{\lambda_g(t)}^{f^*(t)} g^*(s)\,\mathrm{d}s \\ &\leqslant (f^*(t) - \lambda_g(t))(g^*(\lambda_g(t)) - t) \leqslant 0,\end{aligned}$$

as desired. □

This last result will also be clear once we prove, in theorem 2.10, that (1.5) is equivalent to the existence of a set $E \subset X \times Y$ with f and g as its cross-sections. Changing E by its transpose set $\widetilde{E} = \{(y, x) : Y \times X, \text{ such that } (x, y) \in E\}$ and applying proposition 1.4 to $\widetilde{E}$, we finally obtain (1.8).

To finish the section, we are going to work the details, for a couple of concrete and elementary cases, showing that the existence (or, rather, the construction) of the set E is not in general straightforward. Moreover, from these examples we will see that the solution is not, in general, uniquely determined.

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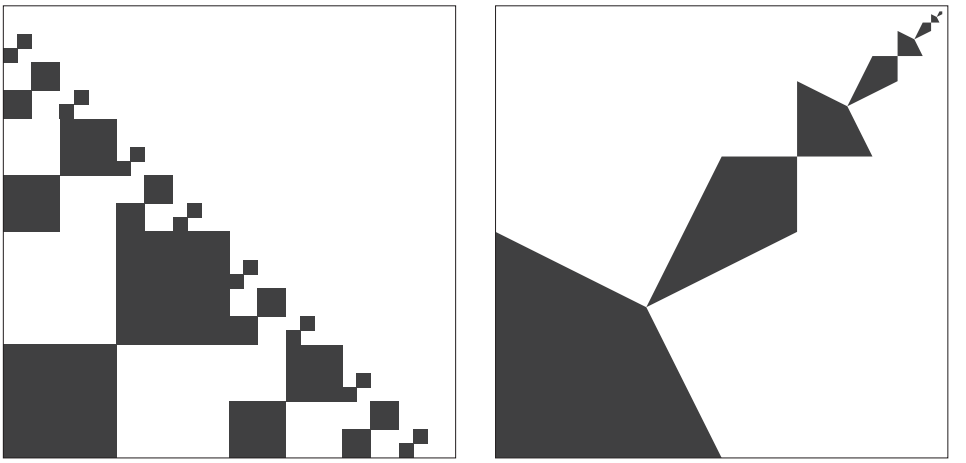


Figure 1. Two different approximations of a set $E \subset [0,1] \times [0,1]$, with cross-sections equal to $f(x) = g(x) = (1-x)/2$.

Examples 1.8.

(i) Let $X = Y = [0,1]$, with the Lebesgue measure, and consider $f(x) = g(x) = (1-x)/2$. Then, $f^*(t) = f(t)\chi_{[0,1]}(t)$ and $\lambda_g(t) = (1-2t)\chi_{[0,1/2]}(t)$. Thus,

$$\int_0^t f(s)\,\mathrm{d}s = \left(\frac{t}{2} - \frac{t^2}{4}\right)\chi_{[0,1]}(t) + \frac{1}{4}\chi_{(1,\infty)}(t)$$

and

$$\int_0^t \lambda_g(s)\,\mathrm{d}s = (t-t^2)\chi_{[0,1/2]}(t) + \frac{1}{4}\chi_{(1/2,\infty)}(t),$$

and (1.5) holds. For this case, it is easy to check that any of the sets on figure 1 give a positive solution to problem 1.3.

(ii) For $0 < a < 1$, let $f(x) = g(x) = a\chi_{[0,1]}(x)$. Then (1.5) holds and two possible sets for which (1.4) is satisfied are shown in figure 2.

2. Existence of the set E with a priori cross-sections

Our main result in this section is theorem 2.10, where we show that the necessary condition (1.5) of proposition 1.4 is actually sufficient to find a set E, in the product space, with given cross-sections. The main geometric tool used for such construction is the notion of swappable squares in definition 2.5, which allows us to horizontally translate the mass of the hypograph of the function g in such a way that in the limit, after suitable iterations for different grids of dyadic squares, we get precisely a set E with the vertical cross-section equal to f. In the discrete case, this idea lies behind the proof of the Gale–Ryser theorem 1.2 given in [**3**].

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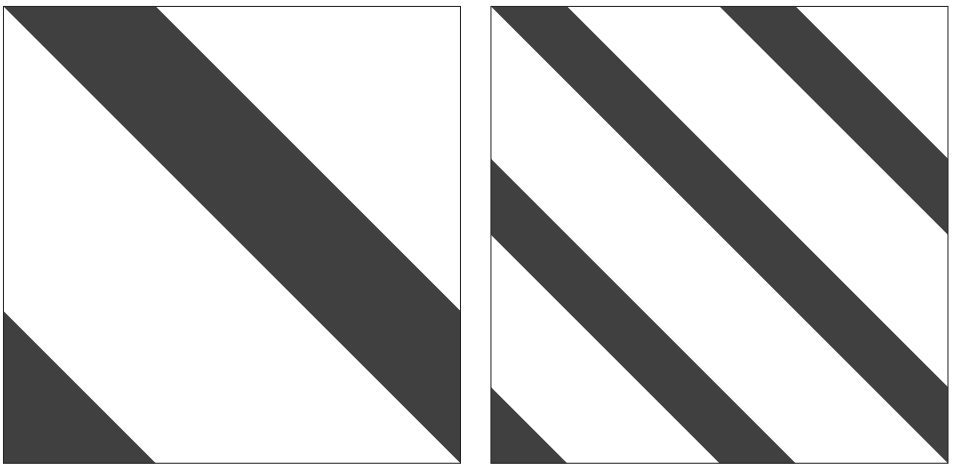


Figure 2. Two different sets $E \subset [0,1] \times [0,1]$, with cross-sections equal to $f(x) = g(x) = a\chi_{[0,1]}(x)$. In this example, $a = 1/3$.

To this end, we will start by proving some interesting properties of this swapping argument, as well as some measure theoretical estimates of the (lower) limit set obtained.

DEFINITION 2.1 Dyadic squares. For $n \in \mathbb{N}$ and $i, j \in \{1, \dots, 2^n\}$ define

$$X_j^n := [2^{-n}(j-1), 2^{-n}j), \qquad Y_i^n := [2^{-n}(i-1), 2^{-n}i)$$

and

$$Q_{ij}^n := X_j^n \times Y_i^n.$$

We call Q_{ij}^n the *dyadic squares of n-th generation* with indices i, j.

DEFINITION 2.2 Shifted set. Let $Q \subset \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$. Then we define

$$Q + (x_0, y_0) := \{(x, y) \in \mathbb{R}^2 : (x - x_0, y - y_0) \in Q\}.$$

DEFINITION 2.3 Cross-sections of a set. Let $A \subset [0,1]^2$ be a measurable set. Define the *vertical cross-section* and the *horizontal cross-section*, respectively, of the set A as

$$v_A(x) := \int_0^1 \chi_A(x, z)\mathrm{d}z, \qquad x \in [0,1],$$

and

$$h_A(y) := \int_0^1 \chi_A(z, y)\mathrm{d}z, \qquad y \in [0,1].$$

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Definition 2.4 Horizontal swapping. Let $A \subset [0,1]^2$ be a set. Let $n \in \mathbb{N}$ and $i,j,k \in \{1,\dots,2^n\}$, $j \neq k$. Then we define the set

$$\sigma^n_{ijk}(A) := (A \backslash (Q^n_{ij} \cup Q^n_{ik})) \cup ((Q^n_{ij} \cap A) + (2^{-n}(k-j),0)) \cup ((Q^n_{ik} \cap A) - (2^{-n}(k-j),0)).$$

In other words, $\sigma^n_{ijk}(A)$ is the set A, whose subsets $Q^n_{ij} \cap A$ and $Q^n_{ik} \cap A$ have "changed their places".

Definition 2.5 Swappable squares. Let $A \subset [0,1]^2$ be a measurable set and $f : [0,1] \to [0,1]$ be a measurable function such that

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t v^*_A(s)\mathrm{d}s, \quad \text{for all } t \in [0,1].$$

Let $n \in \mathbb{N}$ and $i,j,k \in \{1,\dots,2^n\}$, $j \neq k$. Then, given f and $n \in \mathbb{N}$ as above, we say that the dyadic squares Q^n_{ij} and Q^n_{ik} are *swappable* (with respect to f, A and n) if, with

$$A' := \sigma^n_{ijk}(A),$$

the following conditions are satisfied:

$$v_A \geqslant f + 2^{-n} \quad \text{a.e. in } X^n_j, \tag{2.1}$$

$$v_A \leqslant f - 2^{-n} \quad \text{a.e. in } X^n_k, \tag{2.2}$$

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t v^*_{A'}(s)\mathrm{d}s \quad \text{for all } t \in [0,1], \tag{2.3}$$

$$((Q^n_{ik} \cap A) - (2^{-n}(k-j),0)) \subsetneq Q^n_{ij} \cap A \quad \text{and} \quad |A \,\Delta\, A'| > 0. \tag{2.4}$$

Lemma 2.6. *Let the function f, the indices n,i,j,k, the squares Q^n_{ij} and Q^n_{ik}, and the sets A and A' be as in definition 2.5. Then,*

(*i*) $h_A(y) = h_{A'}(y)$, *for every* $y \in [0,1]$, *and hence* $|A| = |A'|$.

(*ii*)

$$|A \setminus A'| = \int_{X^n_j} (v_A(x) - v_{A'}(x))\mathrm{d}x \quad \textit{and} \quad |A' \setminus A| = \int_{X^n_k} (v_{A'}(x) - v_A(x))\mathrm{d}x.$$

(*iii*)

$$\int_0^1 |f(x) - v_{A'}(x)|\mathrm{d}x = \int_0^1 |f(x) - v_A(x)|\mathrm{d}x - |A \,\Delta\, A'|.$$

(*iv*) *If* $l \in \{1,\dots,2^n\}$, *then:*

$$f \geqslant v_{A'} \geqslant v_A \text{ in } X^n_l \quad \text{if } f \geqslant v_A \text{ in } X^n_l,$$
$$f \leqslant v_{A'} \leqslant v_A \text{ in } X^n_l \quad \text{if } f \leqslant v_A \text{ in } X^n_l,$$
$$v_{A'} = v_A \text{ in } X^n_l \quad \text{else.}$$

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Proof. (i) The result is clear if $y \in [0,1] \setminus Y_i^n$. Now, if $y \in Y_i^n$, then

$$\begin{aligned} h_{A'}(y) &= \int_0^1 \chi_{A'}(x,y)\mathrm{d}x = \int_0^1 \chi_{(A\setminus(Q_{ij}^n \cup Q_{ik}^n))}(x,y)\mathrm{d}x \\ &\quad + \int_0^1 \chi_{((Q_{ij}^n \cap A)+(2^{-n}(k-j),0))}(x,y)\mathrm{d}x \\ &\quad + \int_0^1 \chi_{((Q_{ik}^n \cap A)-(2^{-n}(k-j),0))}(x,y)\mathrm{d}x \\ &= \int_0^1 \chi_{(A\setminus(Q_{ij}^n \cup Q_{ik}^n))}(x,y)\mathrm{d}x + \int_0^1 \chi_{(Q_{ij}^n \cap A))}(x,y)\mathrm{d}x + \int_0^1 \chi_{(Q_{ik}^n \cap A)}(x,y)\mathrm{d}x \\ &= \int_0^1 \chi_A(x,y)\mathrm{d}x = h_A(y). \end{aligned}$$

(ii) We prove the first equality (the second one is completely analogous). Now, using (2.4), we have that:

$$\begin{aligned} |A \setminus A'| &= |(A \setminus A') \cap Q_{ij}^n| = \int_{X_j^n}\int_{Y_i^n} \chi_{A\setminus A'}(x,y)\mathrm{d}y\mathrm{d}x \\ &= \int_{X_j^n}\int_{Y_i^n} (\chi_A(x,y) - \chi_{A'}(x,y))\mathrm{d}y\mathrm{d}x = \int_{X_j^n} (v_{A\cap Q_{ij}^n}(x) - v_{A'\cap Q_{ij}^n}(x))\mathrm{d}x \\ &= \int_{X_j^n} (v_A(x) - v_{A'}(x))\mathrm{d}x. \end{aligned}$$

Let us now prove (iii). Taking into account part (ii),

$$\begin{aligned} \int_0^1 |f(x) - v_{A'}(x)|\mathrm{d}x &= \int_{[0,1]\setminus(X_j^n \cup X_k^n)} |f(x) - v_{A'}(x)|\mathrm{d}x \\ &\quad + \int_{X_j^n} |f(x) - v_{A'}(x)|\mathrm{d}x + \int_{X_k^n} |f(x) - v_{A'}(x)|\mathrm{d}x \\ &= \int_{[0,1]\setminus(X_j^n \cup X_k^n)} |f(x) - v_A(x)|\mathrm{d}x + \int_{X_j^n} (v_{A'}(x) - f(x))\mathrm{d}x \\ &\quad + \int_{X_k^n} (f(x) - v_{A'}(x))\mathrm{d}x \\ &= \int_{[0,1]\setminus(X_j^n \cup X_k^n)} |f(x) - v_A(x)|\mathrm{d}x + \int_{X_j^n} (v_A(x) - f(x))\mathrm{d}x \\ &\quad + \int_{X_k^n} (f(x) - v_A(x))\mathrm{d}x \\ &\quad + \int_{X_j^n} (v_{A'}(x) - v_A(x))\mathrm{d}x + \int_{X_k^n} (v_A(x) - v_{A'}(x))\mathrm{d}x \end{aligned}$$

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$$= \int_{[0,1]\setminus(X_j^n \cup X_k^n)} |f(x) - v_A(x)|\mathrm{d}x + \int_{X_j^n} |v_A(x) - f(x)|\mathrm{d}x + \int_{X_k^n} |f(x) - v_A(x)|\mathrm{d}x - |A \setminus A'| - |A' \setminus A|$$

$$= \int_0^1 |f(x) - v_A(x)|\mathrm{d}x - |A \,\Delta\, A'|.$$

(iv) Let us observe that, if we construct $A' = \sigma_{ijk}^n(A)$ from A, we change the content of the square Q_{ij}^n, which is the only change in the j-th column. By (2.4), we obtain that $Q_{ij}^n \cap A' \subseteq Q_{ij}^n \cap A$. Hence we get $v_A \geqslant v_{A'} \geqslant v_A - 2^{-n}$ in X_j^n, where the last inequality holds since 2^{-n} is the height of the square. Then, condition (2.1) gives

$$v_A \geqslant v_{A'} \geqslant v_A - 2^{-n} \geqslant f \text{ in } X_j^n.$$

Similarly, by looking at the k-th column, since $Q_{ik}^n \cap A \subseteq Q_{ik}^n \cap A'$, condition (2.2) gives

$$v_A \leqslant v_{A'} \leqslant v_A + 2^{-n} \leqslant f \text{ in } X_k^n.$$

In any other case, since if neither (2.1) nor (2.2) holds with l in the role of j or k, the l-th column remains unchanged, and thus $v_A = v_{A'}$ in X_l^n. □

REMARK 2.7. Given A, f, and n, let Ω_n be the collection of all sets B obtained from A by a finite number of swappings. It is obvious that Ω_n is finite, since, because of (2.4), no pair of squares can be swapped twice. Then $A_{\mathrm{opt},n}$ is defined as such an element of Ω_n that

$$\|f - v_{A_{\mathrm{opt},n}}\|_1 = \min_{B\in\Omega_n} \|f - v_B\|_1.$$

We need to recall the following classical result:

LEMMA 2.8. *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^d$ and let*

$$E := \bigcup_{m\in\mathbb{N}} \bigcap_{k=m}^{\infty} E_k \tag{2.5}$$

be its lower limit. If $|E| < \infty$ and

$$\sum_{n\in\mathbb{N}} |E_n \setminus E_{n+1}| < \infty, \tag{2.6}$$

then $E_n \to E$ in measure; i.e., $|E_n \,\Delta\, E| \to 0$ as $n \to \infty$.

Similarly, the proof of the following result is straightforward and follows from the standard properties of the nonincreasing rearrangement of a function:

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LEMMA 2.9. *Let $u : [0,1] \to [0,\infty)$ be a measurable function. Assume that there exist $0 < p \leqslant q < 1$ and constants $C_1 \geqslant C_2 > 0$ such that $u(x) \geqslant C_1$, for a.e. $x \in [0,p)$, $C_2 \leqslant u(x) \leqslant C_1$, for a.e. $x \in [p,q)$ and $u(x) \leqslant C_2$, for a.e. $x \in [q,1]$. Then*

$$u^*(t) = (u\chi_{[0,p)})^*(t) \text{ for all } t \in [0,p)$$
$$u^*(t) = (u\chi_{[p,q)})^*(t-p) \text{ for all } t \in [p,q)$$

and

$$u^*(t) = (u\chi_{[q,1]})^*(t-q) \text{ for all } t \in [q,1].$$

Furthermore, as a particular consequence, one has

$$\int_0^p u^*(s)\mathrm{d}s = \int_0^p u(z)\mathrm{d}z, \quad \int_p^q u^*(s)\mathrm{d}s = \int_p^q u(z)\mathrm{d}z, \tag{2.7}$$
$$\text{and} \quad \int_q^1 u^*(s)\mathrm{d}s = \int_q^1 u(z)\mathrm{d}z.$$

We will now prove our main result. To do this, we first address the case of cross-sections on $[0,1]$ (moreover, under some monotone conditions) and the general setting will then follow using Ryff's Theorem and some measure preserving transformations (see [**5**] and [**1**, theorem II.7.5 and corollary II.7.6]).

THEOREM 2.10. *Let (X,μ) and (Y,ν) be two finite resonant measure spaces and let $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ be measurable functions satisfying $\|f\|_{L^1(\mu)} = \|g\|_{L^1(\nu)}$ and*

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t \lambda_g(s)\mathrm{d}s,$$

for all $t > 0$. Then, there exists a measurable set $E \subset X \times Y$ such that $v_E(x) = f(x)$, μ-a.e. $x \in X$ and $h_E(y) = g(y)$, ν-a.e. $y \in Y$.

Proof. We start by considering the case $X = Y = [0,1]$, endowed with the Lebesgue measure, and assume also that f is a nonincreasing function. Now, define E_0 by

$$E_0 := \big\{(x,y) \in [0,1]^2 : x < g(y)\big\}$$

and $E_n = (E_{n-1})_{\mathrm{opt},n}$, for each $n \in \mathbb{N}$, as in remark 2.7.

Observe that since $v_{E_0} = \lambda_g$ in $[0,1]$, hence v_{E_0} is nonincreasing and right-continuous, which in turn implies $v_{E_0} = v_{E_0}^*$ in $[0,1]$. Therefore we get

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t \lambda_g(s)\mathrm{d}s = \int_0^t v_{E_0}^*(s)\mathrm{d}s,$$

for all $t \in [0,1]$. Let $n \in \mathbb{N}$. Then, by the definition of E_n,

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t v_{E_n}^*(s)\mathrm{d}s. \tag{2.8}$$

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Using lemma 2.6 (i), it immediately follows that

$$|E_n| = |E_{n-1}|, \ h_{E_n} = g \quad \text{a.e. in} \quad [0,1], \tag{2.9}$$

and

$$|E_n| = \|v_{E_n}\|_1 = \|h_{E_n}\|_1 = \|g\|_1 = \|f\|_1. \tag{2.10}$$

Furthermore, to fix the notation, suppose now that the set E_n is constructed from E_{n-1} as follows: there exists an $m \in \mathbb{N}$ and sets $A_0, \dots, A_m$ such that $A_0 = E_{n-1}$, $A_m = E_n$ and $A_l = (A_{l-1})'$ for all $l \in \{1, \dots, m\}$. Then, using lemma 2.6 (iii)

$$\begin{aligned} |E_{n-1} \,\Delta\, E_n| &\leqslant \sum_{l=1}^{m} |A_{l-1} \,\Delta\, A_l| = \sum_{l=1}^{m} (\|f - v_{A_{l-1}}\|_1 - \|f - v_{A_l}\|_1) \\ &= \|f - v_{E_{n-1}}\|_1 - \|f - v_{E_n}\|_1. \end{aligned}$$

Since the inequality $\|f - v_{A_{l-1}}\|_1 - \|f - v_{A_l}\|_1 > 0$ holds for all $l \in \{1, \dots, m\}$ (see (2.4) and lemma 2.6), we have also $\|f - v_{E_{n-1}}\|_1 - \|f - v_{E_n}\|_1 > 0$. Thus, we get

$$\begin{aligned} \sum_{n\in\mathbb{N}} |E_{n-1} \,\Delta\, E_n| &= \lim_{N\to\infty} \sum_{n=1}^{N} |E_{n-1} \,\Delta\, E_n| \\ &\leqslant \lim_{N\to\infty} \sum_{n=1}^{N} \left(\|f - v_{E_{n-1}}\|_1 - \|f - v_{E_n}\|_1\right) \\ &= \lim_{N\to\infty} \left(\|f - v_{E_0}\|_1 - \|f - v_{E_N}\|_1\right) \\ &\leqslant \|f - v_{E_0}\|_1 \leqslant 1. \end{aligned}$$

We finally define E by (2.5). Since $E \subset [0,1]^2$ obviously holds, the assumptions of lemma 2.8 are satisfied and it follows that E_n converges to E in measure. Observe that (2.9) and (2.10) give us that $h_E = g$, a.e. and $\|v_E\|_1 = \|f\|_1$.

It remains to prove $\|f - v_E\|_1 = 0$, which will be done in the rest of the proof. At first we need some auxiliary observations. We see that

$$\|v_{E_n} - v_E\|_1 \leqslant \int_0^1 \int_0^1 |\chi_{E_n}(x,y) - \chi_E(x,y)| \mathrm{d}x\mathrm{d}y = |E_n \Delta E| \xrightarrow{n\to\infty} 0. \tag{2.11}$$

For every $x \in [0,1]$ let $\{j_n(x)\}_{n\in\mathbb{N}}$ be the unique sequence of natural numbers such that $x \in X^n_{j_n}$ for each $n \in \mathbb{N}$. Now let $n \in \mathbb{N}$ be fixed. By the construction of E_n, for every $i \in \{1, \dots, 2^n\}$ there exists a unique permutation $\pi_{ni} : \{1, \dots, 2^n\} \to \{1, \dots, 2^n\}$ such that $E_n \cap Q^n_{ij} = E_0 \cap Q^n_{i\pi_{ni}(j)}$ for all $j \in \{1, \dots, 2^n\}$. Hence, for

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$x \in X_j^n$ we have

$$v_{E_n}(x) = \sum_{i=1}^{2^n} v_{E_n \cap Q_{ij}^n}(x) = \sum_{i=1}^{2^n} v_{E_0 \cap Q_{i\pi_{ni}(j)}^n}(x).$$

By the definition of E_0, one has

$$v_{E_0 \cap Q_{i\pi_{ni}(j)}^n}(x) = v_{\{(u,z) \in Q_{i\pi_{ni}(j)}^n : u < g(z)\}}(x) = \int_{Y_i^n} \chi_{\{(u,z) \in Q_{i\pi_{ni}(j)}^n : u < g(z)\}}(x,y)\mathrm{d}y,$$

for every pair $i, j \in \{1, \dots, 2^n\}$. Since the above integrand is nonincreasing on the variable x, then $x \mapsto v_{E_n \cap Q_{ij_n(x)}^n}(x) = v_{E_0 \cap Q_{i\pi_{ni}(j_n(x))}^n}(x)$ is a nonincreasing function on X_j^n, for every i and j, and thus v_{E_n} is also nonincreasing on every X_j^n, $j \in \{1, \dots, 2^n\}$.

It is worth mentioning that, for any $i, j, k \in \{1, \dots, 2^n\}$, the following implication is valid:

$$\begin{aligned} &\lim_{z \to 2^{-n} j-} v_{E_n \cap Q_{ij}^n}(z) > v_{E_n \cap Q_{ik}^n}(2^{-n}(k-1)) \\ &\quad \Longrightarrow (E_n \cap Q_{ik}^n) - (2^{-n}(k-j), 0) \subsetneq E_n \cap Q_{ij}^n. \end{aligned} \tag{2.12}$$

This follows from the representation $E_n \cap Q_{ij}^n = E_0 \cap Q_{i\pi_{ni}(j)}^n$, $E_n \cap Q_{ik}^n = E_0 \cap Q_{i\pi_{ni}(k)}^n$, the definition of E_0, and lemma 2.6 (iii).

Next, we have

$$\begin{aligned} &\sum_{j=1}^{2^n} \left(v_{E_n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_n}(z) \right) \\ &= \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \left(v_{E_n \cap Q_{ij}^n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_n \cap Q_{ij}^n}(z) \right) \\ &= \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \left(v_{E_0 \cap Q_{i\pi_{ni}(j)}^n}(2^{-n}(\pi_{ni}(j)-1)) - \lim_{z \to 2^{-n} \pi_{ni}(j)-} v_{E_0 \cap Q_{i\pi_{ni}(j)}^n}(z) \right) \\ &= \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \left(v_{E_0 \cap Q_{ij}^n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_0 \cap Q_{ij}^n}(z) \right), \end{aligned}$$

where the last equality holds by switching the order of the two sums, rearranging all indices obtained when applying the permutation π_{ni} to the set $j \in \{1, \dots, 2^n\}$, and switching the indices i and j back again. Continuing from here we obtain:

$$\begin{aligned} &\sum_{j=1}^{2^n} \left(v_{E_n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_n}(z) \right) \\ &\quad = \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \left(v_{E_0 \cap Q_{ij}^n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_0 \cap Q_{ij}^n}(z) \right) \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{j=1}^{2^n} \left(v_{E_0}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_0}(z) \right) \\
&= \sum_{j=1}^{2^n} \left(\lambda_g(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} \lambda_g(z) \right) \\
&\leqslant \sum_{j=1}^{2^n} \left(\lambda_g(2^{-n}(j-1)) - \lambda_g(2^{-n} j) \right) \\
&\leqslant \lambda_g(0).
\end{aligned}
$$

For every $n \in \mathbb{N}$ and $x \in [0,1]$ define

$$
\varphi_n(x) := v_{E_n}(2^{-n}(j_n(x)-1)) - \lim_{z \to 2^{-n} j_n(x)-} v_{E_n}(z).
$$

As observed above, v_{E_n} is nonincreasing on each X_j^n, therefore φ_n is nonnegative on $[0,1]$ and we have

$$
\|\varphi_n\|_1 = \sum_{j=1}^{2^n} 2^{-n} \left(v_{E_n}(2^{-n}(j-1)) - \lim_{z \to 2^{-n} j-} v_{E_n}(z) \right) \leqslant 2^{-n} \lambda_g(0) \xrightarrow{n \to \infty} 0.
$$

Hence, there exists a subsequence $\{\varphi_{n_m}\}_{m \in \mathbb{N}}$ converging a.e. in $[0,1]$. In particular, this implies that

$$
\liminf_{n \to \infty} \left(v_{E_n}(2^{-n}(j_n(x)-1)) - \lim_{z \to 2^{-n} j_n(x)-} v_{E_n}(z) \right) = 0
$$

for a.e. $x \in [0,1]$. By the monotonicity of v_{E_n} on each X_j^n, we therefore get

$$
\liminf_{n \to \infty} \sup_{z \in X_{j(x)}^n} |v_{E_n}(x) - v_{E_n}(z)| = 0 \tag{2.13}
$$

for a.e. $x \in [0,1]$. Consequently, using lemma 2.6 (iv), for every $x \in [0,1]$ the sequence $\{v_{E_n}(x)\}_{n \in \mathbb{N}}$ is monotone.

Next, for any $n \in \mathbb{N}$ and $t \in [0,1]$ we have

$$
\begin{aligned}
\int_0^t |v_{E_n}^*(s) - v_E^*(s)| \mathrm{d}s &\leqslant \|v_{E_n}^* - v_E^*\|_1 = \|(v_{E_n}^* - v_E^*)^*\|_1 \\
&\leqslant \|(v_{E_n} - v_E)^*\|_1 = \|v_{E_n} - v_E\|_1,
\end{aligned} \tag{2.14}
$$

where the second inequality follows from the Lorentz–Shimogaki theorem [**1**, theorem III.7.4, p. 169]. Hence, taking the limit as $n \to \infty$ in (2.8), using (2.11) and (2.14) give

$$
\int_0^t f^*(s) \mathrm{d}s \leqslant \int_0^t v_E^*(s) \mathrm{d}s, \tag{2.15}
$$

for any fixed $t \in [0,1]$.

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Again, using (2.11) and the monotonicity of the sequence $\{v_{E_n}(x)\}_{n\in\mathbb{N}}$, then, the sequence of functions v_{E_n} converges to v_E a.e. in $[0,1]$. Moreover, the pointwise limit in fact exists for every $x \in [0,1]$, and hence we may thus assume without loss of generality (modifying E on a subset of measure zero, if necessary) that v_{E_n} converges to v_E everywhere in $[0,1]$. Thus, as a consequence of lemma 2.6 (iv), we obtain

$$f(z) \leqslant v_E(z) \leqslant v_{E_n}(z) \leqslant v_{E_0}(z) \quad \text{or} \quad f(z) \geqslant v_E(z) \geqslant v_{E_n}(z) \geqslant v_{E_0}(z) \tag{2.16}$$

for all $z \in [0,1]$ and $n \in \mathbb{N}$. In particular, this yields

$$\min\{f(x), v_{E_0}(x)\} \leqslant v_E(x) \leqslant \max\{f(x), v_{E_0}(x)\}, \quad \text{for all } x \in [0,1]. \tag{2.17}$$

Define by S the set of all $x \in (0,1)$ such that both f and v_{E_0} are continuous in x and (2.13) holds. Since f and v_{E_0} are nonincreasing, and, as shown before, (2.13) holds for a.e. $x \in [0,1]$, it follows that $|[0,1] \setminus S| = 0$.

Now we can return to the main point. We are going to show that $f(x) = v_E(x)$ for almost all $x \in S$, by contradiction. To do so, assume that the set

$$S_0 := \{x \in S : v_E(x) > f(x)\}$$

has positive measure.

The function v_{E_0} is right-continuous. We may assume that f is right-continuous, otherwise its values may be changed on a null set. Hence, the level set $V_+ := \{x \in [0,1] : v_{E_0}(x) - f(x) > 0\}$ is right-open in the sense that for every $x \in V_+$ we have $(x, x+\delta) \subset V_+$, for some $\delta > 0$. By (2.16), inequality $v_{E_0}(x) > f(x)$ holds whenever $v_E(x) > f(x)$. Hence $S_0 \subset V_+$. Thus, there exists an interval $(a,b) \subset V_+ \subset [0,1]$ such that

$$v_{E_0} \geqslant v_E \geqslant f \text{ in } [a,b), \tag{2.18}$$

$$\lim_{x\to a-} v_{E_0}(x)) \leqslant \lim_{x\to a-} f(x), \tag{2.19}$$

$$v_{E_0}(b) \leqslant f(b), \tag{2.20}$$

$$|S_0 \cap (a,b)| > 0. \tag{2.21}$$

Notice that $(a,b) \subset [0,1]$ and hence $b < 1$. If $a > 0$, let $x \in [0,a)$ be arbitrary. Using (2.17), (2.19) and monotonicity of v_{E_0}, we have

$$v_E(x) \geqslant \min\{f(x), v_{E_0}(x)\} \geqslant \lim_{y\to a-} \min\{f(y), v_{E_0}(y)\} = \lim_{y\to a-} v_{E_0}(y) \geqslant v_{E_0}(a).$$

Similarly, for any $y \in [a,1]$ we have

$$v_E(y) \leqslant \max\{f(y), v_{E_0}(y)\} \leqslant \max\{f(a), v_{E_0}(a)\} = v_{E_0}(a).$$

Analogously, by (2.17), (2.18) and (2.20) we get, for any $y \in [0,b)$,

$$\begin{aligned} v_E(y) &\geqslant \min\{f(y), v_{E_0}(y)\} \geqslant \lim_{z\to b-} \min\{f(z), v_{E_0}(z)\} \\ &= \lim_{z\to b-} f(z) \geqslant f(b) \geqslant v_{E_0}(b). \end{aligned}$$

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For every $z \in [b,1]$ one has

$$v_E(z) \leqslant \max\{f(z), v_{E_0}(z)\} \leqslant \max\{f(b), v_{E_0}(b)\} = f(b).$$

Hence, we have

$$v_E(x) \geqslant v_{E_0}(a) \geqslant v_E(y) \geqslant f(b) \geqslant v_E(z)$$

for all $x \in [0,a)$, $y \in [a,b)$ and $z \in [b,1]$. For the sake of correctness, we note that if $a = 0$, the first term and inequality is simply omitted. Lemma 2.9 now gives

$$v_E^*(t) = (v_E \chi_{[b,1]})^*(t-b) \quad \text{for all} \quad t \in [b,1],$$

and

$$\int_a^b v_E^*(s)\mathrm{d}s = \int_a^b v_E(y)\mathrm{d}y.$$

By (2.21), one has $v_E > f$ on a subset of (a,b) with positive measure. Thus, there exists a $\varrho > 0$ such that

$$\int_a^b v_E(y)\mathrm{d}y > \int_a^b f(y)\mathrm{d}y + 4\varrho = \int_a^b f^*(s)\mathrm{d}s + 4\varrho.$$

From this and (2.15) used with $t = a$, we obtain

$$\begin{aligned}\int_0^b f^*(s)\mathrm{d}s + 4\varrho &= \int_0^a f^*(s)\mathrm{d}s + \int_a^b f^*(s)\mathrm{d}s + 4\varrho \\ &< \int_0^a v_E^*(s)\mathrm{d}s + \int_a^b v_E^*(s)\mathrm{d}s = \int_0^b v_E^*(s)\mathrm{d}s.\end{aligned} \tag{2.22}$$

Define

$$h := \min\left\{t \in (b,1) : \int_0^t v_E^*(s)\mathrm{d}s \leqslant \int_0^t f^*(s)\mathrm{d}s + 3\varrho\right\}.$$

The minimum is indeed attained since the function $t \mapsto \int_0^t v_E^*(s)\,\mathrm{d}s - \int_0^t f^*(s)\,\mathrm{d}s$ is continuous and $\int_0^1 v_E^*(s)\,\mathrm{d}s = \int_0^1 f^*(s)\,\mathrm{d}s$, which is (2.10). Let us show that $f > v_E$ holds on a subset of (b,h) with positive measure. To do so, assume, for contradiction, that $f \leqslant v_E$ a.e. in $[b,h)$. Let $t \in [b,h)$. The assumption implies $(f\chi_{[b,h)})^*(t-b) \leqslant (v_E\chi_{[b,h)})^*(t-b)$. Since f is nonincreasing, by lemma 2.9 we have

$$(f\chi_{[b,h)})^*(t-b) = (f\chi_{[b,1]})^*(t-b) = f^*(t).$$

Consequently,

$$v_E^*(t) = (v_E\chi_{[b,1]})^*(t-b) \geqslant (v_E\chi_{[b,h)})^*(t-b) \geqslant (f\chi_{[b,h)})^*(t-b) = f^*(t).$$

Therefore we have $v_E^*(t) \geqslant f^*(t)$ for all $t \in (b,h)$. However, this inequality together with (2.22) implies

$$\begin{aligned}\int_0^h v_E^*(s)\mathrm{d}s &= \int_0^b v_E^*(s)\mathrm{d}s + \int_b^h v_E^*(s)\mathrm{d}s \\ &> \int_0^b f^*(s)\mathrm{d}s + \int_b^h f^*(s)\mathrm{d}s + 4\varrho = \int_0^h f^*(s)\mathrm{d}s + 4\varrho.\end{aligned}$$

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This contradicts the definition of h.

We have shown that $|\{y \in (b,h) : f(y) > v_E(y)\}| > 0$. Since $|[0,1] \setminus S| = 0$, we have also $|\{y \in (b,h) \cap S : f(y) > v_E(y)\}| > 0$. Thanks to this and (2.21), there exist points $x \in (a,b) \cap S_0$ and $y \in (b,h) \cap S$ such that

$$v_E(x) > f(x) \geqslant f(y) > v_E(y) \geqslant v_{E_0}(y). \tag{2.23}$$

Define

$$\varepsilon := \frac{1}{4} \min\{v_E(x) - f(x),\ f(y) - v_E(y)\}.$$

Since f and v_{E_0} are both continuous in x as well as in y (recall the definition of S), there exists a $\delta > 0$ satisfying

$$\min\{x - a, 2(b - x), y - b, 1 - y\} > \delta > 0 \tag{2.24}$$

such that

$$\begin{aligned} &|v_{E_0}(z) - v_{E_0}(x)| < \varepsilon \text{ and } |f(z) - f(x)| < \varepsilon \qquad \text{for all } z \in (x - \delta, x + \delta), \\ &|v_{E_0}(z) - v_{E_0}(y)| < \varepsilon \text{ and } |f(z) - f(y)| < \varepsilon \qquad \text{for all } z \in (y - \delta, y + \delta). \end{aligned} \tag{2.25}$$

In particular, we get $v_{E_0} > f$ and $f > v_{E_0}$ on $(x - \delta, x + \delta)$ and $(y - \delta, y + \delta)$, respectively.

By (2.14) we have, for $t \in [0,1]$,

$$\int_0^t |v^*_{E_n}(s) - v^*_E(s)| \mathrm{d}s \xrightarrow{n \to \infty} 0$$

and this convergence is uniform in t. Moreover, since $x, y \in S$, both x and y satisfy (2.13) (in the latter case with y instead of x) and f as well as v_{E_0} are continuous in x and y. Based on these properties, there exists a sufficiently large $n \in \mathbb{N}$ such that all the following conditions are satisfied:

$$\begin{aligned} &\sup_{z \in X^n_{j_n(x)}} |v_{E_n}(x) - v_{E_n}(z)| < \varepsilon, \qquad \sup_{z \in X^n_{j_n(y)}} |v_{E_n}(y) - v_{E_n}(z)| < \varepsilon, \\ &2^{-n-1} < \varepsilon, \quad 2^{1-n} < \delta, \quad 2^{1-2n} < \varrho, \end{aligned} \tag{2.26}$$

and

$$\int_0^t |v^*_{E_n}(s) - v^*_E(s)| \mathrm{d}s < \varrho \quad \text{for all } t \in [0,1]. \tag{2.27}$$

In the following, we will write $j := j_n(x)$ and $k := j_n(y)$ since n, x and y remain fixed. By (2.16), we have

$$v_{E_0}(x) \geqslant v_{E_n}(x) \geqslant v_E(x) \geqslant f(x) + 4\varepsilon.$$

Let $z \in X^n_j$ be arbitrary. We have

$$\begin{aligned} v_{E_n}(z) &\geqslant v_{E_n}(x) - |v_{E_n}(x) - v_{E_n}(z)| > v_{E_n}(x) - \varepsilon = f(x) + v_{E_n}(x) - f(x) - \varepsilon \\ &\geqslant f(x) + 3\varepsilon \geqslant f(2^{-n}(j-1)) - |f(2^{-n}(j-1)) - f(x)| + 3\varepsilon \\ &\geqslant f(2^{-n}(j-1)) + 2\varepsilon. \end{aligned}$$

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Taking the limit (or infimum), we get

$$\inf_{z\in X_j^n} v_{E_n}(z) = \lim_{z\to 2^{-n}j-} v_{E_n}(z) \geqslant f(2^{-n}(j-1)) + 2\varepsilon. \tag{2.28}$$

Proceeding analogously regarding the point y, we obtain

$$\inf_{z\in X_k^n} f(z) = \lim_{z\to 2^{-n}k-} f(z) \geqslant v_{E_n}(2^{-n}(k-1)) + 2\varepsilon. \tag{2.29}$$

Taking into account the inequalities

$$f(2^{-n}(j-1)) \geqslant f(b) \geqslant f(y) \geqslant \lim_{z\to 2^{-n}k-} f(z),$$

we have

$$\lim_{z\to 2^{-n}j-} v_{E_n}(z) \geqslant v_{E_n}(2^{-n}(k-1)) + 4\varepsilon. \tag{2.30}$$

Recall that, for any $i \in \{1,\dots,2^n\}$, the functions $v_{E_n\cap Q_{ij}^n}$ and $v_{E_n\cap Q_{ik}^n}$ are nonincreasing on X_j^n and X_k^n, respectively. Inequality (2.30) thus yields

$$\sum_{i=1}^{2^n} \lim_{z\to 2^{-n}j-} v_{E_n\cap Q_{ij}^n}(z) > \sum_{i=1}^{2^n} v_{E_n\cap Q_{ik}^n}(2^{-n}(k-1)).$$

Hence, there exists an $i \in \{1,\dots,2^n\}$ such that

$$\lim_{z\to 2^{-n}j-} v_{E_n\cap Q_{ij}^n}(z) > v_{E_n\cap Q_{ik}^n}(2^{-n}(k-1)). \tag{2.31}$$

If we are able to show that the squares Q_{ij}^n and Q_{ik}^n are swappable with respect to E_n and n, we will obtain the ultimate contradiction with the construction of E_n.

Thus, we set $A := E_n$ and $A' := \sigma_{ijk}^n(A)$ and want to show that (2.1)–(2.4) holds. By (2.31) and (2.12) we immediately obtain condition (2.4). Since $2^{-n-1} < \varepsilon$ was assumed and f is nonincreasing, estimate (2.28) implies

$$\inf_{z\in X_j^n} v_{E_n}(z) > \max_{z\in X_j^n} f(z) + 2^{-n},$$

Therefore (2.1) is satisfied. Analogously, (2.29) yields (2.2).

It remains to prove (2.3). In this part, we will frequently use the inequality

$$\int_0^t v_A^*(s)\mathrm{d}s \geqslant \int_0^t f^*(s)\mathrm{d}s \quad \text{for all } t\in[0,1], \tag{2.32}$$

which obviously holds since $A = E_n$.

Since, by (2.23), $v_{E_0}(y) < f(y)$, we can define (c,d) as the maximal open subinterval of $(b,1)$ such that $y \in (c,d)$ and $v_{E_0} < f$ in (c,d). Moreover, using (2.25) and

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(2.26), we have that

$$X_j^n \subset (a,b) \quad \text{and} \quad X_k^n \subset (c,d). \tag{2.33}$$

Keeping in mind that $A = E_n$, by applying (2.16) and (2.17) to (2.18)–(2.20) we get the estimates

$$v_A \geqslant v_{E_0}(a) \text{ in } [0,a), \quad v_{E_0}(a) \geqslant v_A \geqslant f \geqslant f(b) \text{ in } [a,b), \quad f(b) \geqslant v_A \text{ in } [b,1]. \tag{2.34}$$

Now we may proceed analogously with the interval (c,d). Its maximality and (2.16) guarantee that

$$v_A \geqslant f(c) \text{ in } [0,c), \quad f(c) \geqslant f \geqslant v_A \geqslant v_{E_0}(d) \text{ in } [c,d), \quad v_{E_0}(d) \geqslant v_A \text{ in } [d,1]. \tag{2.35}$$

Notice that (2.34) and (2.35) hold also with A' in place of A. To see this, recall that $A' = \sigma_{ijk}^n(A)$ and then use lemma 2.6 (iv) and (2.33) which implies

$$|v_A - v_{A'}| \leqslant 2^{-n}(\chi_{X_j^n} + \chi_{X_k^n}). \tag{2.36}$$

Based on (2.34), (2.35) and their analogues with A', by lemma 2.9 we obtain:

$$v_A^*(t) = (v_A \chi_{[p,q)})^*(t-p) \quad \text{and} \quad v_{A'}^*(t) = (v_{A'}\chi_{[p,q)})^*(t-p) \quad \text{for} \quad t \in [p,q), \tag{2.37}$$

whenever $[p,q)$ is one of the intervals $[0,a), [a,b), [b,c), [c,d), [d,1)$. If $a = 0$ or $b = c$, the intervals $[0,a)$ or $[b,c)$ are, of course, not considered.

In particular, using (2.37) with $[p,q) = [a,b)$, $A' = \sigma_{ijk}^n(A)$ and $X_j^n \subset (a,b)$, which is (2.33), we have

$$\begin{aligned} \int_0^b v_{A'}^*(s)\mathrm{d}s = \int_0^b v_{A'}(z)\mathrm{d}z &= \int_0^b v_A(z)\mathrm{d}z - |A \setminus A'| \\ &= \int_0^b v_A^*(s)\mathrm{d}s - |A \setminus A'|. \end{aligned} \tag{2.38}$$

Again, using (2.33) we have that $X_j^n \subset (a,b)$ and $X_k^n \subset (c,d)$, and obtain

$$v_A = v_{A'} \text{ in } [0,a) \cup [b,c) \cup [d,1]. \tag{2.39}$$

If $a > 0$, let $t \in [0,a]$. By (2.39), (2.37) with $[p,q) = [0,a)$, and (2.32), we get

$$\int_0^t v_{A'}^*(s)\mathrm{d}s = \int_0^t v_A^*(s)\mathrm{d}s \geqslant \int_0^t f^*(s)\mathrm{d}s. \tag{2.40}$$

Let $t \in [a,b]$. Since $v_{A'} \geqslant f$ in $[a,b)$ (recall that this is (2.34) with A replaced by A'), (2.37) with $[p,q) = [a,b)$, and (2.32) provide

$$\begin{aligned} \int_0^t (v_{A'}^*(s) - f^*(s))\mathrm{d}s = &\int_0^a (v_{A'}^*(s) - f^*(s))\mathrm{d}s \\ &+ \int_a^t ((v_{A'}\chi_{(a,b)})^*(s-a) - (f\chi_{(a,b)})^*(s-a))\mathrm{d}s \geqslant 0, \end{aligned}$$

thus (2.40) holds for $t \in [a,b]$.

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If $b < c$, let $t \in [b, c]$. Since $c < h$, the definition of h and (2.27) yield

$$\int_0^t v_A^*(s)\mathrm{d}s > \int_0^t f^*(s)\mathrm{d}s + 2\varrho.$$

Property (2.39) yields $(v_{A'}\chi_{[b,c)})^* = (v_A\chi_{[b,c)})^*$. Considering (2.37) with $[p, q) = [b, c)$, (2.38), the assumption $\varrho > 2^{-2n}$, and (2.32), one has

$$\begin{aligned}
\int_0^t v_{A'}^*(s)\mathrm{d}s &= \int_0^b v_{A'}^*(s)\mathrm{d}s + \int_b^t v_{A'}^*(s)\mathrm{d}s \\
&= \int_0^b v_{A'}(z)\mathrm{d}z + \int_b^t (v_{A'}\chi_{[b,c)})^*(s-b)\mathrm{d}s \\
&= \int_0^b v_A(z)\mathrm{d}z - |A \setminus A'| + \int_b^t (v_A\chi_{[b,c)})^*(s-b)\mathrm{d}s \\
&= \int_0^t v_A^*(s)\mathrm{d}s - |A \setminus A'| > \int_0^t f^*(s)\mathrm{d}s + 2\varrho - |A \setminus A'| \\
&\geqslant \int_0^t f^*(s)\mathrm{d}s + 2\varrho - 2^{-2n} > \int_0^t f^*(s)\mathrm{d}s + \varrho.
\end{aligned}$$

Hence, (2.40) is satisfied for $t \in [b, c]$. In particular, the previous calculation gives

$$\int_0^c v_{A'}^*(s)\mathrm{d}s = \int_0^c v_A^*(s)\mathrm{d}s - |A \setminus A'|.$$

Moreover, since $A' = \sigma_{ijk}^n(A)$ and $X_k^n \subset (c, d)$, by (2.37) with $[p, q) = [c, d)$ we have

$$\int_c^d v_{A'}^*(s)\mathrm{d}s = \int_c^d v_{A'}(z)\mathrm{d}z = \int_c^d v_A(z)\mathrm{d}z + |A' \setminus A| = \int_c^d v_A^*(s)\mathrm{d}s + |A' \setminus A|.$$

Thanks to these two relations, we have

$$\int_0^d v_{A'}^*(s)\mathrm{d}s = \int_0^c v_{A'}^*(s)\mathrm{d}s + \int_c^d v_{A'}^*(s)\mathrm{d}s = \int_0^d v_A^*(s)\mathrm{d}s. \tag{2.41}$$

Let $t \in [c, d]$. By (2.36) it follows that $f \geqslant v_{A'}$ holds in $[c, d)$, and therefore $(f\chi_{[c,d)})^* \geqslant (v_{A'}\chi_{[c,d)})^*$. Applying this to (2.37) with $[p, q) = [c, d)$, one has $f^* \geqslant v_{A'}^*$ in $[c, d)$. Using this fact, (2.41) and (2.32), we get

$$\begin{aligned}
\int_0^t v_{A'}^*(s)\mathrm{d}s &= \int_0^d v_{A'}^*(s)\mathrm{d}s + \int_t^d (f^*(s) - v_{A'}^*(s))\mathrm{d}s - \int_t^d f^*(s)\mathrm{d}s \\
&\geqslant \int_0^d v_A^*(s)\mathrm{d}s - \int_t^d f^*(s)\mathrm{d}s \geqslant \int_0^d f^*(s)\mathrm{d}s - \int_t^d f^*(s)\mathrm{d}s \\
&= \int_0^t f^*(s)\mathrm{d}s.
\end{aligned}$$

In other words, (2.40) holds for $t \in [c, d]$.

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If $d < 1$, let $t \in [d, 1]$. By (2.39) we have $(v_{A'}\chi_{(d,1)})^* = (v_A\chi_{(d,1)})^*$ which together with (2.37) for $[p, q) = [d, 1)$, (2.41) and (2.32) provides

$$\int_0^t v^*_{A'}(s)\mathrm{d}s = \int_0^d v^*_{A'}(s)\mathrm{d}s + \int_d^t v^*_{A'}(s)\mathrm{d}s = \int_0^d v^*_A(s)\mathrm{d}s + \int_d^t (v_{A'}\chi_{(d,1)})^*(s)\mathrm{d}s$$
$$= \int_0^d v^*_A(s)\mathrm{d}s + \int_d^t (v_A\chi_{(d,1)})^*(s)\mathrm{d}s = \int_0^t v^*_A(s)\mathrm{d}s \geqslant \int_0^t f^*(s)\mathrm{d}s.$$

This shows that (2.40) is satisfied for $t \in [d, 1]$. If $d = 1$, this step is omitted.

Altogether we have shown that (2.40) is valid for all $t \in [0, 1]$, hence condition (2.3) is satisfied. Thus, all conditions (2.1)–(2.4) are met, which means that Q^n_{ij} and Q^n_{ik} are swappable with respect to A and n.

Let us summarize what we have proven at this point. Having started with the assumption $|\{x \in S : v_E(x) > f(x)\}| > 0$, we have found a set E_n such that there exist swappable squares Q^n_{ij}, Q^n_{ik} with respect to E_n and n. This contradicts the construction of E_n, as explained in remark 2.7.

Recalling that $|[0, 1] \setminus S| = 0$, we have therefore $v_E \leqslant f$ a.e. in $[0, 1]$. Since $\|v_E\|_1 = \|f\|_1$ and both functions are nonnegative, it follows that $v_E = f$ a.e. in $[0, 1]$.

The proof is now complete, under the conditions $X = Y = [0, 1]$ and f nonincreasing. If X and Y are general finite resonant measure spaces (we can assume, without loss of generality, that they are probability spaces), we use Ryff's Theorem [**1**, theorem II.7.5 and corollary II.7.6] to find a couple of measures preserving transformations

$$\sigma_\mu : X \to [0, 1] \quad \text{and} \quad \sigma_\nu : Y \to [0, 1],$$

such that $f = f^* \circ \sigma_\mu$, μ-a.e. and $g = g^* \circ \sigma_\nu$, ν-a.e. We now apply the previous case to the functions f^* and g^* on $[0, 1]$ (observe that f^* is nonincreasing and the hypothesis on the primitives of f^* and λ_{g^*} trivially holds), to obtain a set $\widehat{E} \subset [0, 1]^2$ such that $h_{\widehat{E}} = g^*$ and $v_{\widehat{E}} = f^*$, a.e. We now define the measurable set $E \subset X \times Y$ as follows:

$$E = \{(x, y) \in X \times Y : (\sigma_\mu(x), \sigma_\nu(y)) \in \widehat{E}\}.$$

Then, using [**1**, proposition II.7.2]

$$h_E(y) = \int_X \chi_E(x, y)\, d\mu(x) = \int_X \chi_{\widehat{E}}(\sigma_\mu(x), \sigma_\nu(y))\, d\mu(x)$$
$$= \int_0^1 \chi_{\widehat{E}}(s, \sigma_\nu(y))\mathrm{d}s = h_{\widehat{E}}(\sigma_\nu(y)) = g^*((\sigma_\nu(y))) = g(y), \quad \nu\text{-a.e.}$$

Analogously, one can prove that $v_E(x) = f(x)$, μ-a.e. □

Remark 2.11. As direct consequences of theorem 2.10, with the Lebesgue measure on $\mathbb{R}$ and the counting measures on finite sets, we recover Lorentz's result, theorem 1.1, as well as Gale–Ryser's theorem 1.2 for matrices in the discrete setting.

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