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ON THE MULTI-STATE SIGNATURES OF ORDERED SYSTEM LIFETIMES

HE YI,* *Beijing University of Chemical Technology*
NARAYANASWAMY BALAKRISHNAN,** *McMaster University*
LIRONG CUI,*** *Beijing Institute of Technology*

Abstract

In this paper, the signature of a multi-state coherent system with binary-state components is discussed, and then it is extended to the case of ordered system lifetimes arising from a life-test on coherent multi-state systems with the same multi-state system signature. Some properties of the multi-state system signature and the ordered multi-state system signature are also studied. The results established here are finally explained through some illustrative examples.

Keywords: Multi-state coherent system; multi-state system signature; ordered system lifetimes; ordered multi-state system signature.

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1. Introduction

A coherent system is a system whose components are all relevant to the system lifetime and its reliability will not decrease if the performance of at least one component improves. Coherent systems are commonly found in practice, and the system signature was proposed as an important measure to describe the structure of a coherent system and also to facilitate some comparisons between different systems. The concept of signature was first introduced by Samaniego [29] in his research on closure of the IFR class under formation of coherent systems. For a coherent system with m independent and identically distributed (i.i.d.) components, its signature $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ is defined as a vector of length m whose ith element s_i is the probability that the system fails due to the ith ordered component failure.

For coherent systems with the same number of i.i.d. components, preservation results on their signatures were established by Kochar, Mukerjee, and Samaniego [14] for comparisons of systems in three different senses: stochastic ordering, hazard rate ordering, and likelihood ratio ordering. For coherent systems with different numbers of i.i.d. components, Samaniego [30] compared their performance by using equivalent mixed systems, and some further investigation in this regard can be found in Navarro, Samaniego, Balakrishnan, and Bhattacharya [27]. System signatures have many extensions and applications in engineering reliability; see

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* Postal address: School of Economics & Management, Beijing University of Chemical Technology, Beijing, 100029, China. Email address: yihe_buct@163.com
** Postal address: Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada.
*** Postal address: School of Management & Economics, Beijing Institute of Technology, Beijing, 100081, China.

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Samaniego [31] for pertinent details. Navarro and Rychlik [23] extended signatures of coherent systems with i.i.d. components to coherent systems with exchangeable components, and also investigated their reliabilities and expectation bounds based on their signatures.

Navarro, Ruiz, and Sandoval [24] proposed minimal and maximal signatures for coherent systems with exchangeable components, which represented the system lifetime distribution as generalized mixtures of the distributions of series and parallel systems. In the more general case, Navarro, Spizzichino, and Balakrishnan [28] introduced average systems and projected systems of coherent systems, which provided new tools to study the performance of coherent systems with non-exchangeable components and to assess the effect of asymmetry in the joint distribution of component lifetimes on system performance. For two coherent systems with shared components, Navarro, Samaniego, and Balakrishnan [25] defined and studied a new measure, called the joint signature, under the i.i.d. assumption on component lifetimes, and they also investigated some ordering properties. Under the only assumption that the joint distribution of component lifetimes has no ties, Marichal and Mathonet [19] presented an alternative expression for the Barlow–Proschan importance measure, and also provided related discussions for the system signature in terms of the Möbius transform.

Computation of signatures can be quite intensive for complex systems with large sizes, and it may be useful to reduce the computational burden by decomposing those systems into several modules. Gertsbakh, Shpungin, and Spizzichino [12] presented the signature of a coherent system that consists of two coherent modules with known signatures in series/parallel (the two systems may have different numbers of components), and they also investigated the signature of a coherent system that consists of more than two identical coherent modules. Independently of [12], similar results were given by Da, Zheng, and Hu [7] for a coherent system which is the parallel/series system of two modules, and they also considered the signature of system-wise redundancy in terms of the original system and the backup system as an application. Eryilmaz [9] discussed the signature and the minimal signature of a coherent system consisting of two coherent systems in series/parallel. Da, Xia, and Hu [6] considered the signature and the minimal signature of a k-out-of-n coherent system consisting of n modules, and these modules may have different numbers of components. In a general setting, Marichal, Mathonet, and Spizzichino [21] presented precise conditions on the component lifetimes for having a formula expressing the signature of a system in terms of the signatures of its modules. Moreover, the modular method can be used together with many other signature computation methods, for example, the efficient Markov method proposed by Yi and Cui [34], which yielded signatures of several well-known consecutive-type systems.

Inferential methods can also be developed using the system signature based on system lifetime data. For example, Balakrishnan, Ng, and Navarro [4] developed nonparametric statistical inference for the lifetime distribution of the components based on observed lifetimes of coherent systems with the same structure and a known common system signature. Yang, Ng, and Balakrishnan [33] developed expectation–maximization algorithms to deal with system lifetime data without knowledge of the system signature. Based on the scenario considered in [4], Balakrishnan and Volterman [3] proposed a new measure, called the ordered system signature, to describe the signature of the ith failed system in a life-test involving n independent coherent systems with the same signature.

Quite often, systems and their components have more than two states, varying from perfect functioning to complete failure. These systems are referred to as multi-state systems, and some related research can be found in [1], [2], [17], and [22]. The concept of signature was first proposed for binary-state systems, but generalizations of signature to multi-state systems have

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become necessary due to the importance and practicality of multi-state systems in networks, for example. There already exist several works on signature of multi-state systems, but compared to the literature on binary-state systems, the works on multi-state systems are still far fewer in number.

To evaluate the expected damage of complex networks, Levitin, Gertsbakh, and Shpungin [16] first proposed the concept of a multi-dimensional D-spectrum. Without loss of generality, they defined a two-dimensional D-spectrum as $\mathbf{d} = \{w_{i,j}\}$, $i, j = 1, 2, \ldots, L$, for a multi-state network with state space $\{UP, DOWN\,2, DOWN\,3\}$ and binary-state network links, where $w_{i,j}$ is the proportion of permutations such that the network state becomes $DOWN\,2$ at the ith ordered link failure and then becomes $DOWN\,3$ at the jth ordered link failure. In their discussions, the network is in state $DOWN\,2$ if and only if it has two clusters, and in state $DOWN\,3$ if and only if it has three clusters, which means the UP state corresponds to only one cluster and state $DOWN\,3$ is worse than state $DOWN\,2$ with more disconnected links. Gertsbakh and Shpungin [11] subsequently considered multi-dimensional D-spectra and cumulative D-spectra for multi-state systems with binary components, and established their connection with the probability of the multi-state system being in each state. However, in their research, no further theoretical results on multi-state signature were provided. A similar definition of multi-state signature was also presented by Gertsbakh, Shpungin, and Spizzichino [13] for three-state systems with binary-state components in a matrix form, and they then analyzed the multi-state signature of ternary-state systems obtained from generalized series/parallel connections of separate ternary-state modules.

Recently, Marichal, Mathonet, Navarro, and Paroissin [20] extended the concept of joint signature in Navarro, Samaniego, and Balakrishnan [26] to ternary-state systems with binary-state components, and presented formulas in the non-i.i.d. case, which only assumed that the distribution of the component lifetimes has no ties. Liu, Shi, Bai, and Liu [18] presented the definition of generalized survival signature for multi-state systems with multi-state components, and then discussed the stress-strength reliability of these systems. Based on the universal generating function, Kumar and Singh [15] presented a new approach to computing signatures of multi-state systems with binary-state components. This approach, as we will see, can be extended to multi-state systems with multi-state components to calculate the survival signature, which we plan to study in our future research. In the present work, we extend the definition of Da and Hu [5] to general coherent multi-state systems with binary-state components, and then discuss its properties in detail.

As mentioned in Yang, Ng, and Balakrishnan [32], interest often lies in both the system lifetime distribution and the component lifetime distribution when only system lifetimes are observable. When complete system lifetime data are observed, system signature can be used to estimate the parameters of the component lifetime distribution, but the ordered system signature defined in [3] turns out to be more efficient for this purpose, as shown in [32] in their simulation study. Moreover, when the system lifetime data are censored, which is often the case in life-testing experiments, the system signature will not be useful since the observed lifetimes are no longer exchangeable, and in this situation the ordered system signature becomes a very effective tool for the estimation of the component lifetime distribution as well as the prediction of the censored system lifetimes. Furthermore, use of the ordered system signature does not add much complexity to the computation, thus rendering it to be quite useful in inferential problems. To the best of our knowledge, no research has been carried out on the ordered system signature for multi-state systems, even though both the ordered system signature and the multi-state systems have become important in practical applications. For this reason, in

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this work we generalize the ordered system signature in [3] to multi-state systems with binary-state components and discuss some of its properties. Along the lines of [32], we plan to use this ordered system signature in developing parametric and nonparametric inferential methods in our future work. Results of this paper can be used to analyze lifetime data observed from many multi-state systems such as power systems, air conditioning systems, oil and gas production systems, transportation systems, redundant systems, and so on.

Specifically, in this paper we consider a life-test of several multi-state coherent systems with the same structure, and propose a conditional signature for the ith system entering state j or below as the ordered multi-state system signature. The rest of this paper proceeds as follows. In Section 2 we formally define the system signature for multi-state coherent systems, and discuss some related properties. In Section 3 we define the ordered multi-state signature and discuss some of its interesting properties that assist in simplifying the associated computational work. Then, in Section 4, we present some examples to illustrate the developed results. Finally, we provide some concluding remarks in Section 5.

2. The signature of a multi-state coherent system

In this section we consider a multi-state coherent system that has $n+1$ system states and m binary-state components. Denote the state vector of the components as $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, where $x_i (i = 1, 2, \ldots, m)$ is the state of component i with $x_i = 1$. Here, state $x_i = 0$ denotes the failure state of component i while state $x_i = 1$ denotes the working state of component i. Denote the state of the system by x, where $x \in \tilde{S} = \{0, 1, \ldots, n\}$. Here, state n is the perfect functioning state, state 0 is the complete failure state, and the intermediate states $1, 2, \ldots, n-1$ correspond to different ordered degradation levels of the system. The multi-state structure function of the system can be represented by $\phi : S^m \to \tilde{S}$, which means $x = \phi(\mathbf{x})$. We assume that $\phi(\mathbf{x})$ is monotone increasing in each argument and $\phi(\mathbf{0}) = 0$, $\phi(\mathbf{1}) = n$, where $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$.

Let $X_1, X_2, \ldots, X_m$ be i.i.d. random variables with X_i representing the lifetime of the binary-state component i, and let $X_{i:m}$ be the ith smallest component lifetime among $X_1, X_2, \ldots, X_m$. Assume that the m component lifetimes $X_1, X_2, \ldots, X_m$ are i.i.d. from an absolutely continuous distribution $F(x)$, $x \geq 0$, with probability density function $f(x)$, $x \geq 0$, which readily implies that the ties among them occur with zero probability. For the set $\mathbf{P}$ of all permutations of the integers $\{1, 2, \ldots, m\}$, each of its $m! = m \times (m-1) \times \cdots \times 1$ element can be identified by $\boldsymbol{\pi} = (\pi(1), \pi(2), \ldots, \pi(m)) \in \mathbf{P}$, and then the events $A_{\boldsymbol{\pi}} = \{X_{\pi(1)} < X_{\pi(2)} < \cdots < X_{\pi(m)}\}$ can be defined unambiguously to describe orderings of the component lifetimes. From the fact that $X_1, X_2, \ldots, X_m$ are i.i.d., the events $A_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \mathbf{P}$ are all equally likely, namely, with probability $P(A_{\boldsymbol{\pi}}) = 1/m!$. Let T_k $(k = 1, 2, \ldots, n)$ be the first time that the multi-state coherent system enters state $n-k$ or below, and then once the ordering $A_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \mathbf{P}$ of component lifetimes is determined, the system lifetimes $T_1, T_2, \ldots, T_n$ will be equated with particular order statistics $X_{i_1:m}, X_{i_2:m}, \ldots, X_{i_n:m}$ $(1 \leq i_1, i_2, \ldots, i_n \leq m)$, which depend on $i_1, i_2, \ldots, i_n$. Here, we define these discrete random variables for $i_1, i_2, \ldots, i_n$ as $I_1(\phi), I_2(\phi), \ldots, I_n(\phi)$ for this multi-state system with structure function $\phi(x)$. Then the system signature for such a multi-state coherent system with binary-state components can be defined as follows.

Definition 2.1. (*Multi-state signature.*) The signature of a multi-state coherent system with m binary-state components is

$$\mathbf{s} = (s_{i_1, i_2, \ldots, i_n}, 1 \leq i_1, i_2, \ldots, i_n \leq m),$$

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where $s_{i_1,i_2,\ldots,i_n} = \mathbb{P}\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\}$ and $X_{i_1:m}, X_{i_2:m}, \ldots, X_{i_n:m}$ are the i_1th, i_2th, $\ldots$, i_nth order statistics among $X_1, X_2, \ldots, X_m$, respectively.

Remark 2.1.

(a) The multi-state signature $\mathbf{s} = (s_{i_1,i_2,\ldots,i_n}, 1 \le i_1, i_2, \ldots, i_n \le m)$ is a generalized matrix with dimension

$$\underbrace{m \times \cdots \times m}_{n},$$

and in particular for the simplest case when $m = 2$ (binary-state system), the signature $\mathbf{s} = (s_{i_1,i_2}, 1 \le i_1, i_2 \le m)$ is an ordinary matrix with dimension $m \times m$; see Examples 4.1 and 4.3 for details.

(b) Each element $s_{i_1,i_2,\ldots,i_n}$ is non-negative ($s_{i_1,i_2,\ldots,i_n} \ge 0$) in a multi-state signature $\mathbf{s} = (s_{i_1,i_2,\ldots,i_n}, 1 \le i_1, i_2, \ldots, i_n \le m)$, since it is a probability. Note that $s_{i_1,i_2,\ldots,i_n} \ne 0$ only if $1 \le i_1 \le i_2 \le \cdots \le i_n \le m$ since $s_{i_1,i_2,\ldots,i_n}$ corresponds to the event that the multi-state coherent system enters states $n-1$ or below at the i_1th component failure, then enters state $n-2$ or below at the i_2th component failure, and so on.

(c) The multi-state signature $\mathbf{s} = (s_{i_1,i_2,\ldots,i_n}, 1 \le i_1, i_2, \ldots, i_n \le m)$ is the joint probability distribution of the discrete random variables $(I_1(\phi), I_2(\phi), \ldots, I_n(\phi))$, where $I_k(\phi)$ $(k = 1, 2, \ldots, n)$ equals $i_k \in \{1, 2, \ldots, m\}$ if the i_kth smallest component failure causes the system to enter state $n-k$ or below. That is so because, for any $1 \le i_1, i_2, \ldots, i_n \le m$,

$$\begin{aligned} s_{i_1,i_2,\ldots,i_n} &= \mathbb{P}\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\} \\ &= \frac{1}{m!} \sum_{\boldsymbol{\pi} \in \mathbf{P}} \mathbb{P}\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m} \mid A_{\boldsymbol{\pi}}\} \\ &= \frac{1}{m!} \sum_{\boldsymbol{\pi} \in \mathbf{P}} \mathbb{P}\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n \mid A_{\boldsymbol{\pi}}\} \\ &= \mathbb{P}\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n\}. \end{aligned}$$

Here, for given $\boldsymbol{\pi} \in \mathbf{P}$, $P\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n \mid A_{\boldsymbol{\pi}}\}$ equals 1 for only one case of $(i_1, i_2, \ldots, i_n)$ (determined by the ordering $A_{\boldsymbol{\pi}}$ and the system structure ϕ) and equals 0 for all other cases, which leads to the fact that

$$\begin{aligned} &\sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1,i_2,\ldots,i_n} \\ &= \sum_{1 \le i_1, i_2, \ldots, i_n \le m} \frac{1}{m!} \sum_{\boldsymbol{\pi} \in \mathbf{P}} \mathbb{P}\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n \mid A_{\boldsymbol{\pi}}\} \\ &= \frac{1}{m!} \sum_{\boldsymbol{\pi} \in \mathbf{P}} \sum_{1 \le i_1, i_2, \ldots, i_n \le m} \mathbb{P}\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n \mid A_{\boldsymbol{\pi}}\} \\ &= \frac{1}{m!} \sum_{\boldsymbol{\pi} \in \mathbf{P}} 1 = 1. \end{aligned}$$

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With the multi-state system signature defined as above, the joint distribution of lifetimes T_k $(k = 1, 2, \ldots, n)$ can be presented as follows.

Theorem 2.1. *For* $0 \le t_1 \le t_2 \le \cdots \le t_n < \infty$, *we have*

$$\begin{aligned}
&\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} \\
&\quad = \sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \sum_{0 \le j_1 \le i_1 - 1, j_1 \le j_2 \le i_2 - 1, \ldots, j_{n-1} \le j_n \le i_n - 1} F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1} \\
&\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n).
\end{aligned}$$

Proof. As mentioned in Remark 2.1(c), the conditional probabilities $\mathbb{P}\{I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n \mid A_{\boldsymbol{\pi}}\}$ $(1 \le i_1, i_2, \ldots, i_n \le m)$ depend only on the ordering $A_{\boldsymbol{\pi}}$ and the system structure ϕ. Then, for $\mathbb{P}(A_{\boldsymbol{\pi}}) = 1/m!$, $\boldsymbol{\pi} \in \mathbf{P}$, with the effect of $A_{\boldsymbol{\pi}}$ averaged, the distribution of the event $\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\}$ depends only on the structure of the system, and is independent of the component lifetime distribution $F(x)$. Due to this fact, the event $\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\}$ is independent of the order statistics $X_{i_1:m}, X_{i_2:m}, \ldots, X_{i_n:m}$. Then

$$\begin{aligned}
&\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} \\
&\quad = \sum_{1 \le i_1, i_2, \ldots, i_n \le m} \mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n \mid T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\} \\
&\qquad \times \mathbb{P}\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\} \\
&\quad = \sum_{1 \le i_1, i_2, \ldots, i_n \le m} \mathbb{P}\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n \mid \\
&\qquad\qquad T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\} \\
&\qquad \times \mathbb{P}\{T_1 = X_{i_1:m}, T_2 = X_{i_2:m}, \ldots, T_n = X_{i_n:m}\} \\
&\quad = \sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \mathbb{P}\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n\}.
\end{aligned}$$

Note that for $0 \le t_1 \le t_2 \le \cdots \le t_n < \infty$, the event $\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n\}$ is true if and only if $j_1 \in \{1, 2, \ldots, i_1 - 1\}$ components fail in the interval $[0, t_1]$, $j_2 \in \{j_1, j_1 + 1, \ldots, i_2 - 1\}$ components fail in the interval $[0, t_2], \ldots, j_n \in \{j_{n-1}, j_{n-1} + 1, \ldots, i_n - 1\}$ components fail in the interval $[0, t_n]$, and $n - j_n$ components fail in the interval $[t_n, \infty)$. Then

$$\begin{aligned}
&\mathbb{P}\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n\} \\
&\quad = \sum_{0 \le j_1 \le j_2 \le \cdots \le j_n \le m, j_1 < i_1, j_2 < i_2, \ldots, j_n < i_n} F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1} \\
&\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n) \\
&\quad = \sum_{0 \le j_1 \le i_1 - 1, j_1 \le j_2 \le i_2 - 1, \ldots j_{n-1} \le j_n \le i_n - 1} F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1} \\
&\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n).
\end{aligned}$$

□

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Remark 2.2.

(a) For general $0 \le t_1, t_2, \ldots, t_n < \infty$,

$$\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} = \mathbb{P}\{T_1 > \tilde{t}_1, T_2 > \tilde{t}_2, \ldots, T_n > \tilde{t}_n\},$$

where $\tilde{t}_1 = t_1, \tilde{t}_2 = \max(t_1, t_2), \ldots, \tilde{t}_n = \max(t_1, t_2, \ldots, t_n)$ satisfy $0 \le \tilde{t}_1 \le \tilde{t}_2 \le \cdots \le \tilde{t}_n < \infty$.

(b) For lifetimes T_k $(k = 1, 2, \ldots, n)$, let $t_1 = \cdots = t_{k-1} = t_{k+1} = \cdots = t_n = 0$, and then

$$\begin{aligned}
\mathbb{P}\{T_k > t\} &= \sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \mathbb{P}\{X_{i_k:m} > t\} \\
&= \sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \sum_{0 \le j \le i_k - 1} F^j(t)[1 - F(t)]^{m-j} \\
&= \sum_{1 \le i_k \le m} \left(\sum_{1 \le i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \right) \sum_{0 \le j \le i_k - 1} F^j(t)[1 - F(t)]^{m-j} \\
&= \sum_{1 \le i_k \le m} s_{i_k}^{(k)} \sum_{0 \le j \le i_k - 1} F^j(t)[1 - F(t)]^{m-j} \\
&= \sum_{0 \le j \le m-1} \left(\sum_{j+1 \le i_k \le m} s_{i_k}^{(k)} \right) F^j(t)[1 - F(t)]^{m-j},
\end{aligned}$$

where

$$s_{i_k}^{(k)} = \sum_{1 \le i_1, \ldots, i_{k-1}, i_{k+1}, \cdots, i_n \le m} s_{i_1, i_2, \ldots, i_n} = P\{T_k = X_{i_k:m}\},\ i_k = 1, 2, \ldots, m.$$

Here, the vector $\mathbf{s}^{(k)} = (s_1^{(k)}, s_2^{(k)}, \ldots, s_m^{(k)})$ is the same as the ordinary signature of the multi-state system when states $n-k$ or below are regarded as failure states, and states above $n-k$ are regarded as working states. Similarly, the vector $\mathbf{S}^{(k)} = (S_1^{(k)}, S_2^{(k)}, \ldots, S_m^{(k)})$, where $S_j^{(k)} = \sum_{j \le i_k \le m} s_{i_k}^{(k)} (j = 1, 2, \ldots, m)$ can be regarded as the ordinary survival signature in this case.

One of the most important applications of the system signature is its use in comparing the performance of different system structures. As in the case of binary-state systems, preservation results of the system signature also exist for multi-state systems in the sense of stochastic ordering. First of all, the stochastic ordering should be generalized for joint discrete distributions and vectors of non-negative variables. For two joint discrete distributions $\mathbf{s} = (s_{i_1, i_2, \ldots, i_n}, 1 \le i_1, i_2, \ldots, i_n \le m)$ and $\tilde{\mathbf{s}} = (\tilde{s}_{i_1, i_2, \ldots, i_n}, 1 \le i_1, i_2, \ldots, i_n \le m)$ on the integers $\{1, 2, \ldots, m\}$, we write $\mathbf{s} \le^{\mathrm{st}} \tilde{\mathbf{s}}$ if and only if

$$\sum_{j_1 \le i_1 \le m, j_2 \le i_2 \le m, \ldots, j_n \le i_n \le m} s_{i_1, i_2, \ldots, i_n} \le \sum_{j_1 \le i_1 \le m, j_2 \le i_2 \le m, \ldots, j_n \le i_n \le m} \tilde{s}_{i_1, i_2, \ldots, i_n}$$

for all $1 \le j_1, j_2, \ldots, j_n \le m$. For two vectors of non-negative variables $(T_1, T_2, \ldots, T_n)$ and $(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$, we write $(T_1, T_2, \ldots, T_n) \le^{\mathrm{st}} (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$ if and only if

$$\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} \le \mathbb{P}\{\tilde{T}_1 > t_1, \tilde{T}_2 > t_2, \ldots, \tilde{T}_n > t_n\}$$

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for $0 \le t_1, t_2, \ldots, t_n < \infty$. Note that the multivariate stochastic ordering $\le^{\mathrm{st}}$ here is less strong than the commonly used one (which is denoted by $\le_{\mathrm{st}}$) for multivariate stochastic comparison in the applied probability literature, which means it can be referred to as weak stochastic ordering when necessary. Then the relationship between stochastically ordered signatures and system lifetimes can be given for multi-state systems with binary-state components as follows.

Theorem 2.2. *Assume that* $\mathbf{s}$ *and* $\tilde{\mathbf{s}}$ *are signatures of two multi-state systems, each of which has* $n+1$ *system states and* m *binary-state components. If* $\mathbf{s} \le^{\mathrm{st}} \tilde{\mathbf{s}}$, *then the lifetime vectors of the two systems satisfy* $(T_1, T_2, \ldots, T_n) \le^{\mathrm{st}} (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$.

Proof. Due to Remark 2.2(a), we only need to prove

$$P\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} \le P\{\tilde{T}_1 > t_1, \tilde{T}_2 > t_2, \ldots, \tilde{T}_n > t_n\}$$

for $0 \le t_1 \le t_2 \le \cdots \le t_n < \infty$. For arbitrary $0 \le t_1, t_2, \ldots, t_n < \infty$, from Theorem 2.1, we have

$$\begin{aligned}&\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\}\\ &\quad = \sum_{1 \le i_1, i_2, \ldots, i_n \le m} s_{i_1, i_2, \ldots, i_n} \mathbb{P}\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n\},\end{aligned}$$

where

$$\begin{aligned}&\mathbb{P}\{X_{i_1:m} > t_1, X_{i_2:m} > t_2, \ldots, X_{i_n:m} > t_n\}\\ &\quad = \sum_{0 \le j_1 \le i_1 - 1, j_1 \le j_2 \le i_2 - 1, \ldots, j_{n-1} \le j_n \le i_n - 1} F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1}\\ &\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n).\end{aligned}$$

After exchanging the order of summations, we get

$$\begin{aligned}&\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\}\\ &\quad = \sum_{0 \le j_1 \le j_2 \le \cdots \le j_n \le m-1} \left(\sum_{j_1 + 1 \le i_1 \le m, j_2 + 1 \le i_2 \le m, \ldots, j_n + 1 \le i_n \le m} s_{i_1, i_2, \ldots, i_n} \right) F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1}\\ &\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\mathbb{P}\{\tilde{T}_1 > t_1, \tilde{T}_2 > t_2, \ldots, \tilde{T}_n > t_n\}\\ &\quad = \sum_{0 \le j_1 \le j_2 \le \cdots \le j_n \le m-1} \left(\sum_{j_1 + 1 \le i_1 \le m, j_2 + 1 \le i_2 \le m, \ldots, j_n + 1 \le i_n \le m} \tilde{s}_{i_1, i_2, \ldots, i_n} \right) F^{j_1}(t_1)[F(t_2) - F(t_1)]^{j_2 - j_1}\\ &\qquad \times \cdots \times [F(t_n) - F(t_{n-1})]^{j_n - j_{n-1}} F^{m - j_n}(t_n).\end{aligned}$$

From the fact that $\mathbf{s} \le^{\mathrm{st}} \tilde{\mathbf{s}}$, we have

$$\sum_{j_1 \le i_1 \le m, j_2 \le i_2 \le m, \ldots, j_n \le i_n \le m} s_{i_1, i_2, \ldots, i_n} \le \sum_{j_1 \le i_1 \le m, j_2 \le i_2 \le m, \ldots, j_n \le i_n \le m} \tilde{s}_{i_1, i_2, \ldots, i_n}$$

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for all $1 \le j_1, j_2, \ldots, j_n \le m$, which leads to the fact that

$$\mathbb{P}\{T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n\} \le \mathbb{P}\{\tilde{T}_1 > t_1, \tilde{T}_2 > t_2, \ldots, \tilde{T}_n > t_n\}$$

for arbitrary $0 \le t_1, t_2, \ldots, t_n < \infty$, as required. □

3. The ordered multi-state system signature

In this section we will generalize the results about ordered system signature in [3] to multi-state systems with binary-state components, and discuss some of its properties. Consider a life-test in which l independent multi-state systems are tested, and the systems have n binary-state components each and share a multi-state system signature $\mathbf{s}$. The lifetimes X_i^h ($i = 1, 2, \ldots, m$, $h = 1, 2, \ldots, l$) for component i in system h are i.i.d. with a common absolutely continuous distribution function $F(x)$, $x \ge 0$. Let $(T_1^h, T_2^h, \ldots, T_n^h)$ denote the lifetime vector for system h ($h = 1, 2, \ldots, l$), where T_k^h ($k = 1, 2, \ldots, n$) is the first time that system h enters state $n - k$ or below. As is often the case with life-testing experiments, we may not want to wait until all the l systems enter complete failure state 0. In this case, the ordered multi-state system signature defined below becomes more useful than the multi-state system signature itself.

Definition 3.1. (*Ordered multi-state signature.*) The ordered multi-state system signature

$$\mathbf{s}^{qr} = (s^{qr}_{j_1,j_2,\ldots,j_n},\ 1 \le j_1, j_2, \ldots, j_n \le m)$$

for the qth system that enters state $n - r$ or below ($q = 1, 2, \ldots, l$, $r = 1, 2, \ldots, n$) is defined as

$$s^{qr}_{j_1,j_2,\ldots,j_n} = \mathbb{P}\{T_1^p = X^p_{j_1:m}, T_2^p = X^p_{j_2:m}, \ldots, T_n^p = X^p_{j_n:m} \mid T_r^{q:l} = T_r^p\},$$

where $T_r^{q:l}$ is the qth order statistic of $T_r^1, T_r^2, \ldots, T_r^l$ and $X^p_{j_1:m}, X^p_{j_2:m}, \ldots, X^p_{j_n:m}$ are the j_1th, j_2th, …, j_nth order statistics of $X_1^p, X_2^p, \ldots, X_m^p$, respectively.

Remark 3.1.

(a) The event $\{T_r^{q:l} = T_r^p\}$ means that of all the l multi-state systems, the qth one that enters state $n - r$ or below is system p, namely, the qth smallest one among lifetimes $T_r^1, T_r^2, \ldots, T_r^l$ is T_r^p.

(b) $s^{qr}_{j_1,j_2,\ldots,j_n}$ is the conditional probability that system p enters state $n - 1$ or below at the j_1th component failure, then enters state $n - 2$ or below at the j_2th component failure, and so on, given that system p is the qth system that enters state $n - r$ or below. Here, p can be any of $1, 2, \ldots, l$, and does not have anything to do with the value of $s^{qr}_{j_1,j_2,\ldots,j_n}$.

(c) As in the case of multi-state signature, the ordered multi-state signature is also a generalized matrix with dimension

$$\underbrace{m \times \cdots \times m}_{n}$$

which can be regarded as a joint conditional probability distribution of discrete random variables, namely, $s^{qr}_{j_1,j_2,\ldots,j_n} \ge 0$ for any $1 \le j_1, j_2, \ldots, j_n \le m$ and

$$\sum_{1 \le j_1, j_2, \ldots, j_n \le m} s^{qr}_{j_1,j_2,\ldots,j_n} = 1.$$

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(d) Let $l_{i_1,i_2,\ldots,i_n}$ $(1 \le i_1, i_2, \ldots, i_n \le m)$ be the number of systems that satisfy $T_1^h = X_{i_1:m}^h$, $T_2^h = X_{i_2:m}^h, \ldots, T_n^h = X_{i_n:m}^h$ $(h = 1, 2, \ldots, l)$ (i.e. $I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n$, which means the system enters state $n-1$ or below at the component failure, then enters state $n-2$ or below at the i_2th component failure, and so on). Then, from the fact that the l identical multi-state systems are independent of each other, it is evident that $l_{i_1,i_2,\ldots,i_n}$ $(1 \le i_1, i_2, \ldots, i_n \le m)$ follows a multinomial distribution with parameters l and $s_{i_1,i_2,\ldots,i_n}$ $(1 \le i_1, i_2, \ldots, i_n \le m)$, with state space

$$\mathcal{L}_{l,m} = \Bigg\{ \mathbf{l} = (l_{i_1,i_2,\ldots,i_n},\, 1 \le i_1, i_2, \ldots, i_n \le m) : \sum_{1 \le i_1,i_2,\ldots,i_n \le m} l_{i_1,i_2,\ldots,i_n} = l,\, l_{i_1,i_2,\ldots,i_n} = 1, 2, \ldots, l \Bigg\}.$$

See Examples 4.2 and 4.4 for $\mathcal{L}_{l,m}$ with different parameters l and m.

For the ordered multi-state system signature defined above, the following theorem shows that it is determined only by the system structure and does not depend on the component lifetime distribution. Before introducing the theorem, let $p_{j_1,j_2,\ldots,j_n|\mathbf{l}}^{qr}$ denote the conditional probability that the qth system entering state $n-r$ or below (say, system p) satisfies $T_1^p = X_{j_1:m}^p$, $T_2^p = X_{j_2:m}^p, \ldots, T_n^p = X_{j_n:m}^p$, given $\mathbf{l} \in \mathcal{L}_{l,m}$. It needs to be noted that for each given $\mathbf{l} \in \mathcal{L}_{l,m}$, there are $\sum_{1 \le i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n \le m} l_{i_1,i_2,\ldots,i_n}$ of the l multi-state systems that enter state $n-r$ or below at the i_rth $(i_r = 1, 2, \ldots, m)$ component failure.

Theorem 3.1. *The ordered multi-state system signature*

$$\mathbf{s}^{qr} = (s_{j_1,j_2,\ldots,j_n}^{qr},\, 1 \le j_1, j_2, \ldots, j_n \le m)$$

is free of the underlying component lifetime distribution F, and is thus a distribution-free measure.

Proof. From Remark 3.1(d), the probability that there are $l_{i_1,i_2,\ldots,i_n}$ $(1 \le i_1, i_2, \ldots, i_n \le m)$ systems satisfying $T_1^h = X_{i_1:m}^h$, $T_2^h = X_{i_2:m}^h, \ldots, T_n^h = X_{i_n:m}^h$ $(h = 1, 2, \ldots, l)$ is given by

$$\binom{l}{l_{i_1,i_2,\ldots,i_n},\, 1 \le i_1, i_2, \ldots, i_n \le m} \left\{ \prod_{1 \le i_1,i_2,\ldots,i_n \le m} s_{i_1,i_2,\ldots,i_n}^{l_{i_1,i_2,\ldots,i_n}} \right\}.$$

Then, by using the law of total probability, the ordered multi-state system signature can be calculated as follows:

$$s_{j_1,j_2,\ldots,j_n}^{qr} = \sum_{\mathbf{l} \in \mathcal{L}_{l,m}} \binom{l}{l_{i_1,i_2,\ldots,i_n},\, 1 \le i_1, i_2, \ldots, i_n \le m} \left\{ \prod_{1 \le i_1,i_2,\ldots,i_n \le m} s_{i_1,i_2,\ldots,i_n}^{l_{i_1,i_2,\ldots,i_n}} \right\} p_{j_1,j_2,\ldots,j_n|\mathbf{l}}^{qr}.$$

Given that $\mathbf{l} \in \mathcal{L}_{l,m}$, the performance of all the l systems is classified by the conditions $I_1(\phi) = i_1, I_2(\phi) = i_2, \ldots, I_n(\phi) = i_n$ $(1 \le i_1, i_2, \ldots, i_n \le m)$ they satisfy. Specifically, there are

$$\sum_{1 \le i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n \le m} l_{i_1,i_2,\ldots,i_n} \quad (1 \le i_r \le m)$$

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of the l systems that enter state $n-r$ or below at the i_rth component failure at time $X^h_{i_r:m}$ $(h=1,2,\ldots,l)$. This means whether the qth one of them satisfies the conditions $I_1(\phi)=j_1, I_2(\phi)=j_2,\ldots,I_n(\phi)=j_n$ will depend only on the ordering of the l order statistics, $\sum_{1\le i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n\le m} l_{i_1,i_2,\ldots,i_n}$ of which are $X^h_{i_r:m}$. Since X^h_i $(i=1,2,\ldots,m,\ h=1,2,\ldots,l)$ are i.i.d., the orderings of these order statistics are distribution-free, which implies that $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ is distribution-free as well. □

To simplify the calculation of the ordered multi-state system signature and to help explain this newly introduced concept, we discuss some properties of the important term $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ next. For any generalized matrix $\mathbf{l}=(l_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m)$ with dimension

$$\underbrace{m\times\cdots\times m}_{n},$$

let its reverse matrix be defined by

$$rev\,\mathbf{l}:=(\tilde{l}_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m),$$

where $\tilde{l}_{j_1,j_2,\ldots,j_n}=l_{m-j_n+1,m-j_{n-1}+1,\ldots,m-j_1+1}$, that is,

$$(rev\,\mathbf{l})_{j_1,j_2,\ldots,j_n}=(\mathbf{l})_{m-j_n+1,m-j_{n-1}+1,\ldots,m-j_1+1}$$

for all $1\le j_1,j_2,\ldots,j_n\le m$.

Theorem 3.2. *The conditional probabilities $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ satisfy the following properties:*

(1) *If $l_{j_1,j_2,\ldots,j_n}=l$, then $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=1$.*

(2) *If $l_{j_1,j_2,\ldots,j_n}=0$, then $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=0$.*

(3) $\sum_{q=1}^{l} p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=l_{j_1,j_2,\ldots,j_n}$.

(4) $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=p^{(l-q+1)(n-r+1)}_{m-j_n+1,m-j_{n-1}+1,\ldots,m-j_1+1|rev\,\mathbf{l}}$.

Proof. Parts (1) and (2) are trivially true, and (3) is true because there are $l_{j_1,j_2,\ldots,j_n}$ of $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ $(q=1,2,\ldots,l)$ equal to 1 and $l-l_{j_1,j_2,\ldots,j_n}$ of $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ equal to 0 for any permutation of T^h_r $(h=1,2,\ldots,l)$ given $\mathbf{l}$, and the probabilities of these permutations are the same since the X^h_i are i.i.d. For (4), by transformation $X^h_i\to -X^h_i$ and state $r\to n-r+1$ (here $-X^h_i$ means less than a given standard value by X^h_i instead of a negative lifetime), the qth smallest system lifetime before transform will become the $(l-q+1)$th smallest after transform, the j_1th, j_2th, …, j_nth smallest component lifetimes that cause the system to fail at states $n, n-1,\ldots,1$ before transform will become the $(m-j_n+1)$th, $(m-j_{n-1}+1)$th, …, $(m-j_1+1)$th smallest component lifetimes that cause the related system to fail at states $n, n-1,\ldots,1$ after transform, and entering state $n-r$ or below before transform will become entering state $r+1$ or below after transform. Hence, the equality is true since $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}$ is distribution-free. □

From Theorem 3.2, the following corollary describes the connections between the zero elements in the signature and the ordered signature of multi-state systems with binary-state components.

Corollary 3.1. *For any multi-state signature* $\mathbf{s}$ *and ordered multi-state signature* $\mathbf{s}^{qr}$ $(r=1,2,\ldots,n,\ q=1,2,\ldots,l)$, *all $s^{qr}_{j_1,j_2,\ldots,j_n}=0$ if and only if $s_{j_1,j_2,\ldots,j_n}=0$.*

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Proof.

(1) As discussed in Remark 2.1(c), $s_{j_1,j_2,\ldots,j_n}=0$ means that the event $\{I_1(\phi)=j_1, I_2(\phi)=j_2,\ldots,I_n(\phi)=j_n\}$ happens with probability 0. Then, for any life-test involving multi-state systems with signature $\mathbf{s}$ that satisfies $s_{j_1,j_2,\ldots,j_n}=0$, there will be no system that makes $\{I_1(\phi)=j_1, I_2(\phi)=j_2,\ldots,I_n(\phi)=j_n\}$ true, which obviously means that $l_{j_1,j_2,\ldots,j_n}=0$. Then, from part (2) of Theorem 3.2, we have $p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=0$ ($r=1,2,\ldots,n$, $q=1,2,\ldots,l$), which means $s^{qr}_{j_1,j_2,\ldots,j_n}=0$, as required.

(2) If $s^{qr}_{j_1,j_2,\ldots,j_n}=0$, from Theorem 3.1, we have

$$\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\ldots,i_n},\,1\le i_1,i_2,\ldots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=0,$$

which means

$$\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=0$$

for all $\mathbf{l}$. Since

$$p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=1\quad\text{for }\mathbf{l}=(l_{i_1,i_2,\ldots,i_n},\,1\le i_1,i_2,\ldots,i_n\le m),$$

where $l_{j_1,j_2,\ldots,j_n}=l$ and $l_{i_1,i_2,\ldots,i_n}=0$ otherwise, we must have

$$\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=s^{l}_{j_1,j_2,\ldots,j_n}=0,$$

which means $s_{j_1,j_2,\ldots,j_n}=0$, as required. □

Below we present some more properties of ordered multi-state system signatures, to assist in reducing the amount of computation required for ordered multi-state system signature.

Theorem 3.3. *The ordered signatures satisfy*

$$\frac{1}{l}\sum_{q=1}^{l}\mathbf{s}^{qr}=\mathbf{s}\quad\text{and}\quad rev\,\mathbf{s}^{qr}=(rev\,\mathbf{s})^{(l-q+1)(n-r+1)}.$$

Proof.

(1) From Theorem 3.1 and part (3) of Theorem 3.2, we have

$$\begin{aligned}&\sum_{q=1}^{l}s^{qr}_{j_1,j_2,\ldots,j_n}\\&=\sum_{q=1}^{l}\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\ldots,i_n},\,1\le i_1,i_2,\ldots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}p^{qr}_{j_1,j_2,\ldots,j_n|\mathbf{l}}\\&=\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\ldots,i_n},\,1\le i_1,i_2,\ldots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}l_{j_1,j_2,\ldots,j_n},\end{aligned}$$

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which can be regarded as the expected number of systems satisfying $I_1(\phi)=j_1, I_2(\phi)=j_2, \ldots, I_n(\phi)=j_n$. Then, from Remark 2.1(d), we evidently have

$$\sum_{q=1}^{l} s^{qr}_{j_1,j_2,\ldots,j_n} = l \cdot s_{j_1,j_2,\ldots,j_n}.$$

(2) Here, for $\mathbf{s}=(s_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m)$,

$$rev\,\mathbf{s}=(\tilde{s}_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m),\ \tilde{s}_{j_1,j_2,\ldots,j_n}=s_{m-j_n+1,m-j_{n-1}+1,\ldots,m-j_1+1}.$$

Similarly, for $\mathbf{s}^{qr}=(s^{qr}_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m)$,

$$rev\,\mathbf{s}^{qr}=(\tilde{s}^{qr}_{j_1,j_2,\ldots,j_n},\ 1\le j_1,j_2,\ldots,j_n\le m),\ \tilde{s}^{qr}_{j_1,j_2,\ldots,j_n}=s^{qr}_{m-j_n+1,m-j_{n-1}+1,\ldots,m-j_1+1}.$$

From Theorem 3.1 and part (4) of Theorem 3.2, we have

$$\begin{aligned}
&((\,rev\,\mathbf{s})^{(l-q+1)(n-r+1)})_{j_1,j_2,\ldots,j_n}\\
&\quad=\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\ldots,i_n},\ 1\le i_1,i_2,\ldots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}\tilde{s}^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n}\right\}p^{(l-q+1)(n-r+1)}_{j_1,j_2,\ldots,j_n|\mathbf{l}}\\
&\quad=\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\ldots,i_n},\ 1\le i_1,i_2,\ldots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}\right\}\\
&\qquad\times p^{qr}_{m-j_n+1,m-j_2+1,\ldots,m-j_n+1|rev\,\mathbf{l}}.
\end{aligned}$$

Then, by variable substitution, we have

$$\begin{aligned}
&((\,rev\,\mathbf{s})^{(l-q+1)(n-r+1)})_{j_1,j_2,\ldots,j_n}\\
&\quad=\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1},\ 1\le i_1,i_2,\ldots,i_n\le m}\\
&\qquad\times\left\{\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}}_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}\right\}p^{qr}_{m-j_n+1,m-j_2+1,\ldots,m-j_n+1|\mathbf{l}},
\end{aligned}$$

where

$$\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}}_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}=\prod_{1\le i_1,i_2,\ldots,i_n\le m}s^{l_{i_1,i_2,\ldots,i_n}}_{i_1,i_2,\ldots,i_n},$$

and

$$\begin{aligned}
&\binom{l}{l_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1},\ 1\le i_1,i_2,\ldots,i_n\le m}\\
&\quad=\frac{l!}{\prod_{1\le i_1,i_2,\ldots,i_n\le m}l_{m-i_n+1,m-i_{n-1}+1,\ldots,m-i_1+1}!}\\
&\quad=\frac{l!}{\prod_{1\le i_1,i_2,\ldots,i_n\le m}l_{i_1,i_2,\ldots,i_n}!}\\
&\quad=\binom{l}{l_{i_1,i_2,\ldots,i_n},\ 1\le i_1,i_2,\ldots,i_n\le m}.
\end{aligned}$$

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This means that

$$\begin{aligned}
&((\,rev\,\mathbf{s})^{(l-q+1)(n-r+1)})_{j_1,j_2,\ldots,j_n} \\
&\quad = \sum_{\mathbf{l}\in\mathcal{L}_{l,m}} \binom{l}{l_{i_1,i_2,\ldots,i_n},\, 1\le i_1,\, i_2,\,\ldots,\, i_n\le m} \\
&\qquad \times \left\{ \prod_{1\le i_1,i_2,\ldots,i_n\le m} s_{i_1,i_2,\ldots,i_n}^{l_{i_1,i_2,\ldots,i_n}} \right\} p^{qr}_{m-j_n+1,m-j_2+1,\ldots,m-j_n+1|\mathbf{l}} \\
&\quad = (\,rev\,\mathbf{s}^{qr})_{j_1,j_2,\ldots,j_n},
\end{aligned}$$

as required. □

Remark 3.2. The property

$$\frac{1}{l}\sum_{q=1}^{l}\mathbf{s}^{qr}=\mathbf{s}$$

shows that

$$\frac{1}{l}\sum_{q=1}^{l}\mathbf{s}^{qr}$$

does not depend on r, and this is so because it is the average of ordered signatures for all the l multi-state systems with signature $\mathbf{s}$, and that parameter r is only used for ordering the systems entering state $n-r$ or below.

The following corollary can be obtained directly from Theorem 3.3 as a special case.

Corollary 3.2. *If a multi-state system signature is symmetric (i.e.* $\mathbf{s}=rev\,\mathbf{s}$*), then the ordered system signature satisfies* $rev\,\mathbf{s}^{qr}=\mathbf{s}^{(l-q+1)(n-r+1)}$.

Another interesting property presented below in Theorem 3.4 gives a sufficient condition for the ordered multi-state signature to equal the multi-state signature.

Theorem 3.4. *For any* $k\in\{1,2,\ldots,n\}$, *if* $s_{i_1,i_2,\ldots,i_n}=0$ *for all* $1\le i_1,\, i_2,\,\ldots,\, i_n\le m$ *with* $i_k\neq a$, *then* $\mathbf{s}^{1k}=\mathbf{s}^{2k}=\cdots=\mathbf{s}^{lk}=\mathbf{s}$.

Proof. For any $k\in\{1,2,\ldots,n\}$, $s_{i_1,i_2,\ldots,i_n}=0$ for all $i_k\neq a$ means $s_{i_1,i_2,\ldots,i_n}>0$ only if $i_k=a$, that is, the system must enter state $n-k$ or below at the ath component failure. Then all the l multi-state systems with multi-state signature $\mathbf{s}$ must enter state $n-k$ or below at the ath component failure. Since probabilities of orderings of the l i.i.d. $X_{a:m}$ are all the same, from the proof of Theorem 3.1, we immediately have

$$p^{1k}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=p^{2k}_{j_1,j_2,\ldots,j_n|\mathbf{l}}=\cdots=p^{lk}_{j_1,j_2,\ldots,j_n|\mathbf{l}}\quad\text{for all } 1\le j_1,\, j_2,\,\ldots,\, j_n\le m,$$

which means that $\mathbf{s}^{1k}=\mathbf{s}^{2k}=\cdots=\mathbf{s}^{lk}$. Then, by Theorem 3.3, we clearly have $\mathbf{s}^{1k}=\mathbf{s}^{2k}=\cdots=\mathbf{s}^{lk}=\mathbf{s}$. □

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In a life-test in which l independent multi-state systems of the same type are tested, a system that enters state $n-r$ $(r=1, 2, \ldots, n)$ or below earlier indicates a worse performance than the others. This property is discussed in terms of stochastic ordering in the following theorem.

Theorem 3.5.

(1) *For any* $1 \le q_1 < q_2 \le l$, *the ordered signatures satisfy* $\mathbf{s}^{q_1 r} \le^{\mathrm{st}} \mathbf{s}^{q_2 r}$.

(2) *If* $\mathbf{s}^{q_1 r} \ge^{\mathrm{st}} \mathbf{s}^{q_2 r}$ *for any* $1 \le q_1 < q_2 \le l$, *then there is at most one positive number in* $s_{j_1, j_2, \ldots, j_n}$ $(j_r = 1, 2, \ldots, m)$ *for any* $1 \le j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_n \le m$.

Remark 3.3. The proof of Theorem 3.5 is presented in the Appendix, and one may also see Remarks 4.1 and 4.2 for illustrations of these properties.

4. Illustrative examples

In this section we present several examples to illustrate the results established in the preceding sections. In what follows, Examples 4.1 and 4.3 show the computation of the system signatures of two multi-state systems with binary-state components, while Examples 4.2 and 4.4 show the computation of the corresponding ordered multi-state system signatures, respectively.

Example 4.1. Consider a power system that consists of two binary-state components. The production capacity of the system is determined by the highest production capacity of its components, which makes it a multi-state system. Assume that component 1 has a better production capacity than component 2, and that the structure of the system is given by $\phi(x_1, x_2) = \max(2x_1, x_2)$, where $x_1, x_2 \in \{0, 1\}$. Then there are two permutations of X_i $(i=1, 2)$ when component lifetimes X_i $(i=1, 2)$ are i.i.d. with an absolutely continuous distribution $F(x)$, $x \ge 0$. Note that the system would enter states below 2 if component 1 fails, and enter state 0 if all components fail, which means the multi-state signature of this power system is given by

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}.$$

Example 4.2. Assume that there are two power systems that have the same structure as in Example 4.1, and let us denote signatures of the first and second of them that enter state 1 or below as $\mathbf{s}^{11}$ and $\mathbf{s}^{21}$, and that enter state 0 as $\mathbf{s}^{12}$ and $\mathbf{s}^{22}$, respectively. In the following, we will compute the ordered multi-state system signatures $\mathbf{s}^{11}$, $\mathbf{s}^{21}$, $\mathbf{s}^{12}$, $\mathbf{s}^{22}$ of the two systems by their common multi-state system signature $\mathbf{s}$.

(1) To get $\mathbf{s}^{11}$, we first need to get

$$p^{11}_{11|\mathbf{l}}, p^{11}_{12|\mathbf{l}}, p^{11}_{21|\mathbf{l}}, p^{11}_{22|\mathbf{l}},$$

where

$$\mathbf{l} = (l_{12}, l_{22}) \in \mathcal{L}_{2,2} \quad \text{and} \quad \mathcal{L}_{2,2} = \{(l_{12}, l_{22}) \colon l_{12} + l_{22} = 2\} = \{(2, 0), (1, 1), (0, 2)\}.$$

Since there are only two positive elements s_{12}, s_{22} in $\mathbf{s}$, we only need to get $p^{11}_{12|\mathbf{l}}, p^{11}_{22|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,2}$, as given in Table 1. Note that

$$\mathrm{d}F_{1:2}(x) = 2[1 - F(x)]\,\mathrm{d}F(x), \quad \mathrm{d}F_{2:2}(x) = 2F(x)\,\mathrm{d}F(x),$$

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TABLE 1: Conditional probabilities $p^{11}_{12|\mathbf{l}}, p^{11}_{22|\mathbf{l}}, p^{21}_{12|\mathbf{l}}, p^{21}_{22|\mathbf{l}}, p^{12}_{12|\mathbf{l}}, p^{12}_{22|\mathbf{l}}, p^{22}_{12|\mathbf{l}}, p^{22}_{22|\mathbf{l}}$ for Example 4.2.

$\mathbf{l} = (l_{12}, l_{22})$	$(2,0)$	$(1,1)$	$(0,2)$
$p^{11}_{12\|\mathbf{l}}$	1	5/6	0
$p^{11}_{22\|\mathbf{l}}$	0	1/6	1
$p^{21}_{12\|\mathbf{l}}$	1	1/6	0
$p^{21}_{22\|\mathbf{l}}$	0	5/6	1
$p^{12}_{12\|\mathbf{l}}$	1	1/2	0
$p^{12}_{22\|\mathbf{l}}$	0	1/2	1
$p^{22}_{12\|\mathbf{l}}$	1	1/2	0
$p^{22}_{22\|\mathbf{l}}$	0	1/2	1

which means

$$\mathbb{P}\{X_{1:2} < X_{2:2}\} = \int_0^1 \int_0^v 2(1-u) \times 2v \, \mathrm{d}u \, \mathrm{d}v = \frac{5}{6}.$$

Then

$$s^{11}_{12} = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{5}{6} + \left(\frac{1}{2}\right)^2 \times 0 = \frac{2}{3},$$
$$s^{11}_{22} = \left(\frac{1}{2}\right)^2 \times 0 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 = \frac{1}{3},$$

which yields

$$\mathbf{s}^{11} = \begin{pmatrix} 0 & 2/3 \\ 0 & 1/3 \end{pmatrix}.$$

(2) To get $\mathbf{s}^{21}$, we first need to get

$$p^{21}_{11|\mathbf{l}}, p^{21}_{12|\mathbf{l}}, p^{21}_{21|\mathbf{l}}, p^{21}_{22|\mathbf{l}},$$

where

$$\mathbf{l} = (l_{12}, l_{22}) \in \mathcal{L}_{2,2} \quad \text{and} \quad \mathcal{L}_{2,2} = \{(l_{12}, l_{22}) \colon l_{12} + l_{22} = 2\} = \{(2,0), (1,1), (0,2)\}.$$

Since there are only two positive elements s_{12}, s_{22} in $\mathbf{s}$, we only need to get $p^{21}_{12|\mathbf{l}}, p^{21}_{22|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,2}$, as given in Table 1. Then

$$s^{21}_{12} = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 \times 0 = \frac{1}{3},$$
$$s^{21}_{22} = \left(\frac{1}{2}\right)^2 \times 0 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{5}{6} + \left(\frac{1}{2}\right)^2 = \frac{2}{3},$$

which yields

$$\mathbf{s}^{21} = \begin{pmatrix} 0 & 1/3 \\ 0 & 2/3 \end{pmatrix}.$$

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(3) To get $\mathbf{s}^{12}$, we first need to get

$$p^{12}_{11|\mathbf{l}}, p^{12}_{12|\mathbf{l}}, p^{12}_{21|\mathbf{l}}, p^{12}_{22|\mathbf{l}},$$

where

$$\mathbf{l} = (l_{12}, l_{22}) \in \mathcal{L}_{2,2} \quad \text{and} \quad \mathcal{L}_{2,2} = \{(l_{12}, l_{22}) \colon l_{12} + l_{22} = 2\} = \{(2, 0), (1, 1), (0, 2)\}.$$

Since there are only two positive elements s_{12}, s_{22} in $\mathbf{s}$, we only need to get $p^{21}_{12|\mathbf{l}}, p^{21}_{22|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,2}$, as given in Table 1. Then

$$s^{12}_{12} = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 \times 0 = \frac{1}{2},$$
$$s^{12}_{22} = \left(\frac{1}{2}\right)^2 \times 0 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \frac{1}{2},$$

which yields

$$\mathbf{s}^{12} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}.$$

(4) To get $\mathbf{s}^{22}$, we first need to get

$$p^{22}_{11|\mathbf{l}}, p^{22}_{12|\mathbf{l}}, p^{22}_{21|\mathbf{l}}, p^{22}_{22|\mathbf{l}},$$

where

$$\mathbf{l} = (l_{12}, l_{22}) \in \mathcal{L}_{2,2} \quad \text{and} \quad \mathcal{L}_{2,2} = \{(l_{12}, l_{22}) \colon l_{12} + l_{22} = 2\} = \{(2, 0), (1, 1), (0, 2)\}.$$

Since there are only two positive elements s_{12}, s_{22} in $\mathbf{s}$, we only need to get $\mathbf{l} \in \mathcal{L}_{2,2}$ for all $\mathbf{l} \in L_{2,2}$, as given in Table 1. Then

$$s^{22}_{12} = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 \times 0 = \frac{1}{2},$$
$$s^{22}_{22} = \left(\frac{1}{2}\right)^2 \times 0 + 2 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \frac{1}{2},$$

which yields

$$\mathbf{s}^{22} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}.$$

Remark 4.1.

(a) We observe $\mathbf{s}^{11} \leq^{\text{st}} \mathbf{s} \leq^{\text{st}} \mathbf{s}^{21}$, which means the second power system of the two that enters state 0 is better than the original design and the first of them is worse than the original design.

(b) We observe $\mathbf{s}^{12} =^{\text{st}} \mathbf{s} =^{\text{st}} \mathbf{s}^{22}$, which means the first and second power systems of the two that enter state 1 or below are the same as the original design.

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Example 4.3. Consider a power system that consists of three binary-state components. The production capacity of the system is determined by the highest production capacity of its components, which makes it a multi-state system. Assume that component 1 has a better production capacity than components 2 and 3, and that the structure of the system is given by $\phi(x_1, x_2, x_3) = \max(2x_1, x_2, x_3)$, where $x_1, x_2, x_3 \in \{0, 1\}$. Then there are six permutations of X_i $(i = 1, 2, 3)$ when component lifetimes X_i $(i = 1, 2, 3)$ are i.i.d. with an absolutely continuous distribution $F(x)$, $x \geq 0$. Note that the system enters state 1 or below if component 1 fails and enters state 0 if all components fail, which means the multi-state signature of the power system is given by

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Example 4.4. Assume that there are two power systems that have the same structure as in Example 4.3, and let us denote signatures of the first and second systems that enter state 1 as $\mathbf{s}^{11}$ and $\mathbf{s}^{21}$, and that enter state 0 as $\mathbf{s}^{12}$ and $\mathbf{s}^{22}$, respectively. In the following, we will compute the ordered multi-state system signatures $\mathbf{s}^{11}$, $\mathbf{s}^{21}$, $\mathbf{s}^{12}$, $\mathbf{s}^{22}$ of the two systems by their common multi-state system signature $\mathbf{s}$.

(1) To get $\mathbf{s}^{11}$, we first need to get

$$p^{11}_{11|\mathbf{l}}, p^{11}_{12|\mathbf{l}}, p^{11}_{13|\mathbf{l}}, p^{11}_{21|\mathbf{l}}, p^{11}_{22|\mathbf{l}}, p^{11}_{23|\mathbf{l}}, p^{11}_{31|\mathbf{l}}, p^{11}_{32|\mathbf{l}}, p^{11}_{33|\mathbf{l}},$$

where

$$\begin{aligned} \mathbf{l} &= (l_{13}, l_{23}, l_{33}) \in \mathcal{L}_{2,3}, \\ \mathcal{L}_{2,3} &= \{(l_{13}, l_{23}, l_{33}) \colon l_{13} + l_{23} + l_{33} = 2\} \\ &= \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}. \end{aligned}$$

Since there are only three positive elements s_{13}, s_{23}, s_{33} in $\mathbf{s}$, we only need to get $p^{11}_{13|\mathbf{l}}, p^{11}_{23|\mathbf{l}}, p^{11}_{33|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,3}$, as given in Table 2. Note that

$$\begin{aligned} \mathrm{d}F_{1:3}(x) &= 3[1 - F(x)]^2 \, \mathrm{d}F(x), \\ \mathrm{d}F_{2:3}(x) &= 6F(x)[1 - F(x)] \, \mathrm{d}F(x), \\ \mathrm{d}F_{3:3}(x) &= 3F^2(x) \, \mathrm{d}F(x), \end{aligned}$$

which means that

$$\begin{aligned} P\{X_{1:3} < X_{2:3}\} &= \int_0^1 \int_0^v 3(1-u)^2 \times 6v(1-v) \, \mathrm{d}u \, \mathrm{d}v = \frac{4}{5}, \\ P\{X_{1:3} < X_{3:3}\} &= \int_0^1 \int_0^v 3(1-u)^2 \times 3v^2 \, \mathrm{d}u \, \mathrm{d}v = \frac{19}{20}, \\ P\{X_{2:3} < X_{3:3}\} &= \int_0^1 \int_0^v 6u(1-u) \times 3v^2 \, \mathrm{d}u \, \mathrm{d}v = \frac{4}{5}. \end{aligned}$$

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TABLE 2: Conditional probabilities $p^{11}_{13|\mathbf{l}}, p^{11}_{23|\mathbf{l}}, p^{11}_{33|\mathbf{l}}, p^{21}_{13|\mathbf{l}}, p^{21}_{23|\mathbf{l}}, p^{21}_{33|\mathbf{l}}, p^{12}_{13|\mathbf{l}}, p^{12}_{23|\mathbf{l}}, p^{12}_{33|\mathbf{l}}, p^{22}_{13|\mathbf{l}}, p^{22}_{23|\mathbf{l}}, p^{22}_{33|\mathbf{l}}$ for Example 4.4.

$\mathbf{l}=(l_{13}, l_{23}, l_{33})$	(2,0,0)	(0,2,0)	(0,0,2)	(1,1,0)	(1,0,1)	(0,1,1)
$p^{11}_{13\|\mathbf{l}}$	1	0	0	4/5	19/20	0
$p^{11}_{23\|\mathbf{l}}$	0	1	0	1/5	0	4/5
$p^{11}_{33\|\mathbf{l}}$	0	0	1	0	1/20	1/5
$p^{21}_{13\|\mathbf{l}}$	1	0	0	1/5	1/20	0
$p^{21}_{23\|\mathbf{l}}$	0	1	0	4/5	0	1/5
$p^{21}_{33\|\mathbf{l}}$	0	0	1	0	19/20	4/5
$p^{12}_{13\|\mathbf{l}}$	1	0	0	1/2	1/2	0
$p^{12}_{23\|\mathbf{l}}$	0	1	0	1/2	0	1/2
$p^{12}_{33\|\mathbf{l}}$	0	0	1	0	1/2	1/2
$p^{22}_{13\|\mathbf{l}}$	1	0	0	1/2	1/2	0
$p^{22}_{23\|\mathbf{l}}$	0	1	0	1/2	0	1/2
$p^{22}_{33\|\mathbf{l}}$	0	0	1	0	1/2	1/2

Then

$$s^{11}_{13} = \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{4}{5} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{19}{20} + 2 \times \left(\frac{1}{3}\right)^2 \times 0 = \frac{1}{2},$$

$$s^{11}_{23} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{5} + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{4}{5} = \frac{1}{3},$$

$$s^{11}_{33} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{20} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{5} = \frac{1}{6},$$

which yields

$$\mathbf{s}^{11} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/6 \end{pmatrix}.$$

(2) To get $\mathbf{s}^{21}$, we first need to get

$$p^{21}_{11|\mathbf{l}}, p^{21}_{12|\mathbf{l}}, p^{21}_{13|\mathbf{l}}, p^{21}_{21|\mathbf{l}}, p^{21}_{22|\mathbf{l}}, p^{21}_{23|\mathbf{l}}, p^{21}_{31|\mathbf{l}}, p^{21}_{32|\mathbf{l}}, p^{21}_{33|\mathbf{l}},$$

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where

$$\begin{aligned}\mathbf{l} &= (l_{13}, l_{23}, l_{33}) \in \mathcal{L}_{2,3},\\ \mathcal{L}_{2,3} &= \{(l_{13}, l_{23}, l_{33}) \colon l_{13} + l_{23} + l_{33} = 2\}\\ &= \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.\end{aligned}$$

Since there are only three positive elements s_{13}, s_{23}, s_{33} in $\mathbf{s}$, we only need to get $p^{21}_{13|\mathbf{l}}, p^{21}_{23|\mathbf{l}}, p^{21}_{33|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,3}$, as given in Table 2. Then

$$\begin{aligned}s^{21}_{13} &= \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{5} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{20}\\ &\quad + 2 \times \left(\frac{1}{3}\right)^2 \times 0 = \frac{1}{6},\\ s^{21}_{23} &= \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{4}{5} + 2 \times \left(\frac{1}{3}\right)^2 \times 0\\ &\quad + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{5} = \frac{1}{3},\\ s^{21}_{33} &= \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{19}{20}\\ &\quad + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{4}{5} = \frac{1}{2},\end{aligned}$$

which yields

$$\mathbf{s}^{21} = \begin{pmatrix} 0 & 0 & 1/6 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

(3) To get $\mathbf{s}^{12}$, we first need to get

$$p^{12}_{11|\mathbf{l}}, p^{12}_{12|\mathbf{l}}, p^{12}_{13|\mathbf{l}}, p^{12}_{21|\mathbf{l}}, p^{12}_{22|\mathbf{l}}, p^{12}_{23|\mathbf{l}}, p^{12}_{31|\mathbf{l}}, p^{12}_{32|\mathbf{l}}, p^{12}_{33|\mathbf{l}},$$

where

$$\begin{aligned}\mathbf{l} &= (l_{13}, l_{23}, l_{33}) \in \mathcal{L}_{2,3},\\ \mathcal{L}_{2,3} &= \{(l_{13}, l_{23}, l_{33}) \colon l_{13} + l_{23} + l_{33} = 2\}\\ &= \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.\end{aligned}$$

Since there are only three positive elements s_{13}, s_{23}, s_{33} in $\mathbf{s}$, we only need to get $p^{12}_{13|\mathbf{l}}, p^{12}_{23|\mathbf{l}}, p^{12}_{33|\mathbf{l}}$ for all $\mathbf{l} \in \mathcal{L}_{2,3}$, as given in Table 2. Then

$$\begin{aligned}s^{12}_{13} &= \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2}\\ &\quad + 2 \times \left(\frac{1}{3}\right)^2 \times 0 = \frac{1}{3},\end{aligned}$$

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$$s_{23}^{12} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} = \frac{1}{3},$$

$$s_{33}^{12} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} = \frac{1}{3},$$

which yields

$$\mathbf{s}^{12} = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

(4) Finally, to get $\mathbf{s}^{22}$, we first need to get

$$p_{11|\mathbf{l}}^{22}, p_{12|\mathbf{l}}^{22}, p_{13|\mathbf{l}}^{22}, p_{21|\mathbf{l}}^{22}, p_{22|\mathbf{l}}^{22}, p_{23|\mathbf{l}}^{22}, p_{31|\mathbf{l}}^{22}, p_{32|\mathbf{l}}^{22}, p_{33|\mathbf{l}}^{22},$$

where

$$\mathbf{l} = (l_{13}, l_{23}, l_{33}) \in \mathcal{L}_{2,3},$$

$$\mathcal{L}_{2,3} = \{(l_{13}, l_{23}, l_{33}) \colon l_{13} + l_{23} + l_{33} = 2\} = \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.$$

Since there are only three positive elements s_{13}, s_{23}, s_{33} in $\mathbf{s}$, we only need to get $p_{13|\mathbf{l}}^{22}, p_{23|\mathbf{l}}^{22}, p_{33|\mathbf{l}}^{22}$ for all $\mathbf{l} \in \mathcal{L}_{2,3}$, as given in Table 2. Then

$$s_{13}^{22} = \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times 0 = \frac{1}{3},$$

$$s_{23}^{22} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} = \frac{1}{3},$$

$$s_{33}^{22} = \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 0 + \left(\frac{1}{3}\right)^2 \times 1 + 2 \times \left(\frac{1}{3}\right)^2 \times 0 + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} + 2 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} = \frac{1}{3},$$

which yields

$$\mathbf{s}^{22} = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

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Remark 4.2.

(a) We observe $\mathbf{s}^{11} \leq^{\mathrm{st}} \mathbf{s} \leq^{\mathrm{st}} \mathbf{s}^{21}$, which means the second power system of the two that enters state 0 is better than the original design and the first one is worse than the original design.

(b) We observe $\mathbf{s}^{12} =^{st} \mathbf{s} =^{st} \mathbf{s}^{22}$, which means the first and second power systems of the two that enter state 1 or below are the same as the original design.

5. Concluding remarks and discussions

In this paper we have considered multi-state signatures for multi-state systems comprising binary-state components. For a set of multi-state coherent systems that have the same multi-state signature, we have proposed the ordered multi-state system signature and we have then investigated some interesting properties of this new distribution-free measure. Some illustrative examples have also been presented to explain the results established here. The concepts introduced here may be used to develop parametric/nonparametric inference for component lifetimes in multi-state coherent systems. Moreover, some comparisons may also be considered for multi-state systems having different numbers of components. By using the ordered system signature introduced here, statistical estimation similar to that considered in [32] can be developed for multi-state systems with binary-state components. Work on these problems is currently in progress, and we hope to report the findings in a future paper.

Appendix A. Proof of Theorem 3.5

(1) As discussed in Section 2, $\mathbf{s}^{q_1 r} \leq^{\mathrm{st}} \mathbf{s}^{q_2 r}$ if and only if, for all $1 \leq k_1, k_2, \ldots, k_n \leq m$,

$$\sum_{k_1 \leq j_1 \leq m, k_2 \leq j_2 \leq m, \ldots, k_n \leq j_n \leq m} s^{q_1 r}_{j_1, j_2, \ldots, j_n} \leq \sum_{k_1 \leq j_1 \leq m, k_2 \leq j_2 \leq m, \ldots, k_n \leq j_n \leq m} \tilde{s}^{q_2 r}_{j_1, j_2, \ldots, j_n}.$$

From the proof of Theorem 3.1, we have

$$s^{qr}_{j_1, j_2, \ldots, j_n} = \sum_{\mathbf{l} \in \mathcal{L}_{l,m}} \binom{l}{l_{i_1, i_2, \ldots, i_n}, 1 \leq i_1, i_2, \ldots, i_n \leq m} \left\{ \prod_{1 \leq i_1, i_2, \ldots, i_n \leq m} s^{l_{i_1, i_2, \ldots, i_n}}_{i_1, i_2, \ldots, i_n} \right\} p^{qr}_{j_1, j_2, \ldots, j_n | \mathbf{l}},$$

which means $\mathbf{s}^{q_1 r} \leq^{\mathrm{st}} \mathbf{s}^{q_2 r}$ if, for all $1 \leq k_1, k_2, \ldots, k_n \leq m$ and $\mathbf{l} \in L_{l,m}$,

$$\sum_{k_1 \leq j_1 \leq m, k_2 \leq j_2 \leq m, \ldots, k_n \leq j_n \leq m} p^{q_1 r}_{j_1, j_2, \ldots, j_n | \mathbf{l}} \leq \sum_{k_1 \leq j_1 \leq m, k_2 \leq j_2 \leq m, \ldots, k_n \leq j_n \leq m} p^{q_2 r}_{j_1, j_2, \ldots, j_n | \mathbf{l}}.$$

We say that $(T_1^{qr}, T_2^{qr}, \ldots, T_n^{qr}) \in U_{j_1, j_2, \ldots, j_n}$ if the qth system that enters state $n - r$ satisfies

$$I_1(\phi) = j_1, I_2(\phi) = j_2, \ldots, I_n(\phi) = j_n \quad (k_1 \leq j_1, j_2, \ldots, j_n \leq m),$$

which means it stays in state n or above before the k_1th component failure, stays in state $n - 1$ or above before the k_2th component failure, and so on. Similarly, from the fact that $j_1 \leq j_2 \leq \cdots \leq j_n$, we say $(T_1^{qr}, T_2^{qr}, \ldots, T_n^{qr}) \in L^e_{j_1, j_2, \ldots, j_n}$ $(e = 1, 2, \ldots, n)$ if the qth system that enters state $n - r$ satisfies

$$I_1(\phi) = j_1, I_2(\phi) = j_2, \ldots, I_n(\phi) = j_n \quad (1 \leq j_1, \ldots, j_e \leq k_e - 1, k_{e+1} \leq j_{e+1}, \ldots, j_n \leq m).$$

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Let

$$U = \bigcup_{k_1 \le j_1 \le m, k_2 \le j_2 \le m, \ldots, k_n \le j_n \le m} U_{j_1, j_2, \ldots, j_n},$$

$$L = \bigcup_{1 \le j_1 \le k_1 - 1, \ldots, 1 \le j_e \le k_e - 1, k_{e+1} \le j_{e+1} \le m, \ldots, k_n \le j_n \le m} L^e_{j_1, j_2, \ldots, j_n}.$$

Then we have

$$\begin{aligned}
&\sum_{k_1 \le j_1 \le m, k_2 \le j_2 \le m, \ldots, k_n \le j_n \le m} p^{q_1 r}_{j_1, j_2, \ldots, j_n | \mathbf{l}} \\
&\quad = \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U \mid \mathbf{l}\} \\
&\quad = \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in L \mid \mathbf{l}\} \\
&\qquad + \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\} \\
&\quad \le \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in L, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\} \\
&\qquad + \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\} \\
&\quad = \mathbb{P}\{(T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\} \\
&\quad = \sum_{k_1 \le j_1 \le m, k_2 \le j_2 \le m, \ldots, k_n \le j_n \le m} p^{q_2 r}_{j_1, j_2, \ldots, j_n | \mathbf{l}}.
\end{aligned}$$

To prove the inequality, we have

$$\begin{aligned}
&\mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in L \mid \mathbf{l}\} \\
&\quad = \sum_{e=1}^{n} \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in L^e \mid \mathbf{l}\} \\
&\quad = \sum_{e=1}^{n} \sum_{k_1 \le i_1 \le m, k_2 \le i_2 \le m, \ldots, k_n \le i_n \le m} \; \sum_{1 \le j_1 \le k_1 - 1, \ldots, 1 \le j_e \le k_e - 1, k_{e+1} \le j_{e+1} \le m, \ldots, k_n \le j_n \le m} \\
&\qquad \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in U_{i_1, i_2, \ldots, i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in L^e_{j_1, j_2, \ldots, j_n} \mid \mathbf{l}\} \\
&\quad \le \sum_{e=1}^{n} \sum_{k_1 \le i_1 \le m, k_2 \le i_2 \le m, \ldots, k_n \le i_n \le m} \; \sum_{1 \le j_1 \le k_1 - 1, \ldots, 1 \le j_e \le k_e - 1, k_{e+1} \le j_{e+1} \le m, \ldots, k_n \le j_n \le m} \\
&\qquad \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in L^e_{j_1, \ldots, j_e, i_{e+1}, \ldots, i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \\
&\qquad\quad \in U_{i_1, \ldots, i_e, j_{e+1}, \ldots, j_n} \mid \mathbf{l}\} \\
&\quad = \sum_{e=1}^{n} \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in L^e, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\} \\
&\quad = \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \ldots, T_n^{q_1 r}) \in L^e, (T_1^{q_2 r}, T_2^{q_2 r}, \ldots, T_n^{q_2 r}) \in U \mid \mathbf{l}\}.
\end{aligned}$$

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Now we need to prove that, for any $1 \le j_1, \dots, j_e \le k_e - 1$, $k_1 \le j_{e+1}, \dots, j_n$, $i_1, \dots, i_n \le m$,

$$\begin{aligned}&\mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \dots, T_n^{q_1 r}) \in U_{i_1,i_2,\dots,i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \dots, T_n^{q_2 r}) \in L^e_{j_1,j_2,\dots,j_n} \mid \mathbf{I}\}\\ &\quad \le \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \dots, T_n^{q_1 r}) \in L^e_{j_1,\dots,j_e,i_{e+1},\dots,i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \dots, T_n^{q_2 r}) \in U_{i_1,\dots,i_e,j_{e+1},\dots,j_n} \mid \mathbf{I}\}.\end{aligned}$$

For $e = 1, 2, \dots, n-1$, this inequality follows by first conditioning on that the two systems entering state $n-1, n-2, \dots, 0$ or above at times $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ respectively, and then unconditioning in the following way:

$$\begin{aligned}&\mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \dots, T_n^{q_1 r}) \in U_{i_1,i_2,\dots,i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \dots, T_n^{q_2 r}) \in L^e_{j_1,j_2,\dots,j_n} \mid \mathbf{I}\}\\
&\quad = \sum_{S\in\mathcal{C}} \int_{\substack{0<u_1<u_2<\cdots<u_n<\infty\\ 0<v_1<v_2<\cdots<v_n<\infty, u_r<v_r}} C_S F^{i_1-1}(u_1)[F(u_2)-F(u_1)]^{i_2-i_1-1}\\
&\qquad \times \cdots \times [F(u_n)-F(u_{n-1})]^{i_n-i_{n-1}-1}[1-F(u_n)]^{m-i_n} F^{j_1-1}(v_1)[F(v_2)-F(v_1)]^{j_2-j_1-1}\\
&\qquad \times \cdots \times [F(v_n)-F(v_{n-1})]^{j_n-j_{n-1}-1}[1-F(v_n)]^{m-j_n}\, \mathrm{d}F(u_1)\cdots \mathrm{d}F(u_n)\, \mathrm{d}F(v_1)\cdots \mathrm{d}F(v_n)\\
&\quad = \sum_{S\in\mathcal{C}} \int_{\substack{0<u_1<u_2<\cdots<u_n<1\\ 0<v_1<v_2<\cdots<v_n<1, u_r<v_r}} C_S u_1^{i_1-1}(u_2-u_1)^{i_2-i_1-1}\\
&\qquad \times \cdots \times (u_n-u_{n-1})^{i_n-i_{n-1}-1}(1-u_n)^{m-i_n} v_1^{j_1-1}(v_2-v_1)^{j_2-j_1-1}\\
&\qquad \times \cdots \times (v_n-v_{n-1})^{j_n-j_{n-1}-1}(1-v_n)^{m-j_n}\, \mathrm{d}u_1\cdots \mathrm{d}u_n\, \mathrm{d}v_1\cdots \mathrm{d}v_n\\
&\quad \le \sum_{S\in\mathcal{C}} \int_{\substack{0<u_1<u_2<\cdots<u_n<1\\ 0<v_1<v_2<\cdots<v_n<1, u_r<v_r}} C_S u_1^{j_1-1}(u_2-u_1)^{j_2-j_1-1}\\
&\qquad \times \cdots \times (u_e-u_{e-1})^{j_e-j_{e-1}-1}(u_{e+1}-u_e)^{i_{e+1}-j_e-1}(u_{e+2}-u_{e+1})^{i_{e+2}-i_{e+1}-1}\\
&\qquad \times \cdots \times (u_n-u_{n-1})^{i_n-i_{n-1}-1}(1-u_n)^{m-i_n} v_1^{i_1-1}(v_2-v_1)^{i_2-i_1-1}\\
&\qquad \times \cdots \times (v_e-v_{e-1})^{i_e-i_{e-1}-1}(v_{e+1}-v_e)^{j_{e+1}-i_e-1}(v_{e+2}-v_{e+1})^{j_{e+2}-j_{e+1}-1}\\
&\qquad \times \cdots \times (v_n-v_{n-1})^{j_n-j_{n-1}-1}(1-v_n)^{m-j_n}\, \mathrm{d}u_1\cdots \mathrm{d}u_n\, \mathrm{d}v_1\cdots \mathrm{d}v_n\\
&\quad = \sum_{S\in\mathcal{C}} \int_{\substack{0<u_1<u_2<\cdots<u_n<1\\ 0<v_1<v_2<\cdots<v_n<1, u_r<v_r}} C_S F^{j_1-1}(u_1)[F(u_2)-F(u_1)]^{j_2-j_1-1}\\
&\qquad \times \cdots \times [F(u_e)-F(u_{e-1})]^{j_e-j_{e-1}-1}[F(u_{e+1})-F(u_e)]^{i_{e+1}-j_e-1}\\
&\qquad\qquad \times [F(u_{e+2})-F(u_{e+1})]^{i_{e+2}-i_{e+1}-1}\\
&\qquad \times \cdots \times [F(u_n)-F(u_{n-1})]^{i_n-i_{n-1}-1}[1-F(u_n)]^{m-i_n} F^{i_1-1}(v_1)[F(v_2)-F(v_1)]^{i_2-i_1-1}\\
&\qquad \times \cdots \times [F(v_e)-F(v_{e+1})]^{i_e-i_{e+1}-1}[F(v_{e+1})-F(v_e)]^{j_{e+1}-i_e-1}\\
&\qquad\qquad \times [F(v_{e+2})-F(v_{e+1})]^{j_{e+2}-j_{e+1}-1}\\
&\qquad \times \cdots \times [F(v_n)-F(v_{n-1})]^{j_n-j_{n-1}-1}[1-F(v_n)]^{m-j_n}\mathrm{d}F(u_1)\cdots \mathrm{d}F(u_n)\, \mathrm{d}F(v_1)\cdots \mathrm{d}F(v_n)\\
&\quad = \mathbb{P}\{(T_1^{q_1 r}, T_2^{q_1 r}, \dots, T_n^{q_1 r}) \in L^e_{j_1,\dots,j_{e-1},i_e,\dots,i_n}, (T_1^{q_2 r}, T_2^{q_2 r}, \dots, T_n^{q_2 r}) \in U_{i_1,\dots,i_{e-1},j_e,\dots,j_n} \mid \mathbf{I}\},\end{aligned}$$

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where $\mathcal{C}$ represents a set of possible cases such that $q_1 - 1$ of the $l-2$ remaining systems enter state $n-r$ or below in the interval $(0, u_r]$, $q_2 - q_1 - 1$ of them in the interval $(u_r, v_r]$, and $l - q_2$ of them in the interval (v_r, ∞). C_S is simply the product of the multinomial probability corresponding to S and the normalizing constants for appropriate distributions. Note that the integral

$$\int_{\substack{0<u_1<u_2<\cdots<u_n<1 \\ 0<v_1<v_2<\cdots<v_n<1, u_r<v_r}} f(u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)\, \mathrm{d}u_1\, \mathrm{d}u_2 \cdots \mathrm{d}u_n \mathrm{d}v_1\, \mathrm{d}v_2 \cdots \mathrm{d}v_n$$

can be regarded as the expectation of $f(U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n)$, where the independent random variables $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n$ can be described by independent random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ which follow an uniform distribution on $[0, 1]$ and are such that

$$\begin{aligned}
V_n &= Y_n, & U_n &= X_n, \\
V_{n-1} &= V_n Y_{n-1}, & U_{n-1} &= U_n X_{n-1}, \\
&\vdots & &\vdots \\
V_{r+1} &= V_{r+2} Y_{r+1}, & U_{r+1} &= U_{r+2} X_{r+1}, \\
V_r &= V_{r+1} Y_r, & U_r &= \min(U_{r+1}, V_r) X_r, \\
V_{r-1} &= V_r Y_{r-1}, & U_{r-1} &= U_r X_{r-1}, \\
V_{r-2} &= V_{r-1} Y_{r-2}, & U_{r-2} &= U_{r-1} X_{r-2}, \\
&\vdots & &\vdots \\
V_1 &= V_2 Y_1, & U_1 &= U_2 X_1.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{U_1}{U_2 - U_1} \overset{\mathrm{st}}{\leq} \frac{V_1}{V_2 - V_1}, \\
&\frac{U_k - U_{k-1}}{U_{k+1} - U_k} \overset{\mathrm{st}}{\leq} \frac{V_k - V_{k-1}}{V_{k+1} - V_k} \quad (k = 2, 3, \ldots n-1), \\
&\frac{U_n - U_{n-1}}{1 - U_n} \overset{\mathrm{st}}{\leq} \frac{V_n - V_{n-1}}{1 - V_n},
\end{aligned}$$

which yields the following inequality:

$$\begin{aligned}
& u_1^{i_1-1}(u_2-u_1)^{i_2-i_1-1} \cdots (u_e - u_{e-1})^{i_e - i_{e-1}-1}(u_{e+1}-u_e)^{i_{e+1}-i_e-1} \\
&\quad \times v_1^{j_1-1}(v_2-v_1)^{j_2-j_1-1} \cdots (v_e - v_{e-1})^{j_e - j_{e-1}-1}(v_{e+1}-v_e)^{j_{e+1}-j_e-1} \\
&\leq u_1^{j_1-1}(u_2-u_1)^{j_2-j_1-1} \cdots (u_e - u_{e-1})^{j_e - j_{e-1}-1}(u_{e+1}-u_e)^{i_{e+1}-j_e-1} \\
&\quad \times v_1^{i_1-1}(v_2-v_1)^{i_2-i_1-1} \cdots (v_e - v_{e-1})^{i_e - i_{e-1}-1}(v_{e+1}-v_e)^{j_{e+1}-i_e-1} \\
\iff \quad & u_1^{i_1-j_1}(u_2-u_1)^{(i_2-j_2)-(i_1-j_1)} \cdots (u_e - u_{e-1})^{(i_e-j_e)-(i_{e-1}-j_{e-1})}(u_{e+1}-u_e)^{-(i_e-j_e)} \\
& \leq v_1^{i_1-j_1}(v_2-v_1)^{(i_2-j_2)-(i_1-j_1)} \cdots (v_e - v_{e-1})^{(i_e-j_e)-(i_{e-1}-j_{e-1})}(v_{e+1}-v_e)^{-(i_e-j_e)}
\end{aligned}$$

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$$\iff \quad \left(\frac{u_1}{u_2-u_1}\right)^{i_1-j_1}\left(\frac{u_2-u_1}{u_3-u_2}\right)^{i_2-j_2}\cdots\left(\frac{u_e-u_{e-1}}{u_{e+1}-u_e}\right)^{i_e-j_e}$$
$$\le \left(\frac{v_1}{v_2-v_1}\right)^{i_1-j_1}\left(\frac{v_2-v_1}{v_3-v_2}\right)^{i_2-j_2}\cdots\left(\frac{v_e-v_{e-1}}{v_{e+1}-v_e}\right)^{i_e-j_e}.$$

Similar discussions can also be presented for the case $e=n$, but we omit details here for the sake of brevity.

(2) From $\mathbf{s}^{q_1 r}\le^{\mathrm{st}}\mathbf{s}^{q_2 r}$ and $\mathbf{s}^{q_1 r}\ge^{\mathrm{st}}\mathbf{s}^{q_2 r}$ $(1\le q_1<q_2\le l)$, we have $\mathbf{s}^{q_1 r}=^{\mathrm{st}}\mathbf{s}^{q_2 r}$. For any $1\le j_1,\dots,j_{r-1},j_{r+1},\dots,j_n\le m$, let k_r be the largest j_r such that $s_{j_1,j_2,\dots,j_n}\neq 0$, which means $s_{j_1,j_2,\dots,j_n}=0$ for all $j_r\ge k_r+1$. Let $\tilde{k}_r$ be another j_r such that $s_{j_1,\dots,j_{r-1},\tilde{k}_r,j_{r+1},\dots,j_n}\neq 0$; then evidently $\tilde{k}_r<k_r$, and we thus have

$$0=\sum_{0\le j_1\le m,\dots,0\le j_{r-1}\le m,k_r\le j_r\le m,0\le j_{r+1}\le m,\dots,0\le j_n\le m}(s^{q_1 r}_{j_1,j_2,\dots,j_n}-s^{q_2 r}_{j_1,j_2,\dots,j_n})$$
$$=\sum_{\mathbf{l}\in\mathcal{L}_{l,m}}\binom{l}{l_{i_1,i_2,\dots,i_n},\,1\le i_1,i_2,\dots,i_n\le m}\left\{\prod_{1\le i_1,i_2,\dots,i_n\le m}s_{i_1,i_2,\dots,i_n}^{l_{i_1,i_2,\dots,i_n}}\right\}$$
$$\times\sum_{0\le j_1\le m,\dots,0\le j_{r-1}\le m,k_r\le j_r\le m,0\le j_{r+1}\le m,\dots,0\le j_n\le m}(p^{q_1 r}_{j_1,j_2,\dots,j_n|\mathbf{l}}-p^{q_2 r}_{j_1,j_2,\dots,j_n|\mathbf{l}}).$$

As in the case of (1), we say that $(T_1^{qr},T_2^{qr},\dots,T_n^{qr})\in U^r_{j_r}$ if the qth system that enters state $n-r$ satisfies $I_r(\phi)=j_r$ $(k_r\le j_r\le m)$, and $(T_1^{qr},T_2^{qr},\dots,T_n^{qr})\in L^r_{j_r}$ if the qth system that enters state $n-r$ satisfies $I_r(\phi)=j_r$ $(1\le j_r\le k_r-1)$. Let $U=\bigcup_{k_r\le j_r\le m}U^r_{j_r}$ and $L=\bigcup_{1\le j_r\le k_r-1}L^r_{j_r}$. Let $\mathbf{l}$ be such that $l_{j_1,\dots,j_{r-1},\tilde{k}_r,j_{r+1},\dots,j_n}=l-1$ and $l_{j_1,\dots,j_{r-1},k_r,j_{r+1},\dots,j_n}=1$, which means only one system enters state $n-r$ or below at the k_rth component failure and the other $l-1$ systems enter state $n-r$ or below at the $\tilde{k}_r$th component failure $(\tilde{k}_r<k_r)$. Then

$$\sum_{0\le j_1\le m,\dots,0\le j_{r-1}\le m,k_r\le j_r\le m,0\le j_{r+1}\le m,\dots,0\le j_n\le m}(p^{q_1 r}_{j_1,j_2,\dots,j_n|\mathbf{l}}-p^{q_2 r}_{j_1,j_2,\dots,j_n|\mathbf{l}})$$
$$=\sum_{0\le j_1\le m,\dots,0\le j_{r-1}\le m,0\le j_{r+1}\le m,\dots,0\le j_n\le m}(p^{q_1 r}_{j_1,\dots,j_{r-1},k_r,j_{r+1},\dots,j_n|\mathbf{l}}-p^{q_2 r}_{j_1,\dots,j_{r-1},k_r,j_{r+1},\dots,j_n|\mathbf{l}})$$
$$=\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in U^r_{k_r}\mid\mathbf{l}\}-\mathbb{P}\{(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in U^r_{k_r}\mid\mathbf{l}\}$$
$$=\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in U^r_{k_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in L^r_{\tilde{k}_r}\mid\mathbf{l}\}$$
$$+\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in U^r_{k_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in U^r_{k_r}\mid\mathbf{l}\}$$
$$-\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in L^r_{\tilde{k}_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in U^r_{k_r}\mid\mathbf{l}\}$$
$$-\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in U^r_{k_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in U^r_{k_r}\mid\mathbf{l}\}$$
$$=\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in U^r_{k_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in L^r_{\tilde{k}_r}\mid\mathbf{l}\}$$
$$-\mathbb{P}\{(T_1^{q_1 r},T_2^{q_1 r},\dots,T_n^{q_1 r})\in L^r_{\tilde{k}_r},(T_1^{q_2 r},T_2^{q_2 r},\dots,T_n^{q_2 r})\in U^r_{k_r}\mid\mathbf{l}\}$$
$$<0.$$

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Consequently, we have $s^{l-1}_{j_1,\ldots j_{r-1},\tilde{k}_r,j_{r+1},\ldots,j_n} s_{j_1,\ldots j_{r-1},k_r,j_{r+1},\ldots,j_n} = 0$, which contradicts the existence of $\tilde{k}_r$.

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