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MUTUAL INTERPRETABILITY OF WEAK ESSENTIALLY UNDECIDABLE THEORIES

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Abstract. Kristiansen and Murwanashyaka recently proved that Robinson arithmetic, Q, is interpretable in an elementary theory of full binary trees, T. We prove that, conversely, T is interpretable in Q by producing a formal interpretation of T in an elementary concatenation theory QT^+, thereby also establishing mutual interpretability of T with several well-known weak essentially undecidable theories of numbers, strings, and sets. We also introduce a "hybrid" elementary theory of strings and trees, WQT^*, and establish its mutual interpretability with Robinson's weak arithmetic R, the weak theory of trees WT of Kristiansen and Murwanashyaka, and the weak concatenation theory WTC^ϵ of Higuchi and Horihata.

The classic monograph of Tarski, Mostowski, and Robinson [8] isolated two weak formal theories of arithmetic, R and Q, as minimal "basis theories" for metamathematical arguments of foundational significance involving formalizing computation, incompleteness, undecidability, etc. The two theories were singled out as essentially undecidable, in that neither can consistently be extended to a decidable theory. The work introduced a powerful method for establishing incompleteness and undecidability of a wide range of mathematical theories built around the notion of relative interpretability of one theory in another. Roughly, a formula with a single free variable is chosen in the language of the second theory—the interpreting theory—to define the "universe of the interpretation," and suitable definitions for the non-logical vocabulary of the first theory—the interpreted theory—are given in the language of the interpreting theory. Formulae of the interpreted theory are then translated into formulae of the interpreting theory based on those definitions, in such a way that the logical operations are preserved under the translation and, crucially, all occurrences of quantifiers become relativized to the universe of the interpretation. Consequently, deductive relations between formulae are preserved: in particular, theorems of the interpreted theory are translated into theorems of the interpreting theory. In this specific sense reasoning in one theory is formally simulated in another theory, establishing relative consistency of the former in the latter. Once it is shown that R or Q is interpretable in some given theory, it follows from Tarski's methods that the latter is also essentially undecidable.

It was only within the last two decades that some light has been shed on what makes R and Q special, a result of work of many researchers, including (earlier work by) Collins and Halpern, Wilkie, Grzegorczyk, Zdanowski, Švejdar, Ganea, and,

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especially, Visser (see [1] for further references). One approach was to characterize them as mutually interpretable with concatenation theories (theories of strings) or weak subsystems of set theory, each naturally motivated and of independent interest in their own right. Another is to produce a "coordinate-free" characterization independent of a particular axiomatic presentation in some formal language, as, e.g., in the remarkable theorem of Visser [10]: a recursively axiomatizable theory is interpretable in R if and only if it is locally finitely satisfiable, that is, each finite subset of its non-logical axioms has a finite model.

An important new angle on these issues was recently introduced in the work of Kristiansen and Murwanashyaka [6]. They consider two elementary axiomatizations, WT and T, whose intended models are simple inductively generated structures like trees or terms, and rigorously develop a direct and novel approach to formalization of computation by ultra-elementary means.

The theory T is formulated in the language $\mathcal{L}_T = \{0, (\ ), \sqsubseteq\}$ with a single individual constant 0, a binary operation symbol $(,)$ and a 2-place relational symbol $\sqsubseteq$ with the following axioms:

(T1) $\forall\, x, y \neg\ (x, y) = 0$,
(T2) $\forall\, x, y, z, w\ [(x, y) = (z, w) \rightarrow x = z\ \&\ y = w]$,
(T3) $\forall\, x\, [x \sqsubseteq 0 \leftrightarrow x = 0]$,
(T4) $\forall\, x, y, z\ [x \sqsubseteq (y, z) \leftrightarrow x = (y, z)\ \text{v}\ x \sqsubseteq y\ \text{v}\ x \sqsubseteq z]$.

On the other hand, WT is formulated in the same vocabulary, but has infinitely many axioms given by the two schemas

(WT1) $\neg\ (s = t)$ for any distinct variable-free terms s, t of $\mathcal{L}_T$,
(WT2) $\forall\, x\ \left(x \sqsubseteq t \leftrightarrow \vee_{s \in S(t)} x = s\right)$ for each variable-free term t of $\mathcal{L}_T$,

where $S(t)$ is the set of all subterms of t.

The theory WT, which turns out to be contained in T, is proved to be mutually interpretable with R. The stronger theory T, which can be thought of as the basic theory of full binary trees, even though lacking induction, is shown to be sufficiently strong to allow for a formal interpretation of basic arithmetical operations validating the axioms of Q. Kristiansen and Murwanashyaka further conjectured that, conversely, T is also formally interpretable in Q.

In this paper we prove that T is indeed interpretable in Q, by formally interpreting T in a theory of concatenation, QT^+, previously investigated in [1] and established to be mutually interpretable with Q along with a host of other theories whose intended interpretations are natural numbers, strings, or sets. Hence T and Q are mutually interpretable. Further we formulate a weak theory of concatenation, WQT^*, and a "pseudo-concatenation" theory WQT, and establish their mutual interpretability with Robinson's R. (While R is deductively contained, hence also interpretable, in Q, the latter, being finitely axiomatized but having no finite model, by Visser's Theorem is not interpretable in R.)

Several distinct formulations of concatenation theory which have been put forward as standard axiomatizations and as such extensively studied are not deductively co-extensive. Some, like Grzegorczyk's theory TC, are centered around what came to be known as Tarski's Law (or Editor Axiom), and some of the variants include the empty string as a unit element. Others, such as the theory QT^+ used in [1] and here, and a closely related theory F originally introduced by Tarski in [8],

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are on their face more explicitly theories of semi-groups with two generators. Nonetheless, all these theories turn out to be mutually interpretable on account of their mutual interpretability with Q. Our choice of QT^+ is motivated by the "ground-up" approach exemplified in the formula-selection method expounded below in Section 3.

In Sections 1 and 2 we give a preview of our interpretation of T in concatenation theory. In Section 3 we introduce the concatenation theory QT^+, explain the main methodological tool used throughout the paper, the formula selection method applied to tractable strings and string forms, and develop elements of formal concatenation theory QT^+ related to tallies, adding of tallies and parts of strings. In Section 4 we describe the essentials of the coding methods subsequently used in formalization of definitions by string recursion in Section 5. The resulting formal schema of definition is applied to obtain definitions of counting functions α and β which we rely on to construct the formal interpretation introduced in Sections 1 and 2. In Section 6 the interpretation is formally defined, and translations of the axioms of T formally verified. There we state the main result of the paper, the First Mutual Interpretability Theorem of Weak Essentially Undecidable Theories, relating T and QT^+ to a number of well-known theories of numbers, strings, and sets. Finally, in Section 7, we introduce concatenation variants WQT and WQT* of Robinson's theory R and establish the corresponding Second Mutual Interpretability Theorem with the weak theory of trees WT.

Many of our arguments involve construction of specific formulas and tedious verifications of their specific properties. Most of these details can be found in [3]. The entire formal construction ultimately rests on coding of sets of strings by strings within QT^+, which is given in complete detail in [2]. We provide specific references as needed. The author is grateful to the anonymous referee for extremely useful critical comments.

§1. Trees as strings. The intended domain of interpretation of the theory T is the set of variable-free $\mathcal{L}_{\mathrm{T}}$-terms

$$0, (00), \big(0(00)\big), \big((00)0\big), \big((00)(00)\big), \dots . \qquad (*)$$

Alternatively, we may think of the domain as consisting of finite full binary trees—also called 2-trees—trees in which every node other than the end nodes has two immediate descendants. In order to interpret T in concatenation theory, we need some way of representing these objects—terms or trees—by binary strings. We would like to do this directly, without having to rely on a coding of sets or sequences.

For this purpose we will use a variant of Polish notation to read binary strings as codes for inductively generated objects having the structure characteristic of terms or trees. Thus, e.g., the terms in $(*)$ will be coded, respectively, by

$$\mathrm{a}, \mathrm{baa}, \mathrm{babaa}, \mathrm{bbaaa}, \mathrm{bbaabaa}, \dots . \qquad (**)$$

To obtain the string code from a given variable-free $\mathcal{L}_{\mathrm{T}}$-term we proceed from left to right by replacing the left parentheses by b's and 0's by a's, ignoring the right parentheses.

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Looking at the strings that are examples of term codes in (∗∗), we note that they all share the following features:

(c1) the total number of a's in the string exceeds the total number of b's exactly by 1,

(c2) each proper initial segment of the string has at least as many b's as a's.

In other words, each of these strings is its own smallest initial segment in which the number of *a*'s strictly exceeds the number of *b*'s. We will take this to be the defining property of binary term/tree codes. We offer the following as informal justification. Each *b* indicates a branching vertex, incurring a "debt" of two "open places," which need to filled by completing the branchings. That can be done either immediately by simply writing *a*, an end node, or by opening another branching, temporarily increasing the "debt of open places." Each successive *a* reduces the "debt of places" to be filled by one, until all open branchings are completed and the last two remaining "places" filled with *a*'s, resulting in a full binary tree. Ultimately, *b*'s in the binary code track the number of branchings, i.e., non-terminal nodes, and *a*'s the number of terminal nodes in the tree.

To define the domain of the formal interpretation of T in concatenation theory we will need to be able to single out term codes from among arbitrary strings by means of a formula of concatenation theory. The key role in this connection will be played by functions α and β that count the number of occurrences of the letters a, b, resp., in a given binary string. They are defined as follows:

$$\begin{aligned} \alpha(\mathrm{a}) &= 1, & \beta(\mathrm{a}) &= 0, \\ \alpha(\mathrm{b}) &= 0, & \beta(\mathrm{b}) &= 1, \\ \alpha(\mathrm{x}^*\mathrm{a}) &= \alpha(\mathrm{x}) + 1, & \beta(\mathrm{x}^*\mathrm{a}) &= \beta(\mathrm{x}), \\ \alpha(\mathrm{x}^*\mathrm{b}) &= \alpha(\mathrm{x}), & \beta(\mathrm{x}^*\mathrm{b}) &= \beta(\mathrm{x}) + 1. \end{aligned}$$

Call a string x *almost even*, or Æ-string, writing Æ(x), if (c1) $\alpha(\mathrm{x}) = \beta(\mathrm{x})+1$, and (c2) for each proper initial segment u of x, $\alpha(\mathrm{u}) \leq \beta(\mathrm{u})$.

Within concatenation theory the values of α, β will be expressed by b-tallies, i.e., strings of consecutive *b*'s. The functions α and β are additive in that

$$\alpha(\mathrm{x}^*\mathrm{y}) = \alpha(\mathrm{x}) + \alpha(\mathrm{y}) \text{ and } \beta(\mathrm{x}^*\mathrm{y}) = \beta(\mathrm{x}) + \beta(\mathrm{y}).$$

To express and verify these properties in concatenation theory we will need to introduce a suitable operation Addtally having the requisite properties of addition on non-negative integers. But the main problem to be solved is showing that α and β, which are defined by recursion on strings, can actually be defined in formal concatenation theory.

§2. Outline of the interpretation. The language $\mathcal{L}_{\mathrm{C}} = \{\mathrm{a}, \mathrm{b}, {}^*\}$ of concatenation theory has two individual constants a, b, and a single binary operation symbol *. Its intended interpretation Σ^* has as its domain the set of all non-empty finite strings of *a*'s and *b*'s, the constants "a," "b," resp., stand for the digits a, b (or 0, 1, resp.), and, for given strings x, y from the domain of Σ^*, we take $\mathrm{x}^*\mathrm{y}$ to be the string obtained by concatenation (i.e., juxtaposition) of the successive digits of y to the right of the end digit of x. For the purpose of informal exposition of the basic idea behind the interpretation we will avail ourselves, "as a first approximation,"

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of formulations couched in the first-order theory $\mathrm{Th}(\Sigma^*)$ consisting of all true sentences of $\mathcal{L}_C$ in Σ^*. Specifically, at this point, we will simply assume that the graphs of the functions α, β are expressible by some formulas $A^{\#}(x,y)$, $B^{\#}(x,y)$, resp., of $\mathcal{L}_C$ along with the graph of Addtally, and carry on reasoning informally within $\mathrm{Th}(\Sigma^*)$. In subsequent sections we turn to the detailed technical work of actually proving these assumptions by formalizing string recursion in concatenation theory and verifying the corresponding translations into $\mathcal{L}_C$ of the axioms of T, all of which has to be formally carried out within an extremely weak subtheory QT^+ of $\mathrm{Th}(\Sigma^*)$.

First, some abbreviations. Let $xBy \equiv \exists z\, x^*z = y$ and $xEy \equiv \exists z\, z^*x = y$.

Then let $x \subseteq_p y \equiv x = y \vee xBy \vee xEy \vee \exists y_1 \exists y_2 y = y_1 * (x^*y_2)$.

(Often, we shall write xy for x^*y.) Under the assumption that formulae $A^{\#}(x,y)$, $B^{\#}(x,y)$ express in $\mathcal{L}_C$ the graphs of the functions α, β, we shall write, e.g., "$\alpha(x) \leq \beta(y)$" and "$\alpha(x) = \beta(y) + 1$" with the understanding that these expressions abbreviate appropriately chosen $\mathcal{L}_C$ formulas such as "$\exists x_1, y_1 \big(A^{\#}(x, x_1)\ \&\ B^{\#}(y, y_1)\ \&\ (x_1 = y_1 \vee x_1 B y_1)$" and "$\exists x_1, y_1 \big(A^{\#}(x, x_1)\ \&\ B^{\#}(y, y_1)\ \&\ x_1 = by_1\big)$." We can then establish:

2.1. (a) $\Sigma^* \models Æ(x) \rightarrow x = a \vee (bBx\ \&\ aaEx)$.

(b) $\Sigma^* \models Æ(x)\ \&\ x_2Ex \rightarrow \alpha(x_2) \geq \beta(x_2) + 1$.

(c) $\Sigma^* \models Æ(x)\, Æ(u)\ \&\ xy = uv \rightarrow x = u\ \&\ y = v$.

Proof. (a) Clearly, $\Sigma^* \models Æ(a)$. Assume $\Sigma^* \models Æ(x)\ \&\ x \neq a$. Then $\Sigma^* \models \neg\, aBx$, by (c2). So $\Sigma^* \models bBx$. Note that $\Sigma^* \models \neg\, Æ(aa)\ \&\ \neg\, Æ(ab)\ \&\ \neg\, Æ(ba)\ \&\ \neg\, Æ(bb)$.

Hence any x such that $\Sigma^* \models Æ(x)$ must have a (proper) end segment of length 2. Suppose $\Sigma^* \models x = x_1ab \vee x = x_1ba \vee x = x_1bb$, that is, $\Sigma^* \models abEx \vee baEx \vee bbEx$. By (c1) and (c2), we have $\Sigma^* \models \alpha(x) = \beta(x) + 1$, and also $\Sigma^* \models \alpha(x_1) \leq \beta(x_1)$. If $\Sigma^* \models abEx$ or $\Sigma^* \models baEx$, then $\Sigma^* \models \alpha(x) = \alpha(x_1) + 1$ and $\Sigma^* \models \beta(x) = \beta(x_1) + 1$. But then

$$\begin{aligned}\Sigma^* \models \alpha(x) = \beta(x) + 1 &= (\beta(x_1) + 1) + 1\\ &= \beta(x_1) + 2 \geq \alpha(x_1) + 2 > \alpha(x_1) + 1 = \alpha(x),\end{aligned}$$

a contradiction. On the other hand, if $\Sigma^* \models bbEx$, then $\Sigma^* \models \alpha(x) = \alpha(x_1)$ and $\Sigma^* \models \beta(x) = \beta(x_1) + 2$. But then

$$\begin{aligned}\Sigma^* \models \alpha(x) = \beta(x) + 1 &= (\beta(x_1) + 2) + 1\\ &= \beta(x_1) + 3 \geq \alpha(x_1) + 3 = \alpha(x) + 3 > \alpha(x),\end{aligned}$$

a contradiction again. Hence $\Sigma^* \models \neg\, abEx\ \&\ \neg baEx\ \&\ \neg bbEx$. But then we must have $\Sigma^* \models aaEx$.

(b) Assume $\Sigma^* \models Æ(x)\ \&\ x_2Ex$. Then $\Sigma^* \models \exists x_1 x = x_1x_2$, hence $\Sigma^* \models \alpha(x_1) \leq \beta(x_1)$. But $\Sigma^* \models \alpha(x) = \beta(x) + 1$ and

$$\Sigma^* \models \alpha(x) = \alpha(x_1x_2) = \alpha(x_1) + \alpha(x_2),$$

whereas $\Sigma^* \models \beta(x) = \beta(x_1x_2) = \beta(x_1) + \beta(x_2)$. Then

$$\Sigma^* \models \alpha(x_1) + \alpha(x_2) = \beta(x_1) + \beta(x_2) + 1,$$

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whence from $\Sigma^* \models \alpha(x_1) \leq \beta(x_1)$ we have $\alpha(x_2) \geq \beta(x_2) + 1$, as claimed.
(c) Assume $\Sigma^* \models Æ(x)$ & $Æ(u)$ & $xy = uv$. We have that

$$\Sigma^* \models (x = u \;\&\; y = v) \vee xBu \vee uBx.$$

Suppose $\Sigma^* \models xBu$. From $\Sigma^* \models Æ(u)$, we have $\Sigma^* \models \alpha(x) \leq \beta(x)$. From $\Sigma^* \models Æ(x)$, we have $\Sigma^* \models \alpha(x) = \beta(x) + 1$. But then $\Sigma^* \models \beta(x) + 1 \leq \beta(x)$, a contradiction. Likewise $\Sigma^* \models uBx$ yields a contradiction. Hence $\Sigma^* \models x = u \;\&\; y = v$. ⊣

2.2. $\Sigma^* \models Æ(x) \leftrightarrow x = a \vee \exists\, !\, y, z\ (x = b(yz)\ \&\ Æ(y)\ \&\ Æ(z))$.

PROOF. ($\Leftarrow$) Assume $\Sigma^* \models Æ(y)\ \&\ Æ(z)\ \&\ x = byz$. Then

$$\Sigma^* \models \alpha(y) = \beta(y) + 1 \;\&\; \alpha(z) = \beta(z) + 1.$$

Now, we have $\Sigma^* \models \alpha(x) = \alpha(byz) = \alpha(yz) = \alpha(y) + \alpha(z)$ and further $\Sigma^* \models \beta(x) = \beta(byz) = \beta(b) + \beta(yz) = \beta(y) + \beta(z) + 1$. Then

$$\begin{aligned}\Sigma^* \models \alpha(x) &= \alpha(y) + \alpha(z) = (\beta(y) + 1) + (\beta(z) + 1)\\ &= (\beta(y) + \beta(z) + 1) + 1 = \beta(x) + 1,\end{aligned}$$

which verifies (c1). For (c2), assume $\Sigma^* \models uBx$, i.e., that $\Sigma^* \models uBbyz$.
Then $\Sigma^* \models u = b \vee uBby \vee u = by \vee \exists\, z_1\ (z_1Bz \;\&\; u = byz_1)$.
To illustrate the proof, we consider the case $\Sigma^* \models \exists\, z_1\ (z_1Bz \;\&\; u = byz_1)$.
Then from $\Sigma^* \models Æ(z)$, we have $\Sigma^* \models \alpha(z_1) \leq \beta(z_1)$. From $\Sigma^* \models Æ(y)$, we have $\Sigma^* \models \alpha(y) = \beta(y) + 1$. Then $\Sigma^* \models \alpha(u) = \alpha(byz_1) = \alpha(yz_1) = \alpha(y) + \alpha(z_1)$ whereas $\Sigma^* \models \beta(u) = \beta(byz_1) = \beta(b) + \beta(yz_1) = \beta(y) + \beta(z_1) + 1$. It follows that

$$\begin{aligned}\Sigma^* \models \alpha(u) = \alpha(y) + \alpha(z_1) &= (\beta(y) + 1) + \alpha(z_1)\\ &\leq (\beta(y) + 1) + \beta(z_1) = \beta(y) + \beta(z_1) + 1 = \beta(u).\end{aligned}$$

Thus $\Sigma^* \models \alpha(u) \leq \beta(u)$. This completes the proof of (c2). So $\Sigma^* \models Æ(x)$.
($\Rightarrow$) Assume $\Sigma^* \models Æ(x)$ & $x \neq a$. Then, by 2.1(a), $\Sigma^* \models bBx \;\&\; aaEx$, that is,

$$\Sigma^* \models \exists\, x_1 x = bx_1 \;\&\; \exists\, x_2 x = x_2 aa.$$

So $\Sigma^* \models bx_1 = x_2 aa$. We may assume that $\Sigma^* \models bBx_2$, for if $\Sigma^* \models x_2 = b$, then $\Sigma^* \models x = b(aa)$ and we may take $y = a$ and $z = a$. So $\Sigma^* \models \exists\, x_3 x_2 = bx_3$, and $\Sigma^* \models x = bx_1 = x_2(aa) = bx_3(aa)$, whence $\Sigma^* \models x_1 = x_3(aa)$. Let y_j be a proper initial segment of x_1, and z_j the corresponding end segment of x_1 such that $\Sigma^* \models y_j z_j = x_1$. At least one y_j has the property

$$\Sigma^* \models \alpha(y_j) = \beta(y_j) + 1. \qquad (*)$$

Consider, e.g., $x_3 a$. From hypothesis $\Sigma^* \models Æ(x)$ we have $\Sigma^* \models \alpha(x) = \beta(x) + 1$.
But $\Sigma^* \models \alpha(x) = \alpha(b((x_3 a)\, a)) = \alpha(b) + \alpha(x_3 a) + \alpha(a) = \alpha(x_3 a) + 1$ and

$$\Sigma^* \models \beta(x) = \beta(b((x_3 a)\, a)) = \beta(b) + \beta(x_3 a) + \beta(a) = 1 + \beta(x_3 a).$$

Then $\Sigma^* \models \alpha(x_3 a) = \alpha(x) - 1 = \beta(x) = \beta(x_3 a) + 1$, as needed.
Let y_i be the shortest initial segment of x_1 with the property $(*)$. Then

$$\Sigma^* \models x_1 = y_i z_i \;\&\; \alpha(y_i) = \beta(y_i) + 1.$$

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We claim that (i) $\Sigma^* \models \alpha(z_i) = \beta(z_i) + 1$, (ii) $\Sigma^* \models \forall\, u\,(uBy_i \rightarrow \alpha(u) \leq \beta(u))$, and (iii) $\Sigma^* \models \forall\, v\,(vBz_i \rightarrow \alpha(v) \leq \beta(v))$.

For (i) we have $\Sigma^* \models \alpha(x) = \alpha(bx_1) = \alpha(x_1) = \alpha(y_iz_i) = \alpha(y_i) + \alpha(z_i)$ and $\Sigma^* \models \beta(x) = \beta(bx_1) = 1 + \beta(x_1) = 1 + \beta(y_iz_i) = 1 + \beta(y_i) + \beta(z_i)$.

Then $\Sigma^* \models \alpha(y_i) + \alpha(z_i) = (1 + \beta(y_i) + \beta(z_i)) + 1$, and from $\Sigma^* \models \alpha(y_i) = \beta(y_i) + 1$ we obtain $\Sigma^* \models \alpha(z_i) = \beta(z_i) + 1$.

For (ii), suppose $\Sigma^* \models uBy_i$. Since $\Sigma^* \models x_1 = y_iz_i$, we then have $\Sigma^* \models uBx_1$. But then, by the choice of y_i, $\Sigma^* \models \alpha(u) \leq \beta(u)$. For (iii), suppose $\Sigma^* \models vBz_i$. Then $\Sigma^* \models \exists\, w\, z_i = vw$, whence $\Sigma^* \models wEx$. From $\Sigma^* \models Æ(x)$, by 2.1(b), $\Sigma^* \models \alpha(w) \geq \beta(w) + 1$. But $\Sigma^* \models \alpha(z_i) = \alpha(v) + \alpha(w)$ and $\Sigma^* \models \beta(z_i) = \beta(v) + \beta(w)$. By (i), $\Sigma^* \models \alpha(v) + \alpha(w) = \beta(v) + \beta(w) + 1$.

Then from $\Sigma^* \models \alpha(w) \geq \beta(w) + 1$, we have $\Sigma^* \models \alpha(v) \leq \beta(v)$.

From (i)–(iii) we have that $\Sigma^* \models Æ(y_i)$ & $Æ(z_i)$. The uniqueness of y, z follows from 2.1(c). ⊣

2.3. $\Sigma^* \models Æ(x)$ & $Æ(y)$ & $Æ(z) \rightarrow \big(x \subseteq_p byz \rightarrow x = byz \vee x \subseteq_p y \vee x \subseteq_p z\big)$.

PROOF. Assume $\Sigma^* \models x \subseteq_p byz$ where $\Sigma^* \models Æ(x)$ & $Æ(y)$ & $Æ(z)$. Now, we have that

$$\Sigma^* \models x = byz \vee x = b \vee x \subseteq_p yz \vee \exists\, u\,(uByz \,\&\, x = bu).$$

Suppose that $\Sigma^* \models \exists\, u\,(uByz \,\&\, x = bu)$. From $\Sigma^* \models Æ(y)$ & $Æ(z)$, by 2.2, $\Sigma^* \models Æ(byz)$. From $\Sigma^* \models uByz$, we have $\Sigma^* \models \exists\, v\, uv = yz$, whence $\Sigma^* \models buBb(yz)$. Thus $\Sigma^* \models xBb(yz)$. But from $\Sigma^* \models Æ(byz)$, we also have $\Sigma^* \models \alpha(x) \leq \beta(x)$, which contradicts $\Sigma^* \models Æ(x)$. So $\Sigma^* \models \exists\, u\,(uByz \,\&\, x = bu)$ is ruled out. By 2.1(a), so is $\Sigma^* \models x = b$. So we are left with $\Sigma^* \models x \subseteq_p byz \rightarrow x = byz \vee x \subseteq_p yz$. Supposing $\Sigma^* \models x \subseteq_p yz$, we have that

$$\Sigma^* \models x = yz \vee x \subseteq_p y \vee x \subseteq_p z \vee \exists\, y_1\,(y_1Ey \,\&\, x = y_1z) \vee \exists\, z_1\,(z_1Bz \,\&\, x = yz_1) \vee \exists\, y_1, z_1\,(y_1Ey \,\&\, z_1Bz \,\&\, x = y_1z_1).$$

Assume $\Sigma^* \models x = yz$. Then from $\Sigma^* \models Æ(y)$ & $Æ(z)$, we have $\Sigma^* \models \alpha(y) = \beta(y) + 1$ and $\alpha(z) = \beta(z) + 1$. But $\Sigma^* \models \alpha(yz) = \alpha(y) + \alpha(z)$, so

$$\Sigma^* \models \alpha(yz) = (\beta(y) + 1) + (\beta(z) + 1) = \beta(y) + \beta(z) + 2.$$

On the other hand, $\Sigma^* \models \beta(yz) = \beta(y) + \beta(z)$. Thus $\Sigma^* \models \alpha(yz) = \beta(yz) + 2$, whence from $\Sigma^* \models x = yz$, we derive $\Sigma^* \models \alpha(x) = \beta(x) + 2$, contradicting $\Sigma^* \models Æ(x)$. So $\Sigma^* \models x = yz$ is ruled out.

Suppose now that $\Sigma^* \models \exists\, y_1\,(y_1Ey \,\&\, x = y_1z)$, so $\Sigma^* \models y_1Bx$. From $\Sigma^* \models Æ(x)$, we have $\Sigma^* \models \alpha(y_1) \leq \beta(y_1)$. But from $\Sigma^* \models Æ(y)$ & y_1Ey, we also obtain, by 2.1(b), $\Sigma^* \models \alpha(y_1) \geq \beta(y_1) + 1$, a contradiction.

Suppose that $\Sigma^* \models \exists\, z_1\,(z_1Bz \,\&\, x = yz_1)$, so $\Sigma^* \models yBx$. But then from $\Sigma^* \models Æ(x)$, we have $\Sigma^* \models \alpha(y) \leq \beta(y)$. From $\Sigma^* \models Æ(y)$, we have $\Sigma^* \models \alpha(y) = \beta(y) + 1$, again a contradiction. If $\Sigma^* \models \exists\, y_1, z_1\,(y_1Ey \,\&\, z_1Bz \,\&\, x = y_1z_1)$, we derive a contradiction by reasoning as in either of the two preceding cases. This proves that $\Sigma^* \models x \subseteq_p byz \rightarrow x = byz \vee x \subseteq_p y \vee x \subseteq_p z$, as required. ⊣

If we take the domain to consists of Æ-strings, 2.1(c), 2.2, and 2.3 suffice to give the "first approximation" of our interpretation of T in concatenation theory:

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translations of (T1)–(T4) will be validated in Σ^* if we model the term/tree-building operation $x, y \mapsto (xy)$ by bxy, the subterm/subtree relation $\sqsubseteq$ by the substring relation $\subseteq_p$ between Æ-strings, and the digit a is taken to stand for the simple term 0. The entire project, however, hinges on definability of the counting functions α and β in concatenation theory. Showing that the latter contains resources needed to formally justify definitions by elementary recursion on strings requires, first, that we precisely formulate concatenation theory as a formal theory, and second, that we introduce codings for ordered pairs of strings, sequences of such, etc., and verify in that formal theory their properties relevant to the argument. We now turn to that task. In the process we shall make crucial use of the method of formula selection explained in [1].

§3. Formal concatenation theory. We shall work within a first-order theory formulated in $\mathcal{L}_C = \{a, b, *\}$, with the universal closures of the following conditions as axioms:

(QT1) $x^*(y^*z) = (x^*y) * z$,
(QT2) $\neg\ (x^*y = a)\ \&\ \neg\ (x^*y = b)$,
(QT3) $(x^*a = y^*a \to x = y)\ \&\ (x^*b = y^*b \to x = y)$
$\&\ (a^*x = a^*y \to x = y)\ \&\ (b^*x = b^*y \to x = y)$,
(QT4) $\neg\ (a^*x = b^*y)\ \&\ \neg\ (x^*a = y^*b)$,
(QT5) $x = a \vee x = b \vee (\exists y\,(a^*y = x \vee b^*y = x)\ \&\ \exists z\,(z^*a = x \vee z^*b = x))$.

On account of (QT1), we sometimes omit parentheses and * when writing (x^*y).

It is convenient to have a function symbol for a successor operation on strings:

(QT6) $Sx = y \leftrightarrow ((x = a\ \&\ y = b) \vee (\neg\ x = a\ \&\ x^*b = y))$.

Since (QT6) is basically a definition, adding it to the rest results in an inessential (i.e., conservative) extension. We call this theory QT^+.

Let $xRy \equiv (x = a\ \&\ \neg\ y = a)\ \vee\ xBy$.

Provably in QT^+, $xRy \vee x = y$ is a discrete preordering of strings (see [1]).

We shall call a formula $I(x)$ in the language of QT^+ a string form if

$$QT^+ \vdash I(a), QT^+ \vdash I(b), QT^+ \vdash I(x) \to I(x^*a), \text{ and } QT^+ \vdash I(x) \to I(x^*b).$$

(Note: in [1, 2] such formulae were called string concepts.) String forms allow us to restrict our attention, systematically step-by-step, to strings that satisfy conditions expressible by specifically selected formulas provided the latter can be proved in QT^+ to apply to "sufficiently many" strings. We say that a string form J is stronger than I if $QT^+ \vdash \forall x\,(J(x) \to I(x))$ and write $J \subseteq I$.

As a theory of strings of symbols, or "texts," QT^+ is quite weak: it can be shown, e.g., that "$\forall x\ \neg xBx$" is not deducible in QT^+, so in QT^+ we cannot rule out the possibility that strings may turn out to be proper initial segments of themselves (see [1, Section 3]). This seems to suggest that QT^+ lacks a workable notion of occurrence for objects it deals with, precluding development in QT^+ of basic syntax of strings thought of as words in some alphabet. We deal with this problem, firstly, by noting that $QT^+ \vdash \neg aBa\ \&\ \neg bBb$, and, secondly, by showing that a formula expressing the relevant property in $\mathcal{L}_C$ is a string form.

Let $I_0(x) \equiv \forall y\,(yRx \vee y = x \to\ \neg\ yRy)$. We call I_0 strings tractable.

We write $x < y$ for $I_0(x)\ \&\ I_0(y)\ \&\ xRy$. As usual, $x \leq y$ stands for $x < y \vee x = y$.

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3.1. (a) $I_0(x)$ is a string form

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash J(x) \ \& \ J(y) \to J(x^*y).$$

(c) For any string form $I \subseteq I_0$ there is a string form $J_{\leq} \subseteq I$ such that

$$QT^+ \vdash J_{\leq}(x) \ \& \ y \leq x \to J_{\leq}(y).$$

(d) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall x \in J\big(y \subseteq_p x \to J(y)\big).$$

(e) For any string form $I \subseteq I_0$ there is a string form $J \equiv I_{LC} \subseteq I$ such that

$$QT^+ \vdash \forall z \in J(z^*x = z^*y \to x = y).$$

(f) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall z \in J(x^*z = y^*z \to x = y).$$

For proofs, see [1], [2, (3.2), (3.3), (3.6), (3.7), and (3.13)]. ⊣

Parts (b–d) tell us that when establishing that a given string form I may be strengthened to a string form J with another property, we can always strengthen the string form J to one that is also closed with respect to * or downward closed with respect to $\leq$ or $\subseteq_p$.

We define $\mathrm{Tally}_a(x) \equiv \forall y \subseteq_p x\,(\mathrm{Digit}(y) \to y = a)$ and $\mathrm{Tally}_b(x) \equiv \forall y \subseteq_p x\,(\mathrm{Digit}(y) \to y = b)$ where $\mathrm{Digit}(x) \equiv x = a \vee x = b$.

The following properties of tallies are easily established:

3.2. (a) $QT^+ \vdash \mathrm{Tally}_b(y) \to \mathrm{Tally}_b(Sy)$.

(b) $QT^+ \vdash \mathrm{Tally}_b(y) \leftrightarrow y = b \vee \exists y_1\,(\mathrm{Tally}_b(y_1) \ \& \ y = Sy_1)$.
(c) $QT^+ \vdash \mathrm{Tally}_b(v) \ \& \ u < v \to Su \leq v$.
(d) $QT^+ \vdash \mathrm{Tally}_b(y) \to (x < y \leftrightarrow Sx < Sy)$.

For some further properties we have to resort to string forms:

3.3. (a) For any string form $I \subseteq I_0$ there is a string form $J \equiv I_{CTC} \subseteq I$ such that.

$$QT^+ \vdash \forall z \in J(\mathrm{Tally}_b(y) \ \& \ \mathrm{Tally}_b(z) \to \mathrm{Tally}_b(y^*z)).$$

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall z \in J(\mathrm{Tally}_b(x) \ \& \ \mathrm{Tally}_b(z) \to x \leq z \vee z \leq x).$$

(c) For any string form $I \subseteq I_0$ there is a string form $J \equiv I_{3.3(c)} \subseteq I$ such that

$$QT^+ \vdash \forall u \in J(\mathrm{Tally}_b(u) \to u^*b = b^*u).$$

(d) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall y \in J(\mathrm{Tally}_b(x) \ \& \ \mathrm{Tally}_b(y) \to Sx^*y = x^*Sy = S(x^*y)).$$

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(e) For any string form $I \subseteq I_{3.3(c)}$ there is a string form $J \equiv I_{COMM} \subseteq I$ such that

$$QT^+ \vdash \forall\, u, v \in J\,(\mathrm{Tally}_b(u)\ \&\ \mathrm{Tally}_b(v) \to u^*v = v^*u).$$

For proofs, see [2, (4.5), (4.6), (4.8), and (4.10)]. ⊣

Let Addtally(x,y,z) abbreviate the formula

$$\begin{aligned}\big(\mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y)\ \&\ \big(&(x = b\ \&\ z = y)\ \vee\ (y = b\ \&\ z = x)\\ &\vee\ \exists\, x_1, y_1\,(\mathrm{Tally}_b(x_1)\ \&\ x = Sx_1\ \&\ \mathrm{Tally}_b(y_1)\ \&\ y = Sy_1\ \&\ z = x^*y_1))\\ &\vee\ ((\neg\ \mathrm{Tally}_b(x)\ \vee \neg\ \mathrm{Tally}_b(y))\ \&\ z = b).\end{aligned}$$

We want to show that, provably in QT^+, Addtally(x,y,z) behaves like the graph of addition function on natural numbers. The following are immediate consequences of definitions:

3.4. (a) $QT^+ \vdash \mathrm{Addtally}\,(x, y, v)\ \&\ \mathrm{Addtally}\,(x, y, w) \to v = w.$

(b) $QT^+ \vdash \mathrm{Tally}_b(x) \to \mathrm{Addtally}\,(x, b, x).$ ("$x + 0 = x$")
(c) $QT^+ \vdash \mathrm{Tally}_b(y) \to \mathrm{Addtally}\,(b, y, y).$ ("$0 + y = y$")
(d) $QT^+ \vdash \mathrm{Tally}_b(x) \to \mathrm{Addtally}\,(x, bb, Sx).$ ("$x + 1 = Sx$")
(e) $QT^+ \vdash \mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y) \to (\mathrm{Addtally}\,(x, y, z) \to \mathrm{Addtally}\,(x, yb, zb)).$
("$x + Sy = S\,(x + y)$")

We also have:

3.5. (a) For any string form $I \subseteq I_0$ there is a string form $J \equiv I_{Add} \subseteq I$ such that.

$$QT^+ \vdash \forall\, x, y \in J\ \exists\, !\, z \in J\,(\mathrm{Tally}_b(z)\ \&\ \mathrm{Addtally}\,(x, y, z)).$$

(b) $$\begin{aligned}QT^+ \vdash \forall\, z \in I_0\big(&\mathrm{Tally}_b(u)\ \&\ \mathrm{Tally}_b(v)\\ &\&\ \mathrm{Addtally}\,(x, u, y)\ \&\ \mathrm{Addtally}\,(x, v, z)\ \&\ u \le v \to y \le z\big).\end{aligned}$$
("$u \le v \to x + u \le x + v$")

(c) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y \in J\ \big(\mathrm{Tally}_b(y) \to \mathrm{Addtally}\,(bb, y, Sy).\qquad (\text{“}1 + y = Sy\text{”})$$

(d) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y \in J\,(\mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y)\ \&\ \mathrm{Addtally}\,(x, y, z) \to \mathrm{Addtally}\,(xb, y, zb)).$$
("$Sx + y = S\,(x + y)$")

(e) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$\begin{aligned}QT^+ \vdash \forall\, x \in J\big(&\mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y)\ \&\ \mathrm{Tally}_b(z)\\ &\to (\mathrm{Addtally}\,(x, y, v)\ \&\ \mathrm{Addtally}\,(x, z, v) \to y = z)\big).\end{aligned}$$
("$x + y = x + z \to y = z$")

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(f) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y \in J\,(\mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y) \rightarrow (x \leq y \leftrightarrow\ \exists\, z\,(\mathrm{Tally}_b(z)\ \&\ \mathrm{Addtally}\,(z, x, y)))).$$
$$(\text{“}x \leq y \leftrightarrow\ \exists\, z\ z + x = y\text{”})$$

(g) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x, y \in J\,(\mathrm{Addtally}\,(x, y, z) \rightarrow \mathrm{Addtally}\,(y, x, z)). \quad (\text{“}x + y = y + x\text{”})$$

(h) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x, y, z \in J\,(\mathrm{Tally}_b(x)\ \&\ \mathrm{Tally}_b(y)\ \&\ \mathrm{Tally}_b(z)\ \&\ \mathrm{Addtally}\,(x, y, u)\ \&\ \mathrm{Addtally}\,(u, z, v_1)\ \&\ \mathrm{Addtally}\,(y, z, w)\ \&\ \mathrm{Addtally}\,(x, w, v_2) \rightarrow v_1 = v_2).$$
$$(\text{“}(x + y) + z = x + (y + z)\text{”})$$

(i) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash\ \forall\, x_2, y_1, y_2 \in J\big(\mathrm{Tally}_b(x_2)\ \&\ \mathrm{Tally}_b(y_1)\ \&\ \mathrm{Tally}_b(y_2)\ \&\ \mathrm{Addtally}\,(x_1, x_2, z_1)\ \&\ \mathrm{Addtally}\,(y_1, y_2, z_2)\ \&\ x_1 \leq y_1\ \&\ z_1 = Sz_2 \rightarrow Sy_2 \leq x_2\big).$$
$$(\text{“}x_1 + x_2 = (y_1 + y_2) + 1\ \&\ x_1 \leq y_1 \rightarrow y_2 + 1 \leq x_2\text{”})$$

Proof. For (a), let $J \equiv I_{CTC}$ from 3.3(a). For (c) and (d), let J be as in 3.3(c). For (e), let $J \equiv I_{LC}$ from 3.1(e). For (f) and (g), let $J \equiv I_{COMM}$ from 3.3(e). For (h), let $J \equiv J_1\ \&\ J_2$ where J_1 is I_{CTC} and J_2 as in 3.3(c). Finally, for (i), let $J \equiv I_{LC}\ \&\ I_{CTC}\ \&\ I_{3.3(c)}\ \&\ I_{COMM}$ and see [3]. ⊣

We now turn to the part-of relation $\subseteq_p$ between strings. To prevent unpleasant surprises, we want to make sure that this relation has natural properties we would normally expect it to have.

3.6. (a) $QT^+ \vdash x \subseteq_p y\ \&\ y \subseteq_p z \rightarrow x \subseteq_p z$.

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\ \neg\, xEx.$$

(c) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\ \neg\ \exists\, x_1, x_2\,(x_1 x x_2 = x).$$

(d) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\,\big(x \subseteq_p y\ \&\ y \subseteq_p x \rightarrow x = y\big).$$

(e) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\,\big(\neg\, xy \subseteq_p x\ \&\ \neg\, yx \subseteq_p x\big).$$

Proof. For (b) and (c), see [2, (3.4) and (3.5)]. For (d) and (e), see [2, (3.11) and (3.12)]. ⊣

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We now specifically consider proper initial segments and end segments. The initial segments of arbitrary strings can be totally ordered by the initial-segment-of relation B, rendering the partial ordering $<$ in which a is the least element tree-like:

3.7. (a) For any string form $I \subseteq I_0$ there is a string form $J \equiv J_{LOIS} \subseteq I$ such that.

$$QT^+ \vdash \forall\, x \in J\,(uBx \,\&\, vBx \rightarrow u = v \vee uBv \vee vBu).$$

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y, z \in J\,(xByz \leftrightarrow xBy \vee x = y \vee \exists\, w\,(wBz \,\&\, yw = x)).$$

(c) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x, y \in J\,(uBb\,(xy) \rightarrow u = b \vee uBbx \vee u = bx \vee \exists\, y_1\,(y_1By \,\&\, u = bxy_1)).$$

(d) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\,(uEx \,\&\, vEx \rightarrow u = v \vee uEv \vee vEu).$$

(e) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y, z \in J\,(xEyz \leftrightarrow xEz \vee x = z \vee \exists\, w\,(wEy \,\&\, wz = x)).$$

(f) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y, z \in J\big(x_1xx_2 = yz \rightarrow x \subseteq_p y \vee x \subseteq_p z \vee \exists\, y_1, z_1\,(y_1Ey \,\&\, z_1Bz \,\&\, x = y_1z_1)\big).$$

(g) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y, z \in J\,\big(x \subseteq_p yz \rightarrow x = yz \vee x \subseteq_p y \vee x \subseteq_p z \vee \exists\, y_1\,(y_1Ey \,\&\, x = y_1z) \vee \exists\, z_1\,(z_1Bz \,\&\, x = yz_1) \vee \exists\, y_1, z_1\,(y_1Ey \,\&\, z_1Bz \,\&\, x = y_1z_1)\big).$$

(h) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, y, z \in J\big(x \subseteq_p b\,(yz) \rightarrow x = byz \vee x = b \vee x \subseteq_p yz \vee \exists\, u_2\,(u_2Byz \,\&\, x = bu_2)\big).$$

Proof. For (a), see [2, (3.8)]. For (b) and (c), let $J \equiv I_{LC} \,\&\, I_{LOIS}$. For (d), see [2, (3.10)], and then (e) is proved analogously to (b). For (f) take J as in (b), and (g) follows from (b)–(f). Then (h) is obtained as a special case of (g). ⊣

§4. Coding sequences and pairs of strings by strings. Formalizing recursion requires coding of sequences, and since the kind of recursion used to define the counting functions α and β proceeds on strings, to carry out the formalization of such definitions in concatenation theory, we will need to be able to code sequences of strings by strings. The general idea behind the coding goes back to Quine [7], and more recently to Visser [9], but the key for our purposes is to show that the relevant properties of the coding are provable in QT^+. We make use of the coding scheme described in [2, pp. 86–88] and summarized in [1, Sections 7 and 8]. To code a

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sequence $w_1,w_2,\ldots$ of arbitrary binary strings, we use the string $w = t_1aw_1at_2aw_2at_3\ldots$, where $t_1,t_2,t_3,\ldots$ are b-tallies that will serve as markers to frame the members $w_1,w_2,\ldots$ of the sequence. It will suffice for t_1 not to occur in w_1, for t_2 not to occur in w_2, etc., provided that the markers $t_1,t_2,t_3,\ldots$ strictly increase in length. The coding works like a step-ladder: starting with the b-tally that precedes the first occurrence of the letter a in w, each next longer b-tally is a successive step of the ladder marking off a frame that corresponds to another member of the coded sequence.

To express the coding in $\mathcal{L}_C$, we use a number of auxiliary predicates.

Let $\mathrm{MaxT_b}(t, w) \equiv \mathrm{Tally_b}(t)\ \&\ \forall\, t'\,(\mathrm{Tally_b}(t')\ \&\ t' \subseteq_p w \rightarrow t' \subseteq_p t)$.

With $\mathrm{Max^+T_b}(t, x) \equiv \mathrm{MaxT_b}(t, x)\ \&\ \neg\, t \subseteq_p x$, we then define when a string u is a *preframe* indexed by t:

$$\mathrm{Pref}\,(u, t) \equiv\ \exists\, y \subseteq_p u\,(aya = u\ \&\ \mathrm{Max^+T_b}(t, u));$$

when t_1ut_2 is (the) *first frame* in the string x, Firstf(x,t_1,u,t_2):

$$\mathrm{Pref}\,(u, t_1)\ \&\ \mathrm{Tally_b}(t_2)\ \&\ ((t_1 = t_2\ \&\ t_1ut_2 = x)\ \mathrm{v}\ (t_1 < t_2\ \&\ (t_1ut_2a)\,Bx));$$

when t_1ut_2 is (the) *last frame* in x, Lastf(x,t_1,u,t_2):

$$\mathrm{Pref}\,(u, t_1)\ \&\ t_1 = t_2\ \&\ (t_1ut_2 = x\ \mathrm{v}\ \exists\, w\,(wat_1ut_2 = x\ \&\ \mathrm{Max^+T_b}(t_1, w)));$$

and when t_1ut_2 is an *intermediate frame in* x *immediately following an initial segment* w of x, Intf(x,w,t_1,u,t_2):

$$\mathrm{Pref}\,(u, t_1)\ \&\ \mathrm{Tally_b}(t_2)\ \&\ t_1 < t_2\ \&\ \exists\, w_1\,(wat_1ut_2aw_1 = x)\ \&\ \mathrm{Max^+T_b}(t_1, w).$$

Then we may define when a string u is t_1,t_2-framed in x:

$$\mathrm{Fr}\,(x, t_1, u, t_2) \equiv \mathrm{Firstf}\,(x, t_1, u, t_2)\ \mathrm{v}\ \exists\, w\ \mathrm{Intf}\,(x, w, t_1, u, t_2)\ \mathrm{v}\ \mathrm{Lastf}\,(x, t_1, u, t_2).$$

We say that t_1 is the *initial*, and t_2 *terminal tally marker in* the frame.

Next we define "t envelops x," Env(t,x), to be the conjunction of the following five conditions:

(a) $\mathrm{MaxT_b}(t, x)$ "t is a longest b-tally in x,"

(b) $\exists\, u \subseteq_p x\ \exists\, t_1, t_2 \mathrm{Firstf}\,(x, t_1, u, t_2)$ "x has a first frame,"

(c) $\exists\, u \subseteq_p x\ \mathrm{Lastf}\,(x, t, u, t)$ "x has a last frame with t as its initial and terminal marker,"

(d) $\forall\, u \subseteq_p x\ \forall\, t_1, t_2, t_3, t_4\,(\mathrm{Fr}\,(x, t_1, u, t_2)\ \&\ \mathrm{Fr}\,(x, t_3, u, t_4) \rightarrow t_1 = t_3)$
"different initial tally markers frame distinct strings,"

(e) $\forall\, u_1, u_2 \subseteq_p x\ \forall\, t', t_1, t_2\,(\mathrm{Fr}\,(x, t', u_1, t_1)\ \&\ \mathrm{Fr}\,(x, t', u_2, t_2) \rightarrow u_1 = u_2)$
"distinct strings are framed by different initial tally markers."

Now we say x is a *set code* if x is aa or else x is enveloped by some b-tally:

$$\mathrm{Set}\,(x) \equiv x = aa\ \mathrm{v}\ \exists\, t \subseteq_p x\ \mathrm{Env}\,(t, x).$$

Finally, we say that a string y *is a member of the set coded by string* x if x is enveloped by some b-tally t and the juxtaposition of the string y with single tokens of digit a is framed in x:

$$y\ \varepsilon\ x \equiv\ \exists\, t \subseteq_p x\,(\mathrm{Env}\,(t, x)\ \&\ \exists\, u \subseteq_p x\ \exists\, t_1, t_2\,(\mathrm{Fr}\,(x, t_1, u, t_2)\ \&\ u = aya)).$$

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The formal machinery needed to demonstrate that, modulo the methodology of formula selection, all of the reasoning needed to verify that the coding works as intended can indeed be carried out in QT^+ is presented in detail in [2, pp. 89–263]. In particular, we can establish:

4.1. (a) SINGLETON LEMMA. For any string form $I \subseteq I_0$ there is a string form $I_{SNGL} \subseteq I$ such that.

$$QT^+ \vdash \forall\, x \in I_{SNGL}\big(\mathrm{Set}\,(x)\ \&\ \mathrm{Firstf}\,(x, t_1, aua, t_2)\ \&\ x = t_1 a u a t_2 \rightarrow \forall\, w\ \ (w\ \varepsilon\ x \leftrightarrow w = u)\big).$$

(b) APPENDING LEMMA. For any string form $I \subseteq I_0$ there is a string form $I_{APP} \subseteq I$ such that

$$QT^+ \vdash \forall\, x, y \in I_{APP}\big(\mathrm{Env}\,(t_2, x)\ \&\ \mathrm{Env}\,(t, y)\ \&\ (t_3 a)\,\mathrm{By}\ \&\ \mathrm{Tally}_b(t_3)\ \&\ t_2 < t_3 \ \&\ \neg\,\exists\, u\,(u\ \varepsilon\ x\ \&\ u\ \varepsilon\ y) \rightarrow\ \exists\, z \in I_{APP}\,(\mathrm{Env}\,(t, z)\ \&\ \forall\, u\,(u\ \varepsilon\ z \leftrightarrow u\ \varepsilon\ x\ v\ u\ \varepsilon\ y)).$$

(c) DOUBLETON LEMMA. For any string form $I \subseteq I_0$ there is a string form $I_{DBL} \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in I_{DBL}\big(\mathrm{Pref}\,(aua, t_1)\ \&\ \mathrm{Pref}\,(ava, t_2)\ \&\ t_1 < t_2\ \&\ t_2 = t_3\ \&\ u \neq v\ \&\ x = t_1 a u a t_2 a v a t_3 \rightarrow \mathrm{Set}\,(x)\ \&\ \forall\, w\,(w\ \varepsilon\ x \leftrightarrow (w = u\ v\ w = v)).$$

PROOF. See [2, (5.21), (5.46), and (5.58)]. ⊣

To use the coding of sets to code sequences of strings, we need to populate the coded sets with ordered pairs of arbitrary strings.

Let $\mathrm{Pair}\,[x, y, z] \equiv\ \exists\, t \subseteq_p z\,\big(z = taxatayat\ \&\ \mathrm{MinMax}^+T_b(t, xay)\big)$,
where $\mathrm{MinMax}^+T_b(t, u) \equiv \mathrm{Max}^+T_b(t, u)\ \&\ \forall\, t'\,\big(\mathrm{Max}^+T_b(t', u) \rightarrow t \leq t'\big)$.

We then have:

4.2. PAIRING LEMMA. (a) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x, y \in J\ \exists\, z \in J\,\big(\mathrm{Pair}\,[x, y, z]\ \&\ \forall\, z'\,\big(\mathrm{Pair}\,[x, y, z'] \rightarrow z' = z\big)\big).$$

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, z \in J\,(\mathrm{Pair}\,[x_1, y_1, z]\ \&\ \mathrm{Pair}\,[x_2, y_2, z] \rightarrow x_1 = x_2\ \&\ y_1 = y_2).$$

In (a), choose J as in [2, (6.8)]. For (b), referring to [2], let $J \equiv I_{3.6}\ \&\ I_{4.20}\ \&\ I_{4.23b}$. ⊣

§5. String recursion. Let p, q be strings, and f_1, f_2 be functions on strings. Informally, we say that h is defined by *string recursion* from f_1, f_2 if

$$h\,(a) = p, \qquad h\,(b) = q,$$

$$h\,(y^*a) = f_1\,(y, h\,(y)), \quad h\,(y^*b) = f_2\,(y, h\,(y)).$$

We want to justify definitions of this sort in QT^+.

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Let $I^{\diamond}$ be the string form that is the conjunction of the string forms used to obtain the Singleton Lemma, the Appending Lemma, the Doubleton Lemma, and the Pairing Lemma.

String Recursion Theorem. Let $F_1(y,z,u)$ and $F_2(y,z,u)$ be $\mathcal{L}_C$ formulae, and let $I \subseteq I^{\diamond}$ be closed under * and downward closed under $\subseteq_p$. Suppose that

$$QT^+ \vdash I(p) \ \& \ I(q),$$

$$QT^+ \vdash \forall\, y, z \in I\ \exists\,!\, u \in I\ F_1(y,z,u), \text{ and } QT^+ \vdash \forall\, y, z \in I\ \exists\,!\, u \in I\ F_2(y,z,u).$$

Then there is an $\mathcal{L}_C$ formula $H(y,z)$ and a string form $J \subseteq I$ such that

(i) $QT^+ \vdash \forall\, y \in J\ \exists\,!\, z \in I\ H(y,z)$,
(iia) $QT^+ \vdash \forall\, y \in I\ (H(a,y) \leftrightarrow y = p)$,
(iib) $QT^+ \vdash \forall\, y \in I\ (H(b,y) \leftrightarrow y = q)$,
(iiia) $QT^+ \vdash \forall\, y \in J\ \forall\, u, z \in I\,(H(y,u) \rightarrow (H(y^*a,z) \leftrightarrow F_1(y,u,z)))$,
(iiib) $QT^+ \vdash \forall\, y \in J\ \forall\, u, z \in I\ (H(y,u) \rightarrow (H(y^*b,z) \leftrightarrow F_2(y,u,z)))$.

(We read "$\exists!x{\in}J\ (\ldots)$" as "$\exists x\,(J(x)\ \&\ (\ldots)\ \&\ \forall y(J(y)\ \&\ (\ldots) \rightarrow y{=}x))$").

Proof. Let Comp(u,m) abbreviate

$$\begin{aligned}
&\mathrm{Set}(u)\ \&\ \big(a \le m \rightarrow\ \exists\, v \subseteq_p u\,(\mathrm{Pair}[a,p,v]\ \&\ v\ \varepsilon\ u)\big)\\
&\quad\ \&\ \big(b \le m \rightarrow\ \exists\, v \subseteq_p u\,(\mathrm{Pair}[b,q,v]\ \&\ v\ \varepsilon\ u)\big)\\
&\quad\ \&\ \forall\, z < m\ \forall\, u_1, u_2, v_1\big(\mathrm{Pair}[z,u_1,v_1]\ \&\ v_1\ \varepsilon\ u\ \&\ F_1(z,u_1,u_2)\\
&\qquad\qquad \rightarrow\ \exists\, v_2 \subseteq_p u\ (\mathrm{Pair}[z^*a,u_2,v_2]\ \&\ v_2\ \varepsilon\ u)\big)\\
&\&\ \forall\, z < m\ \forall\, u_1, u_2, v_1\big(\mathrm{Pair}[z,u_1,v_1]\ \&\ v_1\ \varepsilon\ u\ \&\ F_2(z,u_1,u_2)\\
&\qquad\qquad \rightarrow\ \exists\, v_2 \subseteq_p u\ (\mathrm{Pair}[z^*b,u_2,v_2]\ \&\ v_2\ \varepsilon\ u)\big)\\
&\&\ \forall\, z, u_1, u_2, v_1, v_2\big(\mathrm{Pair}[z,u_1,v_1]\ \&\ \mathrm{Pair}[z,u_2,v_2]\ \&\ v_1\ \varepsilon\ u\ \&\ v_2\ \varepsilon\ u\\
&\qquad\qquad \rightarrow u_1 = u_2\ \&\ v_1 = v_2\big).
\end{aligned}$$

Then let MinComp(u,m) abbreviate

$$\begin{aligned}
&\mathrm{Comp}(u,m)\ \&\ \forall\, u'\,(\mathrm{Comp}(u',m) \rightarrow \forall\, y\ \ (y\ \varepsilon\ u \rightarrow y\ \varepsilon\ u'))\\
&\quad\ \&\ \forall\, z, v, w\ \big(\mathrm{Pair}[z,v,w]\ \&\ w\ \varepsilon\ u \rightarrow (m = a\ \&\ z = a)\ v\ (m = b\ \&\ z = b)\\
&\qquad\qquad v\ \exists\, n < m\,(z \le na\ v\ z \le nb)\big).
\end{aligned}$$

Let J(m) abbreviate

$$I(m)\ \&\ \exists\,!\, y \in I\ \exists\, u \in I\ \exists\, w \subseteq_p u(\mathrm{MinComp}(u,m)\ \&\ \mathrm{Pair}[m,y,w]\ \&\ w\ \varepsilon\ u).$$

Finally, let H(m,y) abbreviate

$$\exists\, u, w\,(\mathrm{MinComp}(u,m)\ \&\ \mathrm{Pair}[m,y,w]\ \&\ w\ \varepsilon\ u).$$

For detailed verification that J and H have the desired properties see [3]. ⊣

We are now in the position to define the counting functions α and β.
Let p = bb, q = b, $F_1(y,z,u) \equiv y = y\ \&\ z = Su$ and $F_2(y,u,z) \equiv y = y\ \&\ z = u$.
Then the principal hypothesis of the String Recursion Theorem holds trivially.

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Applying the Theorem we obtain a formula $A^{\#}(y,z)$ and a string form $I_\alpha \subseteq I$ such that

(i^α) $QT^+ \vdash \forall\, y \in I_\alpha\ \exists\,!\, z \in I\ A^{\#}(y,z)$,
(iia^α) $QT^+ \vdash \forall\, z \in I\,\big(A^{\#}(a,z) \leftrightarrow z = bb\big)$,
(iib^α) $QT^+ \vdash \forall\, z \in I\,\big(A^{\#}(b,z) \leftrightarrow z = b\big)$,
$(iiia^\alpha)$ $QT^+ \vdash \forall\, y \in I_\alpha\ \forall\, u,z \in I\,\big(A^{\#}(y,u) \rightarrow \big(A^{\#}(y^*a,z) \rightarrow z = u^*b\big)\big)$,
$(iiib^\alpha)$ $QT^+ \vdash \forall\, y \in I_\alpha\ \forall\, u,z \in I\,\big(A^{\#}(y,u) \rightarrow \big(A^{\#}(y^*b,z) \rightarrow z = u\big)\big)$.

Informally speaking, $A^{\#}(y,z)$ defines the graph of the function α.

Exactly analogously, by letting p, q, and F_1, F_2, respectively, exchange places, we apply the Theorem to obtain a formula $B^{\#}(y,z)$ defining the graph of the function β and a string form $I_\beta \subseteq I$ such that

(i^β) $QT^+ \vdash \forall\, y \in I_\beta\ \exists\,!\, z \in I\ B^{\#}(y,z)$,
(iia^β) $QT^+ \vdash \forall\, z \in I\,\big(B^{\#}(a,z) \leftrightarrow z = b\big)$,
(iib^β) $QT^+ \vdash \forall\, z \in I\,\big(B^{\#}(b,z) \leftrightarrow z = bb\big)$,
$(iiia^\beta)$ $QT^+ \vdash \forall\, y \in I_\beta\ \forall\, u,z \in I\,\big(B^{\#}(y,u) \rightarrow \big(B^{\#}(y^*a,z) \rightarrow z = u\big)\big)$,
$(iiib^\beta)$ $QT^+ \vdash \forall\, y \in I_\beta\ \forall\, u,z \in I\,\big(B^{\#}(y,u) \rightarrow \big(B^{\#}(y^*b,z) \rightarrow z = u^*b\big)\big)$.

We can then prove that α and β correctly count b's in b-tallies:

5.1. (a) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash A^{\#}(a, Sb)\ \&\ \forall\, x \in J\ \forall\, y \in I\ \big(\mathrm{Tally}_b(x)\ \&\ A^{\#}(x,y) \rightarrow y = b\big) \quad \text{and}$$
$$QT^+ \vdash \forall\, x \in J\ \forall\, y \in I\,\big(\mathrm{Tally}_a(x)\ \&\ B^{\#}(x,y) \rightarrow y = b\big).$$

(That is, "$\alpha(a)=1$" and "$\mathrm{Tally}_b(x) \rightarrow \alpha(x) = 0$," and "$\mathrm{Tally}_a(x) \rightarrow \beta(x) = 0$.")

(b) For any string form $I \subseteq I_0$ there is a string form $J \subseteq I$ such that

$$QT^+ \vdash \forall\, x \in J\ \forall\, y \in I\,\big(\mathrm{Tally}_b(x)\ \&\ B^{\#}(x,y) \rightarrow y = x^*b\big).$$

Informally, $\mathrm{Tally}_b(x) \rightarrow \beta(x) = \mathrm{length}(x)$.

We now verify that the functions α and β are indeed additive. Let I_{Add} be as in 3.5(a).

5.2. (a) For any string form I such that $I \subseteq I_\alpha$ and $I \subseteq I_{\mathrm{Add}}$ there is a string form $J \equiv I_{\mathrm{Add}\alpha} \subseteq I$ such that

$$QT^+ \vdash \forall\, x,y \in J\ \forall\, u,v,w\,\big(A^{\#}(x,u)\ \&\ A^{\#}(y,v)\ \&\ \mathrm{AddTally}(u,v,w) \rightarrow A^{\#}(x^*y,w)\big).$$
("$\alpha(x^*y) = \alpha(x) + \alpha(y)$")

(b) For any string form $I \subseteq I_\beta$ and $I \subseteq I_{\mathrm{Add}}$ there is a string form $J \equiv I_{\mathrm{Add}\beta} \subseteq I$ such that

$$QT^+ \vdash \forall\, x,y \in J\ \forall\, u,v,w\,\big(B^{\#}(x,u)\ \&\ B^{\#}(y,v)\ \&\ \mathrm{AddTally}(u,v,w) \rightarrow B^{\#}(x^*y,w)\big).$$
("$\beta(x^*y) = \beta(x) + \beta(y)$")

PROOF. See [3]. ⊣

§6. Formal construction of the interpretation. Let $I_{\mathrm{Add}\alpha}$ be the string form obtained from I_0 by the series of modifications described in Sections 3–5 up to and including 5.2(a). Analogously for and $I_{\mathrm{Add}\beta}$ and 5.2(b).

Let $J^*(x) \equiv I_{\mathrm{Add}\alpha}(x)\ \&\ I_{\mathrm{Add}\beta}(x)$.

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Then $J^* \subseteq I_{Add\alpha}$ and $J^* \subseteq I_{Add\beta}$ and $J^* \subseteq I_{Add}$ as well as $J^* \subseteq I^\diamond$. We may also assume that J^* is closed under *, and downward closed under $\leq$ and $\subseteq_p$. Hence it may be assumed that the string form J^* is also closed under Addtally and the functions α and β.

We then formally define $Æ(x)$ as

$$\exists\, y, z \left(A^{\#}(x, y) \;\&\; B^{\#}(x, z) \;\&\; y = Sz\right) \\ \&\; \forall\, u, v, w \left(uBx \;\&\; A^{\#}(u, v) \;\&\; B^{\#}(u, w) \rightarrow v \leq w\right).$$

(These are conditions (c1) and (c2) from Section 1.)

We set $\quad I^*(x) \equiv Æ(x) \;\&\; J^*(x)$.

The formula $I^*(x)$ will formally define in QT^+ the domain of interpretation of theory T. We now proceed to formally verify the translations of the axioms of T by derivations in QT^+.

6.1. (a) $QT^+ \vdash I^*(x) \rightarrow x = a \vee (bBx \;\&\; aaEx)$.

(b) $QT^+ \vdash I^*(x) \;\&\; x_2Ex \rightarrow \forall\, u, v \left(A^{\#}(x_2, u) \;\&\; B^{\#}(x_2, v) \rightarrow Sv \leq u\right)$.
(c) $QT^+ \vdash I^*(x) \;\&\; I^*(y) \;\&\; z = bxy \rightarrow I^*(z)$.
(d) $QT^+ \vdash I^*(x) \;\&\; I^*(u) \;\&\; bxy = buv \rightarrow x = u \;\&\; y = v$.
(e) $QT^+ \vdash I^*(x) \rightarrow \left(x \subseteq_p a \leftrightarrow x = a\right)$.
(f) $QT^+ \vdash I^*(x) \;\&\; I^*(y) \;\&\; I^*(z) \rightarrow \left(x \subseteq_p byz \leftrightarrow x = byz \vee x \subseteq_p y \vee x \subseteq_p z\right)$.

Proof. See [3]. We give some details of the proof of (f) to illustrate the flavor of the type of formal argument used. Let M be an arbitrary model of QT^+. Assume $M \models x \subseteq_p byz$ where $M \models I^*(\{x\}) \;\&\; I^*(y) \;\&\; I^*(z)$.

Then $M \models J^*(x) \;\&\; J^*(y) \;\&\; J^*(z)$ and also $M \models Æ(x) \;\&\; Æ(y) \;\&\; Æ(z)$.

By (i^α) and (i^β), $M \models \exists\,!\, x_1 \in J^*A^{\#}(x, x_1) \;\&\; \exists\,!\, x_2 \in J^*B^{\#}(x, x_2)$.

From $M \models x \subseteq_p byz$ by 3.7(h) we have that

$$M \models x = byz \vee x = b \vee x \subseteq_p yz \vee \exists\, u\,(uByz \;\&\; x = bu). \qquad \dashv$$

We distinguish the cases:

(1) $M \models \exists\, u\,(uByz \;\&\; x = bu)$.

Then by (QT2), $M \models x \neq a$. From $M \models I^*(y) \;\&\; I^*(z)$, by 6.1(c), $M \models I^*(byz)$.

From $M \models uByz$, we have $M \models \exists\, v\; uv = yz$, hence $M \models b(uv) = b(yz)$, and therefore $M \models (bu)v = b(yz)$. Thus $M \models buBb(yz)$, hence $M \models xBb(yz)$. Now, from $M \models I^*(byz)$, we have $M \models Æ(byz)$, whence $M \models x_1 \leq x_2$. But from $M \models Æ(x)$, we obtain $M \models x_1 = Sx_2$, and thus $M \models x_1 \leq x_2 < Sx_2 = x_1$, contradicting $M \models I_0(x_1)$. Hence (1) is ruled out.

(2) $M \models x = b$.

Then by (QT4), $M \models x \neq a$, and from $M \models Æ(x)$, we have, by 6.1(a), $M \models bBx$. But then $M \models bBb$, contradicting (QT2). Hence (2) is also ruled out.

(3) $M \models x \subseteq_p yz$.

By 3.7(g), $\quad M \models x = yz \vee x \subseteq_p y \vee x \subseteq_p z \vee \exists\, y_1\,(y_1Ey \;\&\; x = y_1z)$
$\vee\, \exists\, z_1\,(z_1Bz \;\&\; x = yz_1) \vee \exists\, y_1, z_1\,(y_1Ey \;\&\; z_1Bz \;\&\; x = y_1z_1)$.

(3i) $M \models x = yz$.

By (i^α) and (i^β), $M \models \exists\,!\, y_1 \in J^*A^{\#}(y, y_1) \;\&\; \exists\,!\, y_2 \in J^*B^{\#}(y, y_2)$ and further $M \models \exists\,!\, z_1 \in J^*A^{\#}(z, z_1) \;\&\; \exists\,!\, z_2 \in J^*B^{\#}(z, z_2)$.

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From $M \models Æ(y)$, we have $M \models y_1 = Sy_2$, and from $M \models Æ(z)$, we have $M \models z_1 = Sz_2$. By 3.5(a), $M \models \exists!\, p_1 \in J^*(\mathrm{Tally}_b(p_1)$ & $\mathrm{Addtally}(y_1,z_1,p_1))$ and $M \models \exists!\, p_2 \in J^*(\mathrm{Tally}_b(p_2)$ & $\mathrm{Addtally}(y_2,z_2,p_2))$. Then from $M \models A^{\#}(y, y_1)$ & $A^{\#}(z,z_1)$, by 5.2(a), $M \models A^{\#}(y^*z, p_1)$, and from $M \models B^{\#}(y, y_2)$ & $B^{\#}(z,z_2)$, by 5.2(b), $M \models B^{\#}(y^*z, p_2)$. Thus, from $M \models y_1 = Sy_2$ & $z_1 = Sz_2$, we obtain $M \models \mathrm{Addtally}(Sy_2,Sz_2,p_1)$.

On the other hand, from $M \models \mathrm{Addtally}(y_2,z_2,p_2)$, by 3.4(e), $M \models \mathrm{Addtally}(y_2,Sz_2, Sp_2)$, whence by 3.5(d), $M \models \mathrm{Addtally}(Sy_2,Sz_2,SSp_2)$. By single-valuedness of Addtally, we then have $M \models p_1 = SSp_2$.

From hypothesis $M \models x = y^*z$ & $A^{\#}(y^*z,p_1)$ & $B^{\#}(y^*z,p_2)$, we have

$$M \models A^{\#}(x, p_1) \ \& \ B^{\#}(x, p_2).$$

Hence from $M \models A^{\#}(x, x_1)$ & $B^{\#}(x,x_2)$, by single-valuedness of $A^{\#}$ and $B^{\#}$,

$$M \models p_1 = x_1 \ \& \ p_2 = x_2.$$

Thus from $M \models p_1 = SSp_2$, we have $M \models x_1 = SSx_2$. But from $M \models Æ(x)$ we have $M \models x_1 = Sx_2$, whence $M \models x_1 = Sx_1$. But then from $M \models x_1 < Sx_1$ we obtain $M \models x_1 < x_1$, contradicting $M \models I_0(x_1)$. Hence (3i) is ruled out.

(3ii) $M \models \exists\, y_1\, (y_1Ey \ \& \ x = y_1z)$.

Then $M \models y_1Bx$. By (i^{α}) and (i^{β}),

$$M \models \exists!\, u_1 \in J^*A^{\#}(y_1, u_1) \exists!\, u_2 \in J^*B^{\#}(y_2, u_2).$$

From $M \models Æ(x)$ & y_1Bx, we have $M \models u_1 \leq u_2$, whereas from $M \models I^*(y)$ & y_1Ey, by 6.1(b), we also have $M \models Su_2 \leq u_1$. But then $M \models u_2 < Su_2 \leq u_2$, contradicting $M \models I_0(u_2)$. This rules out (3ii). Similar formalized calculations rule out

(3iii) $M \models \exists\, z_1\, (z_1Bz \ \& \ x = yz_1)$, and (3iv) $M \models \exists\, y_1, z_1\, (y_1Ey \ \& \ z_1Bz \ \& \ x = y_1z_1)$.

(See [3].) We conclude that

$$M \models x \subseteq_p yz \rightarrow x \subseteq_p y \vee x \subseteq_p z$$

and further that $M \models x \subseteq_p byz \rightarrow x = byz \vee x \subseteq_p y \vee x \subseteq_p z$.

The converse is immediate from the definition of $\subseteq_p$. ⊣

Taking the formula $Æ(x)$ from Section 6 to define the domain, and interpreting the non-logical vocabulary $\mathcal{L}_T = \{0, (\ ), \sqsubseteq\}$ of T by a, bxy, and $\subseteq_p$, resp., as explained in Section 2, we have that 6.1(c)–(f), along with the fact that $QT^+ \vdash bxy \neq a$, suffice to establish formal interpretability of T in QT^+. On the other hand, from [1], building on previous work of Halpern and Collins, Wilkie, Visser, Grzegorczyk, and Ganea, we have that

$$TC \equiv_I QT^+ \equiv_I AST \equiv_I AST + EXT \equiv_I Q.$$

Since, by [6], $Q \leq_I T$, this suffices to establish

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$$T \equiv_I QT^+ \equiv_I TC \equiv_I Q \equiv_I AST.$$

In addition, each of the theories above is mutually interpretable with AST+EXT, and Buss's theory S_2^1 (for the latter see [4]).

§7. R and its variants. We now consider the expanded vocabulary $\mathcal{L}_{C,\sqsubseteq *} = \{a, b, *, \sqsubseteq^*\}$ with two individual constants—the digits a, b—a single binary operation symbol *, and a 2-place relational symbol $\sqsubseteq^*$. Recalling the theory WT described in the introduction formulated in $\mathcal{L}_T = \{0, (\,), \sqsubseteq\}$, we shall single out $\mathcal{L}_{C,\sqsubseteq *}$-terms that represent tree-like strings obtained from variable-free terms of $\mathcal{L}_T$ as described in Section 1. Let S be the set of variable-free $\mathcal{L}_T$-terms. For each $v \in S$, we associate a unique $\mathcal{L}_{C,\sqsubseteq *}$-term v^τ as follows:

$$0^\tau \equiv a \qquad (u, v)^\tau \equiv b^* (u^\tau * v^\tau).$$

The $\mathcal{L}_{QT,\sqsubseteq *}$-term v^τ is an Æ-string that codes v.

Let $\Sigma^\tau = \{ t^\tau \mid t \text{ is a variable-free } \mathcal{L}_T\text{-term}\}$, and, for a variable-free $\mathcal{L}_T$-term t, let $\Sigma(t) = \{ u^\tau \mid u \text{ is a subterm of } t\}$. A straightforward induction on the complexity of $\mathcal{L}_T$-terms establishes that the mapping τ is 1–1.

Let WQT be the first-order theory formulated in $\mathcal{L}_{C,\sqsubseteq *}$ with the following axioms:

(WQT1) $\neg\ (s = t)$ for any distinct terms $s, t \in \Sigma^\tau$,

(WQT2) $\forall\, z\ (z \sqsubseteq^* b^*(s^* t) \leftrightarrow z = b^*(s^* t)\ \vee\ z \sqsubseteq^* s \vee z \sqsubseteq^* t)$ for all terms $s, t \in \Sigma^\tau$,

(WQT3) $\forall\, z\ (z \sqsubseteq^* a \leftrightarrow z = a)$.

Here, (WQT1) and (WQT2) are axiom schemas with infinitely many instances.

We now define a formal interpretation (τ) of WT in WQT. Let the formula

$$T^*(x) \equiv x = a \vee \exists\, y, z\ x = b^*(y^* z)$$

define the domain. Interpret 0 by a, the binary term building operation $(,)$ of $\mathcal{L}_T$ by $b^*(x^* y)$, and $\sqsubseteq$ by $\sqsubseteq^*$. We then have immediately

$$WQT \vdash T^*(0^\tau),$$

$$WQT \vdash T^*(y)\ \&\ T^*(z) \rightarrow T^*(b^*(y^* z)).$$

A trivial induction on the complexity of $\mathcal{L}_T$-terms verifies that each $v \in S$ is interpreted by $v^\tau \in \Sigma^\tau$ in WQT. Since the map τ is 1–1, we have that

$$WQT \vdash (\neg\ (u = v))^{(\tau)},$$

$\neg\ (u^\tau = v^\tau)$ being the translations $(\neg\ (u = v))^{(\tau)}$ of the instances of axiom schema (WT1) of WT, for distinct $u, v \in S$.

Consider now an instance of the schema (WT2), for some $v \in S$:

$$\forall\, x\ \big(x \sqsubseteq v \leftrightarrow \vee_{u \in S(v)} x = u\big).$$

If v is the atomic term 0, we have that $S(v) = \{0\}$. Hence the formula in question is $\forall\, x\ (x \sqsubseteq 0 \leftrightarrow x = 0)$.

But, by (WQT3), we have $WQT \vdash \forall\, x\ (x \sqsubseteq^* a \leftrightarrow x = a)$.

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Hence, a fortiori, $\mathrm{WQT} \vdash \forall\, x\ (T^*(x) \to (x \sqsubseteq^* a \leftrightarrow x = a))$, which is the (τ)-translation of the above instance of (WT2).

Consider now $t \in S$ of the form (u,v). Note that $S(t) = S(u) \cup S(v) \cup \{t\}$. Hence

$$\Sigma(t) = \Sigma(u) \cup \Sigma(v) \cup \{t^\tau\}. \qquad (\ddagger)$$

Assume now that

$$\mathrm{WQT} \vdash \left[\forall\, z\ \left(z \sqsubseteq u \leftrightarrow \vee_{s \in S(u)} z = s\right)\right]^{(\tau)} \text{ and}$$
$$\mathrm{WQT} \vdash \left[\forall\, z\ \left(z \sqsubseteq v \leftrightarrow \vee_{s \in S(v)} z = s\right)\right]^{(\tau)}.$$

Then $\mathrm{WQT} \vdash \forall\, z\ \left(T^*(z) \to \left(z \sqsubseteq^* u^\tau \leftrightarrow \vee_{s \in \Sigma(u)} z = s\right)\right)$
and $\mathrm{WQT} \vdash \forall\, z\ \left(T^*(z) \to \left(z \sqsubseteq^* v^\tau \leftrightarrow \vee_{s \in \Sigma(v)} z = s\right)\right)$.

Let M be a model of WQT. Assume $M \models T^*(x)$ and consider $M \models x \sqsubseteq^* t$.

We have that t^τ is in fact $b^*(u^\tau\ ^*\ v^\tau)$. Hence, by (WQT2), $M \models x \sqsubseteq^* b^*(u^\tau * v^\tau)$ holds just in case $M \models x = b^*(u^\tau * v^\tau)\ \vee\ x \sqsubseteq^* u^\tau\ \vee\ x \sqsubseteq^* v^\tau$, which, in turn, holds exactly when $M \models x = b^*(u^\tau * v^\tau)\ \vee\ \vee_{s \in \Sigma(u)} x = s\ \vee\ \vee_{s \in \Sigma(v)} x = s$, that is, $M \models \vee_{s \in \Sigma(t)} x = s$ using $(\ddagger)$. Therefore,

$$\mathrm{WQT} \vdash \forall\, x\ \left(T^*(x) \to \left(x \sqsubseteq^* t^\tau \leftrightarrow \vee_{s \in \Sigma(t)} x = s\right)\right),$$

namely, $\mathrm{WQT} \vdash \left[\forall\, x\ \left(x \sqsubseteq t \leftrightarrow \vee_{s \in S(t)} x = s\right)\right]^{(\tau)}$.

Hence the (τ)-translation of each instance of (WT2) is also provable in WQT.

We conclude that

7.1. $\mathrm{WT} \leq_I \mathrm{WQT}$.

The theory WQT is not recognizably a concatenation theory: the axioms make no substantive assumptions about the binary operation *, not even associativity. On that account, it might be considered at best as a "pseudo-concatenation" notational variant of WT. We now consider another first-order theory, WQT*, formulated in the same vocabulary $\mathcal{L}_{C,\sqsubseteq^*} = \{a, b, *, \sqsubseteq^*\}$ as WQT, with the following axioms: for each variable-free term t of $\mathcal{L}_{C,\sqsubseteq^*}$,

(WQT*1) $\forall\, x, y, z\ \left(x^*(y^*z) \sqsubseteq_p t\ \vee\ (x^*y) * z \sqsubseteq_p t \to x^*(y^*z) = (x^*y) * z\right)$,
(WQT*2) $\forall\, x, y\ \left(x^*y \sqsubseteq_p t \to\ \neg\ (x^*y = a)\ \&\ \neg\ (x^*y = b)\right)$,
(WQT*3) $\forall\, x, y\ \Big(\left(a^*x \sqsubseteq_p t\ \&\ a^*y \sqsubseteq_p t \to (a^*x = a^*y \to x = y)\right)$
$\&\ \left(b^*x \sqsubseteq_p t\ \&\ b^*y \sqsubseteq_p t \to (b^*x = b^*y \to x = y)\right)$
$\&\ \left(x^*a \sqsubseteq_p t\ \&\ y^*a \sqsubseteq_p t \to (x^*a = y^*a \to x = y)\right)$
$\&\ \left(x^*b \sqsubseteq_p t\ \&\ x^*y \sqsubseteq_p t \to (x^*b = y^*b \to x = y)\right)\Big)$,
(WQT*4) $\forall\, x, y\ \Big(\left(a^*x \sqsubseteq_p t\ \&\ b^*y \sqsubseteq_p t \to\ \neg\ (a^*x = b^*y)\right)$
$\&\ \left(x^*a \sqsubseteq_p t\ \&\ y^*b \sqsubseteq_p t \to\ \neg\ (x^*a = y^*b)\right)\Big)$,
(WQT*5) $\forall\, x \sqsubseteq_p t\ (x = a\ \vee\ x = b\ \vee\ ((aBx\ \vee\ bBx)\ \&\ (aEx\ \vee\ bEx)))$,
(WQT*6) $\forall\, y, z\ (b^*(y^*z) \sqsubseteq^* t$
$\to \forall\, x\ (x \sqsubseteq^* b^*(y^*z) \leftrightarrow x = b^*(y^*z)\ \vee\ x \sqsubseteq^* y\ \vee\ x \sqsubseteq^* z))$,
(WQT*7) $\forall\, z\ (z \sqsubseteq^* a \leftrightarrow z = a)$,
(WQT*8) $\forall\, x, y\ (x \sqsubseteq^* y\ \&\ y \sqsubseteq^* x \to x = y)$,
(WQT*9) $\forall\, x, y\ (x \sqsubseteq^* y\ \&\ y \sqsubseteq^* z \to y \sqsubseteq^* z)$.

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Here we use the following abbreviations:

$$xBy \equiv \exists z\, y = x^*z, \qquad xEy \equiv \exists z\, y = z^*x,$$

and $x \sqsubseteq_p y \equiv x = y \vee xBy \vee xEy \vee \exists z_1, z_2\, y = z_1 * (x^* z_2) \vee \exists z_1, z_2\, y = (z_1 * x) * z_2$. Then $\forall x \sqsubseteq_p u\, \varphi \equiv \forall x\, (x \sqsubseteq_p u \to \varphi)$, where x does not occur in the term u. Also, $\forall x \sqsubseteq^* u\, \varphi \equiv \forall x\, (x \sqsubseteq^* u \to \varphi)$.

(WQT*1)–(WQT*6) are axiom schemas with infinitely many instances, one for each variable-free term t. The schemas (WQT*1)–(WQT*5) are "bounded" versions of the axioms (QT1)–(QT5) of QT^+. Schema (WQT*6) is a "bounded" generalization of schema (WQT2) of WQT. In light of that, WQT* may be naturally interpreted as a hybrid basic theory of finite strings and trees: the intended domain are the finite non-empty strings over the alphabet {a, b}, the binary operation symbol * is interpreted as the concatenation operation, and $\sqsubseteq^*$ as the substring relation between Æ-strings.

Now, WQT* is an extension of WQT. First, note the following:

7.2. For any distinct terms $s, t \in \Sigma^\tau$, $WQT^* \vdash \neg (s = t)$.

PROOF. We argue by (meta-theoretic) induction on the number of digits in s, t. If either one of s or t is the single digit a, this is immediate by (WQT*2). If neither s nor t are single digits, let $s_1 \ldots s_m$ and $t_1 \ldots t_n$ be their successive digits (ignoring parentheses), and let $s_i \neq t_i$ be the leftmost digit where s and t differ. Then $s = s_1 \ldots s_{i-1} s_i s^-$ and $t = t_1 \ldots t_{i-1} t_i t^-$ where $s^- = s_{i+1} \ldots s_m$ and $t^- = t_{i+1} \ldots t_n$. By (WQT*4), $WQT^* \vdash \neg (s_i s^- = t_i t^-)$. By repeatedly applying (WQT*1) and (WQT*3) we obtain $WQT^* \vdash \neg (s_1 \ldots s_{i-1} s_i s^- = t_1 \ldots t_{i-1} t_i t^-)$, that is, $WQT^* \vdash \neg (s = t)$, as required. ⊣

Hence in particular all instances of schema (WQT1) are provable in WQT*. Consider an instance of (WQT2) for terms $s, t \in \Sigma^\tau$,

$$\forall z\, (z \sqsubseteq^* b^*(s^*t) \leftrightarrow z = b^*(s^*t) \vee z \sqsubseteq^* s \vee z \sqsubseteq^* t).$$

Now, we have $WQT^* \vdash b^*(s^*t) = b^*(s^*t)$, so $WQT^* \vdash b^*(s^*t) \sqsubseteq_p b^*(s^*t)$. From (WQT*6), we have

$$WQT^* \vdash \forall x\, (x \sqsubseteq^* b^*(s^*t) \leftrightarrow x = b^*(s^*t) \vee x \sqsubseteq^* s \vee x \sqsubseteq^* t).$$

Hence each instance of (WQT2) is provable in WQT*. Given that (WQT3) is (WQT*7), this is enough to establish that WQT* is an extension of WQT.

On the other hand, we also have:

7.3. WQT* is locally finitely satisfiable.

That is, each finite subset of its non-logical axioms has a finite model.

PROOF. See [3]. ⊣

By Visser's Theorem, it follows that WQT* is interpretable in R.

Since by [6], $R \leq_I WT$, we then have

WEAK ESSENTIALLY UNDECIDABLE THEORIES: SECOND MUTUAL INTERPRETABILITY THEOREM. $R \equiv_I WTC^{-\varepsilon} \equiv_I WT \equiv_I WQT \equiv_I WQT^*$.

For definition of the theory $WTC^{-\varepsilon}$, see [5].

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