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Iwasawa Theory for $K_{2n}\mathcal{O}_F$

by

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Abstract

Given a number field F and a prime number p, let F_n denote the cyclotomic extension with $[F_n : F] = p^n$, and let $\mathcal{O}_{F_n}$ denote its ring of integers. We establish an analogue of the classical Iwasawa theorem for the orders of $K_{2i}(\mathcal{O}_{F_n})\{p\}$.

Key Words: Higher K-groups, $\mathbb{Z}_p$-extension, Iwasawa theory.

Mathematics Subject Classification 2010: 11R18, 11R23, 11R70, 19F27.

1. Introduction

For a number field F, let $\mathcal{O}_F$ be its ring of integers. Given a positive integer n, it is well known that the groups $K_n(\mathcal{O}_F)$ are finitely generated [6], and their ranks are determined by infinite places of F [1]. However, their torsion subgroups, hence the whole groups $K_n(\mathcal{O}_F)$, were impossible to describe until the 1997 (starting with [12]) works of Suslin, Voevodsky, Weibel, Rost and others. For any odd integer n, the group $K_n(\mathcal{O}_S)$ is completely determined. In this paper, we focus on $K_{2i}(\mathcal{O}_S)$ with F being a totally imaginary number field. A deep result on the group $K_{2i}(\mathcal{O}_S)$ is described by the cohomology group as follows:

Theorem 1.1 ([7],[11]) *Let $\mathcal{O}_S$ be the ring of S-integers in a number field F with some set S of finite places of F. Assume that $n = 2i \geq 2$. Then the following results hold.*

(i) *If p is an odd prime, then $K_n(\mathcal{O}_S)\{p\} \cong H^2(\mathcal{O}_S[\frac{1}{p}], \mathbb{Z}_p(i+1))$;*

(ii) *If $p = 2$, then for a totally imaginary number field F, $K_n(\mathcal{O}_S)\{2\} \cong H^2(\mathcal{O}_S[\frac{1}{2}], \mathbb{Z}_2(i+1))$.*

To determine the structure of $K_{2i}(\mathcal{O}_S)$, we have to determine the p-primary part $K_{2i}(\mathcal{O}_S)\{p\}$ of $K_{2i}(\mathcal{O}_S)$ for any prime p. By the above theorem, it suffices to compute the (étale) cohomology group $H^2(\mathcal{O}_S[\frac{1}{p}], \mathbb{Z}_p(i+1))$. The aim of this paper is to show that a classical theorem for the p-Sylow subgroup of the ideal class

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group of a number field F in Iwasawa 's theory of $\mathbb{Z}_p$-extensions of F holds also for the cohomology group $H^2(\mathcal{O}_S[\frac{1}{p}],\mathbb{Z}_p(i+1))$, which is isomorphic to the p-Sylow subgroup of $K_{2i}(\mathcal{O}_S)$.

Fix a prime p. We assume that $\mu_p \subset F$ if p is odd and $\mu_4 \subset F$ if $p=2$. Let q_0 be the largest power of p such that $\mu_{q_0} \subset F$ and put $q_n = q_0 p^n$. Write $F_n = F(\mu_{q_n})$ and $F_\infty = \bigcup_{n\geq 0}^{\infty} F_n$. Then F_∞/F is a $\mathbb{Z}_p$-extension and $[F_n : F] = p^n$ for all $n \geq 0$. Let $p^{e(i)_n}$ be the exact power of p dividing $\sharp K_{2i}(\mathcal{O}_{F_n})$. We show the following:

Theorem 3.3 *For any integer $i \geq 1$, there exist integers n_i and v_i such that for all $n \geq n_i$,*

$$e(i)_n = \lambda n + \mu p^n + v_i,$$

where λ and μ are the classical Iwasawa invariants independent of i and n, and v_i is a constant independent of n. Moreover, if $K_{2i}(\mathcal{O}_F)\{p\} = 0$, then $K_{2i}(\mathcal{O}_{F_n})\{p\} = 0$, for all $n \geq 0$.

Remark (1) In the case of $i = 1$, Theorem 3.3 has been proved by Coates ([2]).

(2) In this paper, we use Iwasawa descent to prove our main theorems. Some of our results can also be proved alternatively by the machinery of étale cohomology theory. One may work out the cohomological proofs by consulting Quang Do's paper [5] and P. Schneider's paper [8]. We would like to thank the referee for pointing out references [5] and [8] to us.

2. Preliminaries on Iwasawa Modules

As usual, we write $\Gamma = \mathrm{Gal}(F_\infty/F)$ and $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Let $\Lambda = \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ and let $\kappa : \Gamma \longrightarrow 1+q_0\mathbb{Z}_p$ be the isomorphism determined by $\gamma(\zeta) = \zeta^{\kappa(\gamma)}$, for all $\zeta \in W = \bigcup_{n\geq 0}\mu_{p^n}$ and $\gamma \in \Gamma$. Choose, once and for all, a topological generator γ_0 of Γ, and for simplicity, put $u = \kappa(\gamma_0)^{-1}$. In order to prove our main results, we need the following elementary results on Λ-modules.

Lemma 2.1 *Let Y be a torsion Λ-module with no non-trivial finite submodule, and let*

$$0 \longrightarrow Y \longrightarrow \oplus_{j=1}^{r}\Lambda/f_j(T) \longrightarrow D \longrightarrow 0$$

be an exact sequence of Λ-modules, where D is a finite Λ-module. Set $f(T) = \Pi_{j=1}^{r} f_j(T)$, which is the characteristic polynomial of Y. Then the following conditions are equivalent for all integers m and $n \geq 0$:

$$\text{(i) } Y(m)_{\Gamma_n} \text{ is finite}, \quad \text{(ii) } Y(m)^{\Gamma_n} = 0, \quad \text{(iii) } \prod_{i=0}^{n} f(u^m\zeta_{p^i} - 1) \neq 0.$$

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If these conditions hold, then the order of $Y(m)_{\Gamma_n}$ *is* $\prod_{i=0}^{n} |f(u^m\zeta_{p^i}-1)|_{v_i}^{-1}$.

Let Y be a Λ-module. If Z is a $\mathbb{Z}_p$-module, we always assume that $\mathrm{Hom}_{\mathbb{Z}_p}(Y,Z)$ is endowed with the Λ-module structure defined by

$$(\lambda\varphi)(y)=\varphi(\lambda y),\quad \lambda\in\Lambda,\ y\in Y,\ \varphi\in\mathrm{Hom}_{\mathbb{Z}_p}(Y,Z). \tag{1}$$

In particular, for a discrete Λ-module M, the Pontryagin dual $\widehat{M}=\mathrm{Hom}_{\mathbb{Z}_p}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ is equipped with a Λ-action described by the formula (1).

Lemma 2.2 *Let M be a discrete Λ-module having the property that its Pontryagin dual $\widehat{M}$ is a finitely generated torsion Λ-module with no non-trivial finite Λ-submodule. Hence there is a finite Λ-module D such that the sequence*

$$0\longrightarrow\widehat{M}\longrightarrow\oplus_{j=1}^{r}\Lambda/f_j(T)\longrightarrow D\longrightarrow 0$$

is exact. Set $f(T)=\Pi_{j=1}^{r}f_j(T)$, which is the characteristic polynomial of $\widehat{M}$. Then the following conditions are equivalent for all integers m and $n\geq 0$:

$$\text{(i) } M(m)^{\Gamma_n} \text{ is finite}\quad \text{(ii) } M(m)_{\Gamma_n}=0\quad \text{(iii) } \prod_{i=0}^{n} f(u^m\zeta_{p^i}-1)\neq 0.$$

If these conditions holds, then the order of $M(m)^{\Gamma_n}$ *is* $\prod_{i=0}^{n} |f(u^m\zeta_{p^i}-1)|_{v_i}^{-1}$.

Let I_n (resp. I) be the free abelian group generated by the primes of F_n (resp. F_∞) which do not lie above p. As in [4], there is a well defined surjective homomorphism $\psi:(\mathbb{Q}_p/\mathbb{Z}_p)\otimes_{\mathbb{Z}}F_\infty^\times\longrightarrow(\mathbb{Q}_p/\mathbb{Z}_p)\otimes_{\mathbb{Z}}I$. Define $\mathfrak{M}$ to be its kernel. Then $\mathfrak{M}$ is endowed with the natural structure of a discrete Γ-module.

Let $\mathfrak{O}_0=\mathcal{O}_F[\frac{1}{p}]$ and $\mathfrak{O}_n$ (resp. $\mathfrak{O}$) be the algebraic closure of $\mathfrak{O}_0$ in F_n (resp. F_∞). Soule [9] gives explicit relation between $\mathfrak{M}(i-1)$ and $H^1(\mathfrak{O}_n,W^{(i)})$.

Lemma 2.3 ([9]) *For any integer $i\geq 2$ and $n\geq 0$, there is a finite group $G_{n,i}$ such that the following results hold.*

$$\mathfrak{M}(i-1)=H^1(\mathfrak{O},W^{(i)}),\ \mathfrak{M}(i-1)_{\Gamma_n}=H^2(\mathfrak{O}_n,W^{(i)})=0,$$

$$\mathfrak{M}(i-1)^{\Gamma_n}=H^1(\mathfrak{O}_n,W^{(i)})=(\mathbb{Q}_p/\mathbb{Z}_p)^{p^nd/2}\oplus G_{n,i}.$$

Lemma 2.4 ([8]) *For all integers $n\geq 0$ and $i\geq 2$, we have*

$$H^2(\mathfrak{O}_n,\mathbb{Z}_p(i))\cong H^1(\mathfrak{O}_n,W^{(i)})/H^1(\mathfrak{O}_n,W^{(i)})_{div}.$$

Theorem 2.5 *For any integers $n\geq 0$ and $i\geq 1$, we have*

$$K_{2i}(\mathcal{O}_{F_n})\{p\}\cong G_{n,i+1}, \tag{2}$$

where $G_{n,i+1}$ is the finite group defined by Lemma 2.3.

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Proof: From [3], the natural map $K_{2i}(\mathcal{O}_{F_n})\{p\} \longrightarrow K_{2i}(\mathfrak{O}_n)\{p\}$ is an isomorphism for all $i \geq 1$. By Theorem 1.1, we have an isomorphism:

$$K_{2i}(\mathfrak{O}_n)\{p\} \cong H^2(\mathfrak{O}_n, \mathbb{Z}_p(i+1)).$$

Hence the isomorphism (2) follows immediately from Lemmas 2.3 and 2.4. □

The above results show that, for our purpose, it is sufficient to determine $\sharp G_{n,i}$ for all $i \geq 2$. Let P_n (resp. P) be the subgroup of principal p-ideals in I_n (resp. I), and $C_n = I_n/P_n$ (resp. $C = I/P$). Let $\mathfrak{C}_n$ (resp. $\mathfrak{C}$) denote the p-primary component of C_n (resp. C).

Lemma 2.6 *For each integer $n \geq 0$ and any integer $r \geq 1$, $\mathfrak{C}(r)^{\Gamma_n}$ is a finite group, and the sequence*

$$0 \longrightarrow \mathcal{E}(r)^{\Gamma_n} \longrightarrow \mathfrak{M}(r)^{\Gamma_n} \longrightarrow \mathfrak{C}(r)^{\Gamma_n} \longrightarrow 0$$

is exact, where $\mathcal{E} = \mathcal{U} \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $\mathcal{U}$ is the group of all p-units in F_∞.

Proof: For a proof of the case $n = 0$, see [9]. The proof also works in general for any n. Alternatively, we may apply Tate's lemma to Γ_n (see, *e.g.*, [8] Lemma 8). □

Let Y be the Pontryagin dual of $\mathcal{E}$. Iwasawa ([4], Theorem 15 or [9], pp.379-380) shows that there is an exact sequence

$$0 \longrightarrow Y \longrightarrow \Lambda^{d/2} \oplus M \longrightarrow \Phi \longrightarrow 0, \tag{3}$$

where $d = [F : \mathbb{Q}]$ and Φ is a finite Λ-module, $M = \oplus_{j=1}^{t} \Lambda/g_j(T)$ with $g_j(T) = T$ or $g_j(T) = \frac{(1+T)^{p^{\alpha_j}}-1}{(1+T)^{p^{\alpha_j-1}}-1}$, $\alpha_j \geq 1$ and $\sum_{j=1}^{t} \deg g_j(T) = s(F_\infty/F) - 1$, where $s(F_\infty/F)$ is the number of prime divisors of F_∞ which are ramified for the extension F_∞/F.

Tensoring (3) r times over $\mathbb{Z}_p$ with $\mathbb{Z}_p(1)$, and then using the snake lemma, we obtain the exact sequence:

$$\begin{aligned} 0 \to Y(r)^{\Gamma_n} \to (\Lambda(r)^{\Gamma_n})^{d/2} \oplus M(r)^{\Gamma_n} \to \Phi(r)^{\Gamma_n} \to Y(r)_{\Gamma_n} \\ \to \Lambda(r)_{\Gamma_n}^{d/2} \oplus M(r)_{\Gamma_n} \to \Phi(r)_{\Gamma_n} \to 0. \end{aligned} \tag{4}$$

3. The order of the p-primary part of $K_{2i}(\mathcal{O}_{F_n})$

Let $h(T)$ be the characteristic polynomial of the Pontryagin dual of $\mathfrak{C}$ and let $g(T) = \Pi_{j=1}^{t} g_j(T)$ be the characteristic polynomial of M defined by (3). Let

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$\mathcal{M}$ be the maximal abelian p-extension of F_∞, unramified outside p and let $f(T)$ be the characteristic polynomial of the Λ-module $\mathrm{Gal}(\mathcal{M}/F_\infty)^\bullet$ (the Λ-action on $\mathrm{Gal}(\mathcal{M}/F_\infty)^\bullet$ is described in pp. 277-278 of [4]). By [4], we have $f(T) = g(T)h(T)$.

Write $t(Y)$ for the torsion Λ-submodule of the Λ-module $Y = \widehat{\mathcal{E}}$. From (3) and [4], p.249, there is a finite Λ-module H such that the following sequence is exact:

$$0 \longrightarrow Y/t(Y) \longrightarrow \Lambda^{d/2} \longrightarrow H \longrightarrow 0. \tag{5}$$

Theorem 3.1 *For all integers $n \geq 0$ and $i \geq 1$, we have*

$$\sharp K_{2i}(\mathcal{O}_{F_n})\{p\} = \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |f(u^i\zeta_{p^j} - 1)|_{v_j}^{-1}.$$

Proof: Note that the Tate module $\Lambda(i)$ of Λ is isomorphic to Λ by the change of variable:

$$T \longrightarrow T^*_{(-i)} = u^i(1+T) - 1.$$

Hence, we have $\Lambda(i)_{\Gamma_n} = \Lambda_{\Gamma_n} = \mathbb{Z}_p[[T]]/((1+T)^{p^n} - 1) \cong (\mathbb{Z}_p)^{p^n}$. Since $\prod_{j=1}^{t}\prod_{i=0}^{n} g_j(u^r\zeta_{p^i} - 1) \neq 0$ when $r \geq 1$, the group $M(r)_{\Gamma_n}$ is finite. So by (4), one has

$$\widehat{\mathcal{E}(i)^{\Gamma_n}} = Y(i)_{\Gamma_n} = \Lambda(i)_{\Gamma_n}^{d/2} + (\text{finite group}) = (\mathbb{Z}_p)^{dp^n/2} + (\text{finite group}).$$

Therefore we have

$$\mathcal{E}(i)^{\Gamma_n} = (\mathbb{Q}_p/\mathbb{Z}_p)^{dp^n/2} \oplus B_{n,i}, \tag{6}$$

where $B_{n,i}$ is a finite group. By Lemmas 2.3, 2.6 and since $\mathfrak{C}(i)^{\Gamma_n}$ is a finite group, we have an exact sequence $0 \longrightarrow B_{n,i} \longrightarrow G_{n,i+1} \longrightarrow \mathfrak{C}(i)^{\Gamma_n} \longrightarrow 0$. From (5), we have the following sequence

$$\begin{aligned} 0 \to ((Y/t(Y))(i))^{\Gamma_n} \to (\Lambda(i)^{\Gamma_n})^{d/2} \to H(i)^{\Gamma_n} \\ \to ((Y/t(Y))(i))_{\Gamma_n} \to (\Lambda(i)_{\Gamma_n})^{d/2} \to H(i)_{\Gamma_n} \to 0 \end{aligned}$$

is exact. Note that $\Lambda^{\Gamma_n} = 0$ and $\Lambda(i)_{\Gamma_n} \cong \mathbb{Z}_p^{p^n}$ imply that

$$((Y/t(Y))(i))^{\Gamma_n} = 0 \quad \text{and} \quad ((Y/t(Y))(i))_{\Gamma_n} \cong \mathbb{Z}_p^{dp^n/2} \oplus \Phi_{n,i}, \tag{7}$$

where $\Phi_{n,i} \cong H(i)^{\Gamma_n}$ is a finite group.

From the exact sequence

$$0 \longrightarrow t(Y) \longrightarrow M \longrightarrow \Psi \longrightarrow 0,$$

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where Ψ is a finite group, we obtain an exact sequence

$$0 \longrightarrow \Psi(i)^{\Gamma_n} \longrightarrow t(Y)(i)_{\Gamma_n} \longrightarrow M(i)_{\Gamma_n} \longrightarrow \Psi(i)_{\Gamma_n} \longrightarrow 0.$$

The fact that Ψ is a finite group implies that $\sharp\Psi(i)^{\Gamma_n} = \sharp\Psi(i)_{\Gamma_n}$. Hence, by Lemma 2.1, we obtain

$$\sharp t(Y)(i)_{\Gamma_n} = \sharp M(i)_{\Gamma_n} = \prod_{j=0}^{n} |g(u^i \zeta_{p^j} - 1)|_{v_j}^{-1}. \tag{8}$$

From the exact sequence $0 \longrightarrow t(Y) \longrightarrow Y \longrightarrow Y/t(Y) \longrightarrow 0$, we see that the sequence

$$\begin{aligned} 0 \to t(Y)(i)^{\Gamma_n} \to Y(i)^{\Gamma_n} \to ((Y/t(Y))(i))^{\Gamma_n} \to t(Y)(i)_{\Gamma_n} \\ \to Y(i)_{\Gamma_n} \to ((Y/t(Y))(i))_{\Gamma_n} \to 0 \end{aligned}$$

is exact. By (7), we obtain the exact sequence

$$0 \longrightarrow t(Y)(i)_{\Gamma_n} \longrightarrow Y(i)_{\Gamma_n} \longrightarrow ((Y/t(Y))(i))_{\Gamma_n} \longrightarrow 0. \tag{9}$$

Therefore, by (6), (7) and (9), we have

$$0 \longrightarrow t(Y)(i)_{\Gamma_n} \longrightarrow \widehat{B_{n,i}} \longrightarrow \Phi_{n,i} \longrightarrow 0. \tag{10}$$

Hence, by (8), (10) and Lemma 2.2, we have

$$\begin{aligned} \sharp G_{n,i+1} &= \sharp B_{n,i} \cdot \sharp \mathfrak{C}(i)^{\Gamma_n} = \sharp \widehat{B_{n,i}} \cdot \sharp \mathfrak{C}(i)^{\Gamma_n} \\ &= \sharp \Phi(i)_{n,i} \cdot \sharp t(Y)(i)_{\Gamma_n} \cdot \sharp \mathfrak{C}(i)^{\Gamma_n} \\ &= \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |g(u^i \zeta_{p^j} - 1)|_{v_j}^{-1} \cdot \prod_{j=0}^{n} |h(u^i \zeta_{p^j} - 1)|_{v_j}^{-1} \\ &= \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |f(u^i \zeta_{p^j} - 1)|_{v_j}^{-1}. \end{aligned}$$

Finally, by Theorem 2.5, we obtain

$$\sharp K_{2i}(\mathcal{O}_{F_n})\{p\} = \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |f(u^i \zeta_{p^j} - 1)|_{v_j}^{-1}.$$

□

Corollary 3.2 *The finite group H is trivial if and only if there exists integer $i \geq 1$ such that*

$$\sharp K_{2i}(\mathcal{O}_F)\{p\} = |f(u^i - 1)|_p^{-1}.$$

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Proof: The necessity is plain from Theorem 3.1. On the other hand, if $\sharp K_{2i}(\mathcal{O}_F)\{p\} = |f(u^i - 1)|_p^{-1}$, we deduce from Theorem 3.1 that $TH = H$. But H, being finite, is certainly a compact Λ-module, and hence, is trivial. □

Remark Theorem 3.1 illustrates again how the group $K_{2i}(\mathcal{O}_F)$ is connected with the fundamental problems of the theory of cyclotomic fields. On the one hand, Theorem 3.1 gives a criterion for the vanishing of H in terms of $K_{2i}(\mathcal{O}_F)$; on the other hand, the results by Coates ([2]) give a criterion for the vanishing of H in terms of K_2F.

Theorem 3.3 (1) *For any $i \geq 1$, if $K_{2i}(\mathcal{O}_F)\{p\} = 0$, then $K_{2i}(\mathcal{O}_{F_n})\{p\} = 0$, for all $n \geq 0$.*

(2) *Let λ and μ be the classical Iwasawa invariants of the Λ-module $Gal(\mathcal{M}/F_\infty)$. For any $i \geq 1$, let $p^{e(i)_n} = \sharp K_{2i}(\mathcal{O}_{F_n})\{p\}$. There exist integers n_i and a constant v_i independent of n such that, for all $n \geq n_i$,*

$$e(i)_n = \lambda n + \mu p^n + v_i.$$

Proof: (1) If $K_{2i}(\mathcal{O}_F)\{p\} = 0$, then

$$\sharp H(i)^\Gamma = 1, \ i.e., \ H(i) = TH(i) \text{ and } \prod_{j=0}^{n} |f(u^i \zeta_{p^j} - 1)|_{v_j}^{-1} = 1.$$

Hence H is trivial and $f(T) = 1$. By Theorem 3.1, we obtain that $K_{2i}(\mathcal{O}_{F_n})\{p\} = 0$, for all $n \geq 0$.

(2) We know that the Λ-module $\mathrm{Gal}(\mathcal{M}/F_\infty)$ is noetherian and has no non-trivial finite Λ-submodule. From ([4]), we have the following exact sequence:

$$0 \longrightarrow \mathrm{Gal}(\mathcal{M}/F_\infty)^\bullet \longrightarrow \Lambda^{d/2} \oplus \oplus_{j=1}^{r} \Lambda/f_j(T) \longrightarrow M_1 \longrightarrow 0,$$

where M_1 is a finite group and $f(T) = \Pi_{j=1}^{r} f_j(T)$ is the characteristic polynomial of the Λ-module $\mathrm{Gal}(\mathcal{M}/F_\infty)^\bullet$. Set

$$N = \oplus_{j=1}^{r} \Lambda/f_j(T).$$

By Lemma 2.1, we have

$$\sharp N(i)_{\Gamma_n} = \prod_{j=0}^{n} |f(u^i \zeta_{p^j} - 1)|_{v_j}^{-1}.$$

Note that the invariants λ and μ of the modules $N(i)$ and $\mathrm{Gal}(\mathcal{M}/F_\infty)$ coincide. It is a basic theorem about noetherian torsion Λ-modules that, for all sufficiently large n, the order of $N(i)_{\Gamma_n}$ is $p^{e(i)_n}$, with $e(i)_n = \lambda n + \mu p^n + v$, and v is an integer which does not depend on n. By Theorem 3.1, for all sufficiently large n, we have $\sharp K_{2i}(\mathcal{O}_{F_n})\{p\} = \sharp H \cdot \sharp N(i)_{\Gamma_n}$. Note that the order of the group H is independent of n. This completes the proof. □

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Remark Theorem 3.3 (2) can be obtained via straightforward computation using the formula of $K_{2i}(\mathcal{O}_{F_n})\{p\}$ given in Theorem 3.1.

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