Ergod. Th. & Dynam. Sys. (2018), **38**, 666–685 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.40

Closest integer polynomial multiple recurrence along shifted primes

ANDREAS KOUTSOGIANNIS

The Ohio State University, Department of Mathematics, 100 Math Tower, 231 West 18th Avenue, Columbus, OH 43210-1174, USA
(e-mail: koutsogiannis.1@osu.edu)

(*Received* 9 *February* 2016 *and accepted in revised form* 30 *March* 2016)

Abstract. Following an approach presented by Frantzikinakis *et al* [The polynomial multidimensional Szemerédi theorem along shifted primes. *Israel J. Math.* **194**(1) (2013), 331–348], we show that the parameters in the multidimensional Szemerédi theorem for closest integer polynomials have non-empty intersection with the set of shifted primes $\mathbb{P}-1$ (or, similarly, of $\mathbb{P}+1$). Using the Furstenberg correspondence principle, we show this result by recasting it as a polynomial multiple recurrence result in measure ergodic theory. Furthermore, we obtain integer part polynomial convergence results by the same method, which is a transference principle that enables one to deduce results for $\mathbb{Z}$-actions from results for flows. We also give some applications of our approach to Gowers uniform sets.

1. *Introduction and main results*

For a subset $E\subset\mathbb{Z}^{\ell}$,we denote its *upper Banach density* by $d^*(E)$, which is defined to be the number

$$d^*(E)=\limsup_{|I|\to\infty}\frac{|E\cap I|}{|I|},$$

where the lim sup is taken over all the parallelepipeds $I\subset\mathbb{Z}^{\ell}$ whose side lengths tend to infinity.

Bergelson and Leibman, in [**4**], proved the following polynomial multidimensional Szemerédi theorem.

THEOREM. *Let* ℓ, $m\in\mathbb{N}$, $\vec{q}_1,\dots,\vec{q}_m:\mathbb{Z}\to\mathbb{Z}^{\ell}$ *be polynomials with* $\vec{q}_i(0)=\vec{0}$ *for* $1\leq i\leq m$ *and let* $E\subseteq\mathbb{Z}^{\ell}$ *with* $d^*(E)>0$. *Then there exists* $n\in\mathbb{N}$ *such that*

$$d^*(E\cap(E+\vec{q}_1(n))\cap\cdots\cap(E+\vec{q}_m(n)))>0.$$

Let $\mathbb{P}$ denote the set of prime numbers. Frantzikinakis *et al* showed, in [**10**], that the parameters of this result can be restricted to the shifted primes $\mathbb{P}-1$ (and, similarly, to $\mathbb{P}+1$), generalizing results due to Sárközy [**18**], who showed that any $E-E$, with $E\subset\mathbb{Z}$,

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

$d^*(E) > 0$, contains a shifted prime $p - 1$ (and, similarly, a shifted prime $p + 1$) for some $p \in \mathbb{P}$; Wooley and Ziegler [**22**], who proved the $\ell = 1$ case, and finally, of Bergelson *et al* [**5**], who got the result for linear polynomials.

If $[[\cdot]]$ denotes the closest integer, that is, $[[x]] = [x + 1/2]$, where $[\cdot]$ is the integer part function, for $\vec{q} = (q_1, \ldots, q_m)$, we set $[[\vec{q}]] = ([[q_1]], \ldots, [[q_m]])$.

Following the approach presented in [**10**] and methods from [**16**], we will prove the analogous result of Frantzikinakis, Host and Kra for the respective closest integer polynomials. Namely, we will prove the following theorem.

THEOREM 1.1. *Let ℓ, $m \in \mathbb{N}$, $\vec{q}_1, \ldots, \vec{q}_m : \mathbb{Z} \to \mathbb{R}^\ell$ be polynomials with $\vec{q}_i(0) = \vec{0}$ for $1 \leq i \leq m$ and let $E \subseteq \mathbb{Z}^\ell$ with $d^*(E) > 0$. Then the set of integers n such that*

$$d^*(E \cap (E + [[\vec{q}_1(n)]]) \cap \cdots \cap (E + [[\vec{q}_m(n)]])) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

In order to obtain this result, we will make use of the Furstenberg correspondence principle (see below). Via this principle, we will get Theorem 1.1, by proving a multiple recurrence result in ergodic theory (see Theorem 1.2 below).

Definition. For $\ell \in \mathbb{N}$, we call the setting $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a *system*, where $T_1, \ldots, T_\ell \colon X \to X$ are invertible commuting measure preserving transformations on the probability space $(X, \mathcal{X}, \mu)$.

THEOREM (Furstenberg correspondence principle [**11**]). *Let $\ell \in \mathbb{N}$ and $E \subseteq \mathbb{Z}^\ell$. There exist a system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and a set $A \in \mathcal{X}$ with $\mu(A) = d^*(E)$ such that*

$$d^*(E \cap (E + \vec{n}_1) \cap \cdots \cap (E + \vec{n}_m)) \geq \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-n_{i,1}}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-n_{i,m}}\right)A\right)$$

for all $m \in \mathbb{N}$ and $\vec{n}_j = (n_{1,j}, \ldots, n_{\ell,j}) \in \mathbb{Z}^\ell$ for $1 \leq j \leq m$.

By the Furstenberg correspondence principle, in order to prove Theorem 1.1, it suffices to prove the following ergodic reformulation of it.

THEOREM 1.2. *Let $\ell, m \in \mathbb{N}$ and $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $q_{i,j}(0) = 0$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$. Then, for every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and any $A \in \mathcal{X}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]}\right)A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Remark. As in [**10**], the arguments will show that the aforementioned intersection has positive measure for a set of positive relative density in the shifted primes (and so the same holds for the conclusion of Theorem 1.1 as well).

However, our method will show that we can have integer part polynomial multiple convergence along shifted primes. Namely, we prove the following theorem.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

THEOREM 1.3. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ be a system, let $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $1 \le i \le \ell$, $1 \le j \le m$ and let $f_1, \dots, f_m \in L^\infty(\mu)$ be functions with $\|f_i\|_\infty \le 1$. Then the averages*

$$\frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(p)]} \right) f_1 \cdot \dots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(p)]} \right) f_m$$

converge in $L^2(\mu)$ as $N \to \infty$, where $\pi(N) = |\mathbb{P} \cap [1, N]|$ denotes the number of primes up to N and $[1, N]$ denotes the set $\{1, \dots, N\}$.

We generalize [**10**, Theorem 1.3] for integer part polynomial iterates and [**19**, Theorem 1.1] for general polynomials and commuting transformations. Also, since $[[x]] = [x + 1/2]$, Theorem 1.3 holds for closest integer polynomial iterates as well.

Remark. We cannot, in general, obtain polynomial multiple recurrence results with iterates given by integer part polynomials (i.e. the conclusion of Theorem 1.2 and, consequently, the conclusion of Theorem 1.1). Since $[-x] = -[x] - 1$ when $x \notin \mathbb{Z}$, $x < 0$, for $a \notin \mathbb{Q}$, we can find examples with $\mu(T^{[an]}A \cap T^{[-an]}A) = 0$ for every $n \in \mathbb{N}$.

Frantzikinakis *et al*, in [**9**], first showed, using Theorem 2.2 for $d = 3$ (see below—this result was a conjecture, the inverse conjecture for Gowers norms, at that time for $d \ge 4$), the respective results of Theorems 1.3 and 1.2 (for the second one, see the proof of Theorem 1.2 and combine it with Lemma 2.1 below) in Theorems 3 and 4, respectively, for $\ell = 2$ commuting transformations (see [**9**, §7]) and $m = 2$ linear iterates of the form n and $2n$.

Wooley and Ziegler [**22**], showed (in [**22**, Theorem 1.1]) Theorem 1.1 for the special case of $\ell = 1$ and integer valued polynomials with no constant terms. Also, in [**22**, Theorem 1.2], they obtained the respective result of Theorem 1.2, for a single transformation and integer valued polynomial iterates with no constant terms, by exploiting the polynomial exhaustion technique (PET) induction scheme from [**1**].

The same authors of [**9**], in [**10**], with similar arguments from [**9**], using the full strength of Theorem 2.2 and the PET induction scheme from [**1**], showed, in [**10**, Theorem 1.3] and [**10**, Theorem 1.2], the respective results of Theorems 1.3 and 1.2 for any integer valued polynomial (with no constant terms on the polynomial iterates in the second case). [**10**, Theorem 1.2] via the Furstenberg correspondence principle, gives [**10**, Theorem 1.1], which is the respective result of Theorem 1.1 for vectors of integer valued polynomials with no constant terms.

Sun, in [**19**], first obtained results for integer part linear iterates. More specifically, [**19**] treated Theorems 1.1–1.3, for $\ell = 1$ and iterates of the form $[j\alpha n]$ for $1 \le j \le m$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, or of the form $j[\alpha n]$ for $1 \le j \le m$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. The proofs of these results will be simplified and extended by our method in the more general setting of commuting transformations. More specifically, the respective result of Theorem 1.3 [**19**, Theorem 1.1], follows immediately from the linear case of Theorem 1.3. The respective result of Theorem 1.2, and so of Theorem 1.1 ([**19**, Theorem 1.2] and [**19**, Corollary 1.3], respectively), follows from Theorem 1.4, which is a multiple recurrence result on integer part monomials with specific coefficients.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

THEOREM 1.4. *Let $\ell, m \in \mathbb{N}$, $a \in \mathbb{R}$, $d_{i,j} \in \mathbb{N}$ be positive integers and $k_{i,j} \in \mathbb{N} \cup \{0\}$ be non-negative integers for $1 \le i \le \ell$, $1 \le j \le m$. Then, for every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and any $A \in \mathcal{X}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[ak_{i,1}n^{d_{i,1}}]}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[ak_{i,m}n^{d_{i,m}}]}\right)A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

As we have already mentioned, in general, our methods follow the approach of Frantzikinakis *et al* [**9**, **10**] using a transference principle (used in [**16**] as well) that allows one to deduce results for integer part (and hence for closest integer) polynomial iterates with real coefficients, from respective results for integer valued polynomial iterates. Also, in order to obtain the respective uniform multiple recurrence result for closest integer polynomial iterates, in Proposition 3.8, which is a crucial intermediate step for proving Theorem 1.2, we use a method presented in [**2**] and used in [**19**] as well.

Finally, we give an application of our approach to some recent work of Franzikinakis and Host [**8**] on Gowers uniform sets. More specifically, we prove the following theorem (see §5 for definitions and details).

THEOREM 1.5. *Any shift of a Gowers uniform set $S \subseteq \mathbb{N}$ is a set of closest integer polynomial multiple recurrence and integer part polynomial multiple mean convergence.*

At this point, we will borrow some examples from [**8**] of Gowers uniform sets, for which we have the conclusion of Theorem 1.5.

Remarks [**8**]. If $\omega(n)$ is the number of distinct prime factors of an integer n and $\Omega(n)$ is the number of prime factors of n counted with multiplicity, then:

(i) any shift of the sets

$$S_{\omega,A,b} := \{n \in \mathbb{N} : \omega(n) \equiv a \pmod b \text{ for some } a \in A\},$$

and, similarly, $S_{\Omega,A,b}$, for every $b \in \mathbb{N}$ and $A \subseteq \{0, \ldots, b-1\}$ non-empty, is Gowers uniform;

(ii) any shift of the sets

$$S_{\omega,A,\alpha} := \{n \in \mathbb{N} : \|\omega(n)\alpha\| \in A\},$$

and, similarly, $S_{\Omega,A,\alpha}$, for every irrational α and Riemann-measurable set $A \subseteq [0, 1/2]$ of positive measure, where $\|x\| := d(x, \mathbb{Z})$ denotes the distance to the closest integer, is Gowers uniform; and

(iii) the same results as (ii) hold if $S_{\omega,A,\alpha}$ and $S_{\Omega,A,\alpha}$ are defined using fractional parts.

Even if it is not stated, without loss of generality, for the bounded functions f_i, which appear in our expressions, we will always assume that $\|f_i\|_\infty \le 1$ for all i.

Notation. We always denote by $[\cdot]$, $\{\cdot\}$, $[[\cdot]]$ and $\|\cdot\|$ the integer part, the fractional part, the closest integer and the distance to the closest integer function, respectively. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ the integers modulo N. When

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

necessary, we identify the set $\mathbb{Z}_N$ with the set $[1, N]$. For a finite set F and $a : F \to \mathbb{C}$, we write $\mathbb{E}_{n\in F} a(n) = (1/|F|) \sum_{n\in F} a(n)$. For a measurable function f on a measure space X with a transformation $T : X \to X$, we denote by Tf the composition $f \circ T$. Given transformations $T_i : X \to X$, $1 \le i \le \ell$, we denote the composition $T_1 \circ \cdots \circ T_\ell$ by $\prod_{i=1}^{\ell} T_i$. We denote the greatest common divisor of a, b by (a, b). A quantity that goes to zero as $N \to \infty$ is denoted by $o_N(1)$, while a quantity that goes to zero as $N \to \infty$ and then $w \to \infty$ is denoted by $o_{N\to\infty;w}(1)$.

2. *Definitions and tools*

In this section, we give the definitions and the main ideas in order to prove Theorem 1.2. For the convenience of the reader, we repeat most of §2 from [**10**].

We start by recalling the definition of the *von Mangoldt function*, $\Lambda : \mathbb{N} \to \mathbb{R}$, where

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and some } k \in \mathbb{N}, \\ 0 & \text{elsewhere.} \end{cases}$$

As in [**10**], it is more natural for us to deal with the function $\Lambda' : \mathbb{N} \to \mathbb{R}$, instead of Λ, where $\Lambda'(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \Lambda(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \log(n)$.

The function Λ', according to the following lemma, will allow us to relate averages along primes with weighted averages over the integers.

LEMMA 2.1. [**9**] *If $a : \mathbb{N} \to \mathbb{C}$ is bounded, then*

$$\left| \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} a(p) - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) \cdot a(n) \right| = o_N(1).$$

Next, we recall the definition of Gowers norms.

Definition. If $a : \mathbb{Z}_N \to \mathbb{C}$, we inductively define

$$\|a\|_{U_1(\mathbb{Z}_N)} = |\mathbb{E}_{n\in\mathbb{Z}_N} a(n)|;$$

and

$$\|a\|_{U_{d+1}(\mathbb{Z}_N)} = \left(\mathbb{E}_{h\in\mathbb{Z}_N} \|a_h \cdot \overline{a}\|_{U_d(\mathbb{Z}_N)}^{2^d}\right)^{1/2^{d+1}},$$

where $a_h(n) = a(n + h)$. As Gowers showed in [**12**], $\|\cdot\|_{U_d(\mathbb{Z}_N)}$ defines a norm on $\mathbb{Z}_N$ for $d \ge 2$.

For $w > 2$, let

$$W = \prod_{p\in\mathbb{P}\cap[1,w-1]} p$$

be the product of primes bounded above by w. For $r \in \mathbb{N}$, let

$$\Lambda'_{w,r}(n) = \frac{\phi(W)}{W} \cdot \Lambda'(Wn + r)$$

be the *modified von Mangoldt function*, where ϕ is the Euler function.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

The next result, which will play an important role in our proof, shows the Gowers uniformity of the modified von Mangoldt function and can be derived from [**13**, **14**] and [**15**]. We write, and we will later use, the formulation that can be found in [**10**].

THEOREM 2.2. [**13**, Theorem 7.2] *For every* $d \in \mathbb{N}$,

$$\lim_{w\to\infty}\Big(\lim_{N\to\infty}\Big(\max_{1\le r\le W,\,(r,W)=1}\|(\Lambda'_{w,r}-1)\cdot\mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})}\Big)\Big)=0.$$

The following uniformity estimate is an important step towards our result. It will allow us to obtain a similar estimate in our setting, in Theorem 3.1, in order to make use of the conclusion of Theorem 2.2.

LEMMA 2.3. [**10**, Lemma 3.5] *Let* $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ *be a system, let* $q_{i,j} \in \mathbb{Z}[t]$ *be polynomials with* $1 \le i \le \ell$, $1 \le j \le m$, $f_1, \dots, f_m \in L^\infty(\mu)$ *and let* $a : \mathbb{N} \to \mathbb{C}$ *be a sequence satisfying* $a(n)/n^c \to 0$ *for every* $c > 0$. *Then there exists* $d \in \mathbb{N}$, *depending only on the maximum degree of the polynomials* $q_{i,j}$ *and the integers* ℓ, m, *and a constant* C_d, *depending on* d, *such that*

$$\begin{aligned}\Big\|\frac{1}{N}\sum_{n=1}^{N}a(n)\cdot\Big(\prod_{i=1}^{\ell}T_i^{q_{i,1}(n)}\Big)f_1\cdot\dots\cdot\Big(\prod_{i=1}^{\ell}T_i^{q_{i,m}(n)}\Big)f_m\Big\|_{L^2(\mu)}\\ \le C_d(\|a\cdot\mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)).\end{aligned}$$

Furthermore, the constant C_d *is independent of the sequence* $(a(n))$ *and the* $o_N(1)$ *term depends only on the integer* d *and on the sequence* $(a(n))$.

In order to prove this result, the authors in [**10**], successively made use of the following variant of van der Corput's estimate.

LEMMA 2.4. *Let* $N \in \mathbb{N}$ *and* $v(1), \dots, v(N)$ *be elements in an inner product space. Then there exists a constant* C *such that*

$$\Big\|\frac{1}{N}\sum_{n=1}^{N}v(n)\Big\|^2\le C\Big(\frac{1}{N^2}\sum_{n=1}^{N}\|v(n)\|^2+\frac{1}{N}\sum_{h=1}^{N}\Big|\frac{1}{N}\sum_{n=1}^{N-h}\langle v(n+h),v(n)\rangle\Big|\Big).$$

Remark. In order to prove Lemma 2.3, we successively use Lemma 2.4, with the van der Corput operation, choosing appropriate polynomials every time in order to reduce the complexity. Actually, the number d that Lemma 2.3 provides corresponds to the $d-1$ steps that we need to do in order for the polynomials to be reduced into constant ones, using the PET induction, introduced in [**1**].

For more information and details on the van der Corput operation and the scheme of the PET induction we are using here, we refer the reader to [**10**].

3. *Main estimates*

In this section, we prove some key estimates needed in the proof of Theorems 1.2–1.5. The main idea is to use a transference principle that allows us to deduce results for integer part (and hence for closest integer) polynomial iterates with real coefficients from respective results for polynomial iterates with integer coefficients.

We first recall the definition of a measure preserving flow.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

Definition. Let $r \in \mathbb{N}$ and $(X, \mathcal{X}, \mu)$ be a probability space. We call a jointly measurable family $(T_t)_{t\in\mathbb{R}^r}$ of measure preserving transformations $T_t\colon X \to X$ a *measure preserving flow* if it satisfies

$$T_{s+t} = T_s \circ T_t$$

for all $s, t \in \mathbb{R}^r$.

Also, we recall the notion of the *lower* and *upper density* for a subset $S \subseteq \mathbb{N}$ to be the number of $\underline{d}(S)$ and $\overline{d}(S)$, respectively, where

$$\underline{d}(S) = \liminf_{N\to\infty} \frac{|S \cap [1, N]|}{N} \quad \text{and} \quad \overline{d}(S) = \limsup_{N\to\infty} \frac{|S \cap [1, N]|}{N}.$$

In the case $\overline{d}(S) = \underline{d}(S)$, we call the common value the *density* of S.

The following important remarks contain essential information that we will use in various proofs in this article.

Remarks. (i) Let $\ell, m \in \mathbb{N}$ and $q_{i,j} \in \mathbb{R}[t]$ be real valued polynomials for $1 \le i \le \ell$, $1 \le j \le m$. Then, for any $\mathbb{R}^{\ell m}$ measure preserving flow $\prod_{i=1}^{\ell} T_{i,s_{i,1}} \cdot \dots \cdot \prod_{i=1}^{\ell} T_{i,s_{i,m}}$, where the transformations $T_{i,s_{i,j}}$ are defined in the probability space $(X, \mathcal{X}, \mu)$, $s_{i,j} \in \mathbb{R}$, and functions $f_1, \dots, f_m \in L^\infty(\mu)$, the sequence of functions

$$\tilde{b}(n) = \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot q_{i,1}(n)}\right) f_1 \cdot \dots \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot q_{i,m}(n)}\right) f_m,$$

where

$$\delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

is the Kronecker delta function (note that $T_{i,0}$ is the identity for all i) satisfies the conclusion of Lemma 2.3, with $\tilde{b}(n)$ in place of the sequence

$$\left(\prod_{i=1}^{\ell} T_i^{q_{i,1}(n)}\right) f_1 \cdot \dots \cdot \left(\prod_{i=1}^{\ell} T_i^{q_{i,m}(n)}\right) f_m.$$

Indeed, if $q(t) = a_r t^r + \dots + a_1 t + a_0 \in \mathbb{R}[t]$, we write $T_{q(t)} = (T_{a_r})^{n^r} \cdot \dots \cdot (T_{a_1})^n \cdot T_{a_0}$ and we use Lemma 2.3 for the invertible commuting measure preserving transformations $S_1 = T_{a_1}, \dots, S_r = T_{a_r}$.

(ii) Real valued polynomials, $q \in \mathbb{R}[t]$, satisfy the condition

$$\lim_{\delta\to 0^+} \overline{d}(\{n \in \mathbb{N}\colon \{q(n)\} \in [1-\delta, 1)\}) = 0,$$

where $\{\cdot\}$ denotes the fractional part.

Indeed, let $q(t) = a_r t^r + \dots + a_1 t + a_0 \in \mathbb{R}[t]$.

If $a_i \notin \mathbb{Q}$ for some $1 \le i \le r$, then we have the condition from Weyl's result, since $(q(n))$ is uniformly distributed (mod 1).

If $a_i \in \mathbb{Q}$ for all $1 \le i \le r$, then the sequence $(q(n))$ is periodic (mod 1) and the conclusion is obvious.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

(iii) If f is Riemann-integrable on $[0, 1)$ with $\int_{[0,1)} f(x)\,dx = c$, then, for every $\varepsilon > 0$, we can find trigonometric polynomials $q_1,\ q_2$, with no constant terms, in order to have the relation

$$q_1(t) + c - \varepsilon \le f(t) \le q_2(t) + c + \varepsilon.$$

The following result is proved via a transference principle that enables one to deduce results for $\mathbb{Z}$-actions from results for flows (see also [**16**] in comparison with [**7**]). This technique was first used in [**17**] by Lesigne, in order to prove that when a sequence of real positive numbers is good for the single term pointwise ergodic theorem, then the respective sequence of its integer parts is also good (see [**6**] as well). This method was later used by Wierdl (in [**21**]) to deal with multiple term averages (see [**16**, Theorem 3.2]).

THEOREM 3.1. *Let $\ell, m \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, let $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $1 \le i \le \ell$, $1 \le j \le m$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$.*

For the sequence of functions

$$b(n) = \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}\right) f_1 \cdot \cdots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}\right) f_m$$

there exists $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that, for every $0 < \delta < 1$, there exists a constant $C_{d,\delta}$, depending on d and δ, such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} (\Lambda'_{w,r}(n) - 1) b(n) \right\|_{L^2(\mu)} \le C_{d,\delta}(\|(\Lambda'_{w,r} - 1) \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1)) + c_\delta(1 + o_{N\to\infty;w}(1))$$

for all $r \in \mathbb{N}$, where $c_\delta \to 0$ as $\delta \to 0^+$ and the term $o_N(1)$ depends on the integer d.

Remark. Due to unpleasant error terms, it seems difficult to adapt the PET induction that was used in the proof of Lemma 2.3 in order to prove the asserted estimate.

Proof of Theorem 3.1. Let $0 < \delta < 1$ and $w, r \in \mathbb{N}$. For the given transformations on X, we define the $\mathbb{R}^{\ell m}$ action $\prod_{i=1}^{\ell} T_{i,s_{i,1}} \cdot \cdots \cdot \prod_{i=1}^{\ell} T_{i,s_{i,m}}$ on the probability space $Y = X \times [0, 1)^{\ell m}$ endowed with the measure $\nu = \mu \times \lambda^{\ell m}$ (where λ is the Lebesgue measure on $[0, 1)$) by

$$\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_{i,s_{i,j}}(x, a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,2}, \ldots, a_{1,m}, \ldots, a_{\ell,m})$$
$$= \Bigg(\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_i^{[s_{i,j}+a_{i,j}]} x, \{s_{1,1} + a_{1,1}\}, \ldots, \{s_{\ell,1} + a_{\ell,1}\}, \ldots, \{s_{1,m} + a_{1,m}\}, \ldots, \{s_{\ell,m} + a_{\ell,m}\}\Bigg).$$

Since the transformations $T_1, \ldots, T_\ell$ are measure preserving and commute, and also since $[x + \{y\}] + [y] = [x + y]$, it is easy to check that the above action defines a measure preserving flow on the product probability space Y.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

If $f_1, \dots, f_m$ are bounded functions on X, we define the Y-extensions of f_j, setting, for every element $(a_{1,1}, \dots, a_{\ell,1}, a_{1,2}, \dots, a_{\ell,2}, \dots, a_{1,m}, \dots, a_{\ell,m}) \in [0,1)^{\ell m}$,

$$\hat{f}_j(x, a_{1,1}, \dots, a_{\ell,1}, a_{1,2}, \dots, a_{\ell,2}, \dots, a_{1,m}, \dots, a_{\ell,m}) = f_j(x), \quad 1 \le j \le m$$

and

$$\hat{f}_0(x, a_{1,1}, \dots, a_{\ell,1}, a_{1,2}, \dots, a_{\ell,m}) = 1_{[0,\delta]^{\ell m}}(a_{1,1}, \dots, a_{\ell,1}, a_{1,2}, \dots, a_{\ell,m}).$$

If, for every $n \in \mathbb{N}$ (see Remark (i) before Theorem 3.1),

$$\tilde{b}(n) = \hat{f}_0 \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot q_{i,1}(n)}\right)\hat{f}_1 \cdot \dots \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot q_{i,m}(n)}\right)\hat{f}_m$$

for every $x \in X$, we define

$$b'(n)(x) = \int_{[0,1)^{\ell m}} \tilde{b}(n)(x, a_{1,1}, \dots, a_{\ell,1}, a_{1,2}, \dots, a_{\ell,2}, \dots, a_{1,m}, \dots, a_{\ell,m})\, d\lambda^{\ell m},$$

where the integration is with respect to the variables $a_{i,j}$.

Then, using the triangle and the Cauchy–Schwarz inequality, if $a(n) = \Lambda'_{w,r}(n) - 1$,

$$\delta^{\ell m}\left\|\frac{1}{N}\sum_{n=1}^{N} a(n)b(n)\right\|_{L^2(\mu)} \le \left\|\frac{1}{N}\sum_{n=1}^{N} a(n)\cdot(\delta^{\ell m}b(n) - b'(n))\right\|_{L^2(\mu)} + \left\|\frac{1}{N}\sum_{n=1}^{N} a(n)\tilde{b}(n)\right\|_{L^2(\nu)}.$$

From part (i) of the previous remark, we can use Lemma 2.3 to find an integer $d \in \mathbb{N}$ that depends only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ, m and a constant C_d, depending on d, such that

$$\left\|\frac{1}{N}\sum_{n=1}^{N} a(n)\tilde{b}(n)\right\|_{L^2(\nu)} \le C_d(\|a\cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1)),$$

where the $o_N(1)$ term depends only on the integer d and the sequence $(a(n))$.

Next we will study the term $\|(1/N)\sum_{n=1}^{N} a(n)\cdot(\delta^{\ell m}b(n) - b'(n))\|_{L^2(\mu)}$.

For every $x \in X$ and $n \in \mathbb{N}$,

$$|\delta^{\ell m}b(n)(x) - b'(n)(x)| = \left|\int_{[0,\delta]^{\ell m}}\left(\prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[q_{i,j}(n)]}x\right) - \prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[q_{i,j}(n)+a_{i,j}]}x\right)\right) d\lambda^{\ell m}\right|.$$

Since all the relevant $a_{i,j}$ in the integrand are less or equal than δ, if the fractional part of all $q_{i,j}(n)$ is less than $1-\delta$, then $T_i^{[q_{i,j}(n)+a_{i,j}]} = T_i^{[q_{i,j}(n)]}$ for all $1 \le i \le \ell$, $1 \le j \le m$. We deal with the case where the fractional part of some $q_{i,j}(n)$ is greater than or equal to $1-\delta$.

For every $1 \le i \le \ell$, $1 \le j \le m$, let $E_\delta^{i,j} := \{n \in \mathbb{N} : \{q_{i,j}(n)\} \in [1-\delta, 1)\}$.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

Then, using the fact that $\mathbf{1}_{E_\delta^{1,1}\cup\ldots\cup E_\delta^{1,m}\cup E_\delta^{2,1}\cup\ldots\cup E_\delta^{\ell,m}} \le \sum_{(i,j)\in[1,\ell]\times[1,m]} \mathbf{1}_{E_\delta^{i,j}}$ and that $\mathbf{1}_{E_\delta^{i,j}}(n) = \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\})$ for $1 \le i \le \ell$, $1 \le j \le m$, $n \in \mathbb{N}$ for every $x \in X$,

$$|\delta^{\ell m} b(n)(x) - b'(n)(x)| \le 2\delta^{\ell m} \cdot \sum_{(i,j)\in[1,\ell]\times[1,m]} \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}),$$

and so, recalling that $a(n) = \Lambda'_{w,r}(n) - 1$,

$$\frac{1}{N}\sum_{n=1}^{N} |a(n)| \cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}) \le \frac{1}{N}\sum_{n=1}^{N} \Lambda'_{w,r}(n) \cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}) + \frac{|E_\delta^{i,j} \cap [1, N]|}{N}.$$

From part (ii) of the previous remark, for small enough δ, the term (and the sum of a finite number of terms of this form) $(|E_\delta^{i,j} \cap [1, N]|)/N$ is as small as we want.

As for the term $(1/N)\sum_{n=1}^{N} \Lambda'_{w,r}(n) \cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\})$, we will show that it goes to zero when $N \to \infty$, and then $w \to \infty$ and finally $\delta \to 0^+$.

If the polynomial $q_{i,j}(n)$ has only rational coefficients (except maybe the constant term), for small δ, the sum will go to zero from periodicity, and so we can assume that the polynomial has at least one irrational coefficient (except the constant term). From part (iii) of the previous remark (for the function $f = \mathbf{1}_{[1-\delta,1)}$), it suffices to show that $(1/N)\sum_{n=1}^{N} \Lambda'_{w,r}(n) e^{2\pi i k q_{i,j}(n)} \to 0$ as $N \to \infty$ and then $w \to \infty$ for all $k \in \mathbb{Z}\backslash\{0\}$.

We write

$$\frac{1}{N}\sum_{n=1}^{N} \Lambda'_{w,r}(n) e^{2\pi i k q_{i,j}(n)} = \frac{1}{N}\sum_{n=1}^{N} (\Lambda'_{w,r}(n) - 1) e^{2\pi i k q_{i,j}(n)} + \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k q_{i,j}(n)}.$$

The first term goes to zero as $N \to \infty$ and then $w \to \infty$ from [**13**], while the second term goes to zero as $N \to \infty$ from Weyl's equidistribution theorem. The result now follows. □

Remark. The crucial fact for the proof of Theorem 3.1 is that the integer part multicorrelation sequence $(b(n))$ can be written as

$$b(n) = \frac{1}{\delta^{\ell m}} b'(n) + \frac{1}{\delta^{\ell m}} (\delta^{\ell m} b(n) - b'(n)) \quad \text{for any } \delta > 0,$$

that is, as a sum of a conditional expectation of a usual multicorrelation sequence and an error function, controlled by terms of the form $\mathbf{1}_{\{n\in\mathbb{N}:\, q(n)\in[1-\delta,1)\}}$.

Using first Theorem 3.1 and then Theorem 2.2, we will now prove a result, in Proposition 3.2 below, which will give us a comparison between averages over primes (via the modified von Mangoldt function) and averages over integers. This result, together with a uniform multiple recurrence result, that we prove in Corollary 3.9 below, reflect the main arguments for proving Theorem 1.2. The proof of Theorem 1.2 will actually make use of Proposition 3.2 for the respective closest integer polynomial iterates. Corollary 3.9 will mainly follow by a uniform multiple recurrence result for polynomial iterates, which follows from the polynomial Szemerédi theorem (from [**4**]) in the same way as Theorem 2.1(ii) and (iii) are proved in [**3**], in order to obtain the respective uniform multiple recurrence result for closest integer polynomial iterates, in Proposition 3.8, the proof of which uses a method presented in [**2**] and used in [**19**] as well.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

PROPOSITION 3.2. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ be a system, let $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $1 \le i \le \ell$, $1 \le j \le m$ and $f_1, \dots, f_\ell \in L^\infty(\mu)$.*

Then

$$\max_{1\le r\le W,\,(r,W)=1}\Bigg\|\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{w,r}(n)-1)\times\Big(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(Wn+r)]}\Big)f_1\cdot\dots\cdot\Big(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(Wn+r)]}\Big)f_m\Bigg\|_{L^2(\mu)}$$

converges to zero as $N\to\infty$ and then $w\to\infty$.

Proof. Using Theorem 3.1, for the polynomials $q_{i,j}(Wn+r)$, we get that for every $0<\delta<1$, there exists $d\in\mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, and a constant $C_{d,\delta}$ depending on d and δ, such that

$$\begin{aligned}&\max_{1\le r\le W,\,(r,W)=1}\Bigg\|\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{w,r}(n)-1)\times\Big(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(Wn+r)]}\Big)f_1\cdot\dots\cdot\Big(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(Wn+r)]}\Big)f_m\Bigg\|_{L^2(\mu)}\\&\le C_{d,\delta}\Big(\max_{1\le r\le W,\,(r,W)=1}\|(\Lambda'_{w,r}-1)\cdot\mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)\Big)+c_\delta(1+o_{N\to\infty;w}(1)),\end{aligned}$$

where $c_\delta\to 0$ as $\delta\to 0^+$. Taking first $N\to\infty$ and then $w\to\infty$ in this expression, by Theorem 2.2, we have that the required limit is bounded above by c_δ. Taking $\delta\to 0^+$, we get the result. □

We will make use of the following uniform multiple recurrence result; it follows from the polynomial Szemerédi [**4**] in the same way as Theorem 2.1(ii) is proved in [**3**].

THEOREM 3.3. [**3**] *Let $\ell, m\in\mathbb{N}$ and $(X,\mathcal{X},\mu,(T_{i,j})_{1\le i\le\ell,1\le j\le m})$ be a system. Then, for any $A\in\mathcal{X}$ with $\mu(A)>0$, there exists a positive constant $c\equiv c_{\ell,m,\mu(A)}>0$ such that*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu\Big(A\cap\Big(\prod_{i=1}^{\ell}T_{i,1}^{-n^i}\Big)A\cap\dots\cap\Big(\prod_{i=1}^{\ell}T_{i,m}^{-n^i}\Big)A\Big)\ge c. \tag{1}$$

Remark. In fact, it is known that the limit in (1) exists from [**20**].

LEMMA 3.4. *Let $\ell, m\in\mathbb{N}$. For every $\delta>0$ there exists a constant $c\equiv c_{\ell,m,\delta}>0$ such that*

$$\underline{d}(\{n\in\mathbb{N}:\|a_{i,j}n^i\|<\delta\ \forall 1\le i\le\ell,\ 1\le j\le m\})\ge c$$

for all $a_{i,j}\in\mathbb{R}$, $1\le i\le\ell$, $1\le j\le m$.

Proof. Let $0<\delta\le 1/2$. For the system $(\mathbb{T},\mathcal{B}(\mathbb{T}),\mu,(T_{i,j})_{1\le i\le\ell,1\le j\le m})$, where

$$T_{i,j}x=x+a_{i,j}(\mathrm{mod}\ 1),\quad x\in\mathbb{T},\ 1\le i\le\ell,\ 1\le j\le m$$

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

and the set $A = [0, \delta)$, use Theorem 3.3 to find a constant $c \equiv c_{\ell,m,\delta} > 0$ so that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_{i,1}^{-n^i}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_{i,m}^{-n^i}\right)A\right) \geq c.$$

To obtain the result, note that if $x \in A \cap \left(\prod_{i=1}^{\ell} T_{i,1}^{-n^i}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_{i,m}^{-n^i}\right)A$, then

$$x + a_{i,j}n^i \,(\mathrm{mod}\ 1) \in [0, \delta) \quad \text{for all } 1 \leq i \leq \ell,\ 1 \leq j \leq m, \tag{2}$$

and, since $x \in [0, \delta)$ itself, (2) gives

$$\{a_{i,j}n^i\} \in [0, \delta - x) \cup [1 - x, 1) \quad \text{for all } 1 \leq i \leq \ell,\ 1 \leq j \leq m,$$

from which we get

$$\|a_{i,j}n^i\| < \delta \quad \text{for all } 1 \leq i \leq \ell,\ 1 \leq j \leq m,$$

and hence the result. □

Remark. The proof of Lemma 3.4 can also follow by a single recurrence argument as well.

We will also need the following lemma.

LEMMA 3.5. *Let $q(n) = a_1 n + \cdots + a_k n^k \in \mathbb{R}[n]$ be a real valued polynomial with no constant term and, for any $r \in \mathbb{N}$, let*

$$S_r := \left\{m \in \mathbb{N} : \|a_i m^i\| < \frac{1}{2kr^k},\ 1 \leq i \leq k\right\}.$$

Then, for any $m \in S_r$ and $1 \leq n \leq r$,

$$[[q(mn)]] = [[a_1 m]]n + \cdots + [[a_k m^k]]n^k.$$

Proof. If $\|x_1\| + \cdots + \|x_k\| < 1/2$, we can easily check that $[[x_1 + \cdots + x_k]] = [[x_1]] + \cdots + [[x_k]]$, which concludes the lemma. □

We will also use the following uniform multiple recurrence result which follows from the polynomial Szemerédi [**4**] in the same way as Theorem 2.1(iii) is proved in [**3**].

THEOREM 3.6. [**3**] *Let $\ell, m \in \mathbb{N}$ and $(X, \mathcal{X}, \mu, (T_{i,j})_{1\leq i\leq \ell, 1\leq j\leq m})$ be a system. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$, there exist a constant $c \equiv c_{\ell,m,\mu(A)} > 0$ and an integer $N_0 \equiv N_0(\ell, m, \mu(A)) \in \mathbb{N}$ such that, for some $1 \leq n \leq N_0$,*

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_{i,1}^{-n^i}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_{i,m}^{-n^i}\right)A\right) \geq c. \tag{3}$$

Using Theorem 3.6, we get the following proposition.

PROPOSITION 3.7. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system and let $q_{i,j} \in \mathbb{Z}[t]$ be polynomials with maximum degree d and $q_{i,j}(0) = 0$ for any $1 \leq i \leq \ell$, $1 \leq j \leq m$. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$ there exist a constant $c \equiv c_{d,m,\mu(A)} > 0$ and an integer $N_0 \equiv N_0(\ell, m, \mu(A)) \in \mathbb{N}$ such that, for some $1 \leq n \leq N_0$,*

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-q_{i,1}(n)}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-q_{i,m}(n)}\right)A\right) \geq c. \tag{4}$$

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

Proof. If $q_{i,j}(n) = a_{1,i,j}n + \cdots + a_{d,i,j}n^d$ for any $1 \le i \le \ell$, $1 \le j \le m$ (we put some zero terms if needed), set

$$S_{k,j} = \prod_{i=1}^{\ell} T_i^{a_{k,i,j}}, \quad 1 \le k \le d,\ 1 \le j \le m,$$

and use Theorem 3.6 for the system $(X, \mathcal{X}, \mu, (S_{k,j})_{1\le k\le d, 1\le j\le m})$. □

PROPOSITION 3.8. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a system and let $q_{i,j} \in \mathbb{R}[t]$ with maximum degree d and $q_{i,j}(0) = 0$ for any $1 \le i \le \ell$, $1 \le j \le m$. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a constant $c \equiv c_{d,\ell,m,\mu(A)} > 0$ such that*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]}\right)A\right) \ge c. \tag{5}$$

Proof. Write $q_{i,j}(n) = a_{1,i,j}n + \cdots + a_{d,i,j}n^d$, $1 \le i \le \ell$, $1 \le j \le m$ (by putting some zero terms if needed). For any $r \in \mathbb{N}$, let

$$S_r = \left\{s \in \mathbb{N} : \|a_{k,i,j}s^k\| < \frac{1}{2dr^d}\ \forall 1 \le k \le d,\ 1 \le i \le \ell,\ 1 \le j \le m\right\}.$$

Use Lemma 3.4 to bound the lower density of S_r below by some constant $c_{d,\ell,m,r} > 0$.

For any $s \in S_r$, $1 \le n \le r$, using Lemma 3.5,

$$[[q_{i,j}(ns)]] = [[a_{1,i,j}s]]n + \cdots + [[a_{d,i,j}s^d]]n^d, \quad 1 \le i \le \ell,\ 1 \le j \le m.$$

Set

$$p_{s,i,j}(n) = [[a_{1,i,j}s]]n + \cdots + [[a_{d,i,j}s^d]]n^d, \quad s \in \mathbb{N},\ 1 \le i \le \ell,\ 1 \le j \le m.$$

Use Proposition 3.7 to find a positive constant $\tilde{c} \equiv \tilde{c}_{d,\mu(A),\ell,m}$ and a positive integer $N_0 \equiv N_0(d, \mu(A), \ell, m)$ such that, for some $1 \le n \le N_0$ and any $s \in \mathbb{N}$,

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-p_{s,i,1}(n)}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-p_{s,i,m}(n)}\right)A\right) \ge \tilde{c}.$$

Then

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]}\right)A\right)\\
&\quad\ge \frac{1}{N_0} \liminf_{N\to\infty} \frac{1}{N} \sum_{s=1}^{[N/N_0]} \sum_{n=1}^{N_0} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(ns)]]}\right)A\right.\\
&\qquad\left. \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(ns)]]}\right)A\right)\\
&\quad\ge \frac{1}{N_0} \liminf_{N\to\infty} \frac{1}{N} \sum_{s\in S_{N_0}\cap\{1,\ldots,[N/N_0]\}} \sum_{n=1}^{N_0} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-p_{s,i,1}(n)}\right)A\right.\\
&\qquad\left. \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-p_{s,i,m}(n)}\right)A\right)\\
&\quad\ge \frac{\tilde{c}_{d,\mu(A),\ell,m}}{N_0^2} \liminf_{N\to\infty} \frac{|S_{N_0} \cap \{1, \ldots, [N/N_0]\}|}{[N/N_0]} \ge \frac{\tilde{c}_{d,\mu(A),\ell,m} \cdot c_{d,N_0,\ell,m}}{N_0^2} > 0,
\end{aligned}$$

from which we have the result. □

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

By using Proposition 3.8, we immediately obtain the following uniform result, Corollary 3.9, the second ingredient we need in order to prove Theorem 1.2. We note, at this point, that the uniformity in W is crucial for the proof of Theorem 1.2.

COROLLARY 3.9. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system and let $q_{i,j} \in \mathbb{R}[t]$ be polynomials with maximum degree d and $q_{i,j}(0) = 0$ for $1 \le i \le \ell$, $1 \le j \le m$. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a constant $c \equiv c_{d,\ell,m,\mu(A)} > 0$ such that, for every $W \in \mathbb{N}$,*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(Wn)]]}\right) A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(Wn)]]}\right) A\right) \ge c. \quad (6)$$

Remark. In fact, it is known that both limits in (5) and (6) exist from [**16**].

4. *Proof of main results*

In this section, we give the proof of Theorems 1.2–1.4. First, we prove Theorem 1.2 for the $\mathbb{P} - 1$ case (the $\mathbb{P} + 1$ case follows similarly).

Proof of Theorem 1.2. Using Proposition 3.2 for the polynomials $q_{i,j}(n-1) + 1/2$ with $r = 1$ and Corollary 3.9, we have (recall that $[[x]] = [x + 1/2]$), for sufficiently large $w \in \mathbb{N}$, that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda'_{w,1}(n) \cdot \mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(Wn)]]}\right) A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(Wn)]]}\right) A\right) > 0,$$

from which we have the required non-empty intersection with $\mathbb{P} - 1$. □

Remark. According to Lemma 2.1, we have that the conclusion of Theorem 1.2, and so of Theorem 1.1, is satisfied for a set of integers n with positive relative density in the shifted primes $\mathbb{P} - 1$. The analogous result, by a similar argument, holds for the set $\mathbb{P} + 1$ as well.

Proof of Theorem 1.3. We borrow the arguments from the proof of Theorem 1.3 from [**10**]. By Lemma 2.1, it suffices to show that the sequence

$$A(N) := \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}\right) f_1 \cdot \cdots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}\right) f_m$$

is Cauchy in $L^2(\mu)$. Using Proposition 3.2, for any $\varepsilon > 0$, if, for $w, r \in \mathbb{N}$, we define

$$B_{w,r}(N) := \frac{1}{N} \sum_{n=1}^{N} \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(Wn+r)]}\right) f_1 \cdot \cdots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(Wn+r)]}\right) f_m,$$

then, for sufficiently large N and some w_0 (which gives us a corresponding W_0),

$$\left\| A(W_0 N) - \frac{1}{\phi(W_0)} \sum_{1 \le r \le W_0,\ (r, W_0)=1} B_{w_0,r}(N) \right\|_{L^2(\mu)} < \varepsilon. \quad (7)$$

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

Using the fact that for any $1 \leq r \leq W_0$ the sequence $(B_{w_0,r}(N))$ is Cauchy in $L^2(\mu)$, which follows from [**16**], as well as the relation (7), we get that, for M, N sufficiently large,

$$\|A(W_0M) - A(W_0N)\|_{L^2(\mu)} < \varepsilon.$$

Then $(A(N))$ is Cauchy in $L^2(\mu)$ since

$$\|A(W_0N + r) - A(W_0N)\|_{L^2(\mu)} = o_N(1),$$

for every $1 \leq r \leq W_0$. □

Proof of Theorem 1.4. We use the same argument as in the proof of Theorem 1.2. The only difference is that we need to prove a variant of Corollary 3.9 for the integer part of the polynomials $q_{i,j}(n) = ak_{i,j}n^{d_{i,j}}$, $1 \leq i \leq \ell$, $1 \leq j \leq m$. Let $d = \max\{d_{i,j} : 1 \leq i \leq \ell, 1 \leq j \leq m\}$, $k = \max\{k_{i,j} : 1 \leq i \leq \ell, 1 \leq j \leq m\}$, $W \in \mathbb{N}$ and, for $r \in \mathbb{N}$, set

$$S_r = \left\{m \in \mathbb{N} : \{aW^{d_{i,j}}m^{d_{i,j}}\} < \frac{1}{kr^d},\ 1 \leq i \leq \ell, 1 \leq j \leq m\right\}$$

(without loss of generality, $k \in \mathbb{N}$). It is sufficient to find a positive lower bound for the lower density of S_r, that is independent of W. Since, for any $m \in S_r$ and $1 \leq n \leq r$,

$$[q_{i,j}(Wmn)] = [ak_{i,j}W^{d_{i,j}}m^{d_{i,j}}n^{d_{i,j}}] = [ak_{i,j}W^{d_{i,j}}m^{d_{i,j}}]n^{d_{i,j}},$$

we can follow the proof of Proposition 3.8 to obtain the result.

If $a = t/s \in \mathbb{Q}$ with $s \in \mathbb{N}$, then $\underline{d}(S_r) \geq 1/s$.

If $a \notin \mathbb{Q}$, then, if $\{d_{i,j} : 1 \leq i \leq \ell,\ 1 \leq j \leq m\} = \{d_1 < \cdots < d_\xi = d\}$, since $(a(nW)^{d_1}, \ldots, a(nW)^{d_\xi})_n$ is equidistributed in $\mathbb{T}^\xi$, we get $d(S_r) = 1/(kr^d)^\xi \geq 1/(kr^d)^{\ell m}$. □

5. *An application on Gowers uniform sets*

In this last section, following [**8**], we will give an application of our approach, on Gowers uniform sets. More specifically, we will prove Theorem 1.5, that is, any shift of a Gowers uniform set is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence (see definitions below). The main ingredients in order to obtain this result will be Theorem 5.2 and a result from [**8**], Lemma 5.3.

Imitating [**8**], we give the following definitions.

Definition.

(i) A set of integers S is *a set of integer part polynomial multiple mean convergence* if for every $\ell, m \in \mathbb{N}$, system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, functions $f_1, \ldots, f_m \in L^\infty(\mu)$ and polynomials $q_{i,j} \in \mathbb{R}[t]$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$, the averages

$$\frac{1}{|S \cap [1, N]|} \sum_{n \in S \cap [1,N]} \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}\right) f_m$$

converge in $L^2(\mu)$ as $N \to \infty$.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

(ii) A set of integers S is *a set of closest integer polynomial multiple recurrence* if for every $\ell, m \in \mathbb{N}$, system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, set $A \in \mathcal{X}$ with $\mu(A) > 0$ and polynomials $q_{i,j} \in \mathbb{R}[t]$ with $q_{i,j}(0) = 0$, $1 \le i \le \ell$, $1 \le j \le m$,

$$\mu\left(A \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]}\right)A \cap \cdots \cap \left(\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]}\right)A\right) > 0$$

for a set of $n \in S$ with positive lower relative density in S.

Remark. The aforementioned definitions are equivalent to the ones in [**8**] for sets of positive density in $\mathbb{N}$.

The following result is a corollary of Theorems 1.2 and 1.3.

THEOREM 5.1. *The set of shifted primes $\mathbb{P} - 1$ (and, similarly, $\mathbb{P} + 1$) is a set of closest integer polynomial multiple recurrence and the set of primes, $\mathbb{P}$, is a set of integer part polynomial multiple mean convergence.*

We will now prove that $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence, which we will use in the proof of Theorem 1.5.

THEOREM 5.2. *The set of positive integers, $\mathbb{N}$, is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence.*

Proof. That $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence follows from Proposition 3.8. The integer part polynomial multiple mean convergence of $\mathbb{N}$ follows from [**16**, Theorem 1.4]. □

In order to prove Theorem 1.5, we also need the following result of Frantzikinakis and Host, which we borrow from [**8**].

LEMMA 5.3. [**8**, Lemma 2.4] *Let $d \ge 2$ be an integer and let $\varepsilon, \kappa > 0$. Then there exist $\eta > 0$ and $N_0 \in \mathbb{N}$ such that, for all integers N, $\tilde{N}$ with $N_0 \le N \le \tilde{N} \le \kappa N$, every interval $J \subseteq [1, N]$ and $g : \mathbb{Z}_{\tilde{N}} \to \mathbb{C}$ with $|g| \le 1$, the following implication holds*

$$\text{if } \|g\|_{U_d(\mathbb{Z}_{\tilde{N}})} \le \eta, \text{ then } \|g \cdot \mathbf{1}_J\|_{U_d(\mathbb{Z}_N)} \le \varepsilon.$$

We will now recall the definition of a Gowers uniform set.

Definition [**8**]. We say that a set of positive integers S is *Gowers uniform* if there exists a positive constant c such that

$$\lim_{N\to\infty} \|\mathbf{1}_S - c\|_{U_r(\mathbb{Z}_N)} = 0$$

for every $r \in \mathbb{N}$.

Remark. Note that if such a (positive) constant c exists, then, for $r = 1$, it is equal to the density of S, that is,

$$c = \lim_{N\to\infty} \frac{|S \cap [1, N]|}{N}.$$

Using a similar argument to the one of Theorem 3.1, we get the following proposition.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

PROPOSITION 5.4. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ be a system, let $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $1 \le i \le \ell$, $1 \le j \le m$ and let $f_1, \dots, f_m \in L^\infty(\mu)$.*

For the sequence of functions

$$b(n) = \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}\right) f_1 \cdot \dots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}\right) f_m,$$

there exists $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that, for every sequence $(a(n))$ with $\|a\|_\infty \le 1$ and every $0 < \delta < 1$, there exists a constant $C_{d,\delta}$, depending on d and δ, such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \cdot b(n) \right\|_{L^2(\mu)} \le C_{d,\delta}(\|a \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1)) + c_\delta,$$

where $c_\delta \to 0$ as $\delta \to 0^+$ and the term $o_N(1)$ depends only on the integer d.

Remark. As in Theorem 3.1, the proof of Proposition 5.4 also depends on the fact that the integer part multicorrelation sequence can be written as a sum of a conditional expectation of a usual multicorrelation sequence and an error function, which we can control.

Proof of Proposition 5.4. Let $0 < \delta < 1$. For the given transformations on X, we define the same action on $\mathbb{R}^{\ell m}$ as in Theorem 3.1, the function $\hat{f}_0$ and the same Y-extensions $\hat{f}_i$, for $1 \le j \le m$, in the product probability space $Y = X \times [0, 1)^{\ell m}$, endowed with the measure $\nu = \mu \times \lambda^{\ell m}$.

We also define the same sequences of functions $(\tilde{b}(n))$ and $(b'(n))$ in Y and X, respectively, as in the proof of Theorem 3.1. Then, using the triangle and the Cauchy–Schwarz inequality and following the proof of Theorem 3.1,

$$\delta^{\ell m} \left\| \frac{1}{N} \sum_{n=1}^{N} a(n) b(n) \right\|_{L^2(\mu)} \le \left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \cdot (\delta^{\ell m} b(n) - b'(n)) \right\|_{L^2(\mu)} + \left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \tilde{b}(n) \right\|_{L^2(\nu)}.$$

For the second term, we can use Lemma 2.3 to find a positive integer $d = d(\ell, m, \max \deg(q_{i,j}))$ such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \tilde{b}(n) \right\|_{L^2(\nu)} \le C_d(\|a \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1)),$$

where C_d is a constant depending on d. We remark, at this point, that, since the weighted sequence $(a(n))$ is bounded, the $o_N(1)$ term of Lemma 2.3 only depends on d (see [**10**]).

Following the proof of Theorem 3.1, we obtain (since $(a(n))$ is bounded by one) that the first term satisfies

$$\left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \cdot (\delta^{\ell m} b(n) - b'(n)) \right\|_{L^2(\mu)} \le \delta^{\ell m} c_\delta,$$

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

with $c_\delta \to 0$ as $\delta \to 0^+$, since, independently of $x \in X$,

$$\left| \frac{1}{N} \sum_{n=1}^{N} (\delta^{\ell m} b(n)(x) - b'(n)(x)) \right| \leq \delta^{\ell m} c_\delta,$$

with $c_\delta \to 0$ as $\delta \to 0^+$. The result now follows. □

We are now ready to prove Theorem 1.5 and close this article. The main ingredients in order to obtain this result are Proposition 5.4, Lemma 5.3, which we borrow from [**8**], and finally, Theorem 5.2.

Proof of Theorem 1.5. Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, let $f_1, \ldots, f_m \in L^\infty(\mu)$ be functions and let $q_{i,j} \in \mathbb{R}[t]$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$ be polynomials. In order to prove the second conclusion, for any $n \in \mathbb{N}$, let

$$b(n) = \left(\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]} \right) f_1 \cdot \cdots \cdot \left(\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]} \right) f_m. \tag{8}$$

If $S \subseteq \mathbb{N}$ is a Gowers uniform set and c is the respective positive constant from the definition, by Proposition 5.4, we have that there exists a positive integer d, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that, for every $0 < \delta < 1$, there exists a constant $C_{d,\delta}$ depending on d and δ, such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} (\mathbf{1}_S(n) - c) \cdot b(n) \right\|_{L^2(\mu)} \leq C_{d,\delta}(\|(\mathbf{1}_S - c) \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1)) + c_\delta, \tag{9}$$

where $c_\delta \to 0$ as $\delta \to 0^+$ and the term $o_N(1)$ depends only on the integer d.

From Gowers uniformity, for $r = d$, $\lim_{N\to\infty} \|\mathbf{1}_S - c\|_{U_d(\mathbb{Z}_{dN})} = 0$. Then, using Lemma 5.3, for $\kappa = d$, $\tilde{N} = dN$ and $J = [1, N]$, we get

$$\lim_{N\to\infty} \|(\mathbf{1}_S - c) \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} = 0.$$

So, from (9),

$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n \in S \cap [1,N]} b(n) - \frac{c}{N} \sum_{n=1}^{N} b(n) \right\|_{L^2(\mu)} \leq c_\delta,$$

and hence, by taking $\delta \to 0^+$, we get

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n \in S \cap [1,N]} b(n) - \frac{c}{N} \sum_{n=1}^{N} b(n) \right\|_{L^2(\mu)} = 0. \tag{10}$$

Using the remark following the definition of a Gowers uniform set, c is the density of S, so

$$\lim_{N\to\infty} \frac{|S \cap [1, N]|}{N} = c,$$

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

and hence, by writing

$$\frac{1}{N}\sum_{n\in S\cap[1,N]} b(n) - \frac{c}{N}\sum_{n=1}^{N} b(n) = c\Bigg(\frac{1}{|S\cap[1,N]|}\sum_{n\in S\cap[1,N]} b(n) - \frac{1}{N}\sum_{n=1}^{N} b(n)\Bigg) + \Bigg(1 - c\frac{N}{|S\cap[1,N]|}\Bigg)\frac{1}{N}\sum_{n\in S\cap[1,N]} b(n)$$

and recalling from the definition of a Gowers uniform set that $c > 0$, relation (10) implies that

$$\lim_{N\to\infty}\Bigg\|\frac{1}{|S\cap[1,N]|}\sum_{n\in S\cap[1,N]} b(n) - \frac{1}{N}\sum_{n=1}^{N} b(n)\Bigg\|_{L^2(\mu)} = 0. \tag{11}$$

By using (11) and the fact that $\mathbb{N}$ is a set of integer part polynomial multiple mean convergence (from Theorem 5.2), we get the result.

In order to show the first conclusion, for a set $A \in \mathcal{X}$ with $\mu(A) > 0$, similarly, we let, for every $n \in \mathbb{N}$,

$$b(n) = \Bigg(\prod_{i=1}^{\ell} T_i^{[[q_{i,1}(n)]]}\Bigg)\mathbf{1}_A \cdot \cdots \cdot \Bigg(\prod_{i=1}^{\ell} T_i^{[[q_{i,m}(n)]]}\Bigg)\mathbf{1}_A.$$

By using (11), the Cauchy–Schwarz inequality and the fact that $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence (from Theorem 5.2), we get the result. □

Acknowledgements. I would like to express my gratitude to N. Frantzikinakis for his in-depth suggestions and the fruitful discussions we had during the writing procedure of this article. These discussions corrected mistakes from the initial version of this paper and led me to additional (reflected mainly in §5) results.

References

[**1**] V. Bergelson. Weakly mixing PET. *Ergod. Th. & Dynam. Sys.* **7**(3) (1987), 337–349.
[**2**] V. Bergelson and I. J. Håland. Sets of recurrence and generalized polynomials. *Convergence in Ergodic Theory and Probability (Columbus, OH, 1993) (Ohio State University Mathematical Research Institute Publications)*. de Gruyter, Berlin, 1996, vol. 5, pp. 91–110.
[**3**] V. Bergelson, B. Host, R. McCutcheon and F. Parreau. Aspects of uniformity in recurrence. *Colloq. Math.* **84/85**(2) (2000), 549–576.
[**4**] V. Bergelson and A. Leibman. Polynomial extensions of van der Waerden's and Szemerédi's theorems. *J. Amer. Math. Soc.* **9** (1996), 725–753.
[**5**] V. Bergelson, A. Leibman and T. Ziegler. The shifted primes and the multidimensional Szemerédi and polynomial van der Waerden theorems. *C. R. Math. Acad. Sci. Paris.* **349**(3–4) (2011), 123–125.
[**6**] M. Boshernitzan, R. L. Jones and M. Wierdl. Integer and fractional parts of good averaging sequences in ergodic theory. *Convergence in Ergodic Theory and Probability*. Eds. V. Bergelson, P. March and J. Rosenblatt. de Gruyter, Berlin, 1996, pp. 117–132.
[**7**] N. Frantzikinakis. Multiple correlation sequences and nilsequences. *Invent. Math.* **202**(2) (2015), 875–892.
[**8**] N. Frantzikinakis and B. Host. Multiple ergodic theorems for arithmetic sets. *Trans. Amer. Math. Soc.*, to appear.
[**9**] N. Frantzikinakis, B. Host and B. Kra. Multiple recurrence and convergence for sequences related to the prime numbers. *J. Reine Angew. Math.* **611** (2007), 131–144.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press

[**10**] N. Frantzikinakis, B. Host and B. Kra. The polynomial multidimensional Szemerédi theorem along shifted primes. *Israel J. Math.* **194**(1) (2013), 331–348.
[**11**] H. Furstenberg. Ergodic behaviour of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Anal. Math.* **71** (1977), 204–256.
[**12**] W. Gowers. A new proof of Szemerédi's theorem. *Geom. Funct. Anal.* **11** (2001), 465–588.
[**13**] B. Green and T. Tao. Linear equations in primes. *Ann. of Math. (2)* **171** (2010), 1753–1850.
[**14**] B. Green and T. Tao. The Möbius function is strongly orthogonal to nilsequences. *Ann. of Math. (2)* **175** (2012), 541–566.
[**15**] B. Green, T. Tao and T. Ziegler. An inverse theorem for the Gowers $U^{s+1}[N]$-norm. *Ann. of Math. (2)* **176**(2) (2012), 1231–1372.
[**16**] A. Koutsogiannis. Integer part correlation sequences. *Preprint*, 2015, arXiv:1512.01313.
[**17**] E. Lesigne. On the sequence of integer parts of a good sequence for the ergodic theorem. *Comment. Math. Univ. Carolin.* **36**(4) (1995), 737–743.
[**18**] A. Sárközy. On difference sets of sequences of integers, III. *Acta Math. Acad. Sci. Hungar.* **31** (1978), 355–386.
[**19**] W. Sun. Multiple recurrence and convergence for certain averages along shifted primes. *Ergod. Th. & Dynam. Sys.* **35** (2014), 1592–1609.
[**20**] M. Walsh. Norm convergence of nilpotent ergodic averages. *Ann. of Math. (2)* **175**(3) (2012), 1667–1688.
[**21**] M. Wierdl. Personal communication, 2015.
[**22**] T. Wooley and T. Ziegler. Multiple recurrence and convergence along the primes. *Amer. J. Math.* **134**(6) (2012), 1705–1732.

https://doi.org/10.1017/etds.2016.40 Published online by Cambridge University Press