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Summable formal invariant curves of diffeomorphisms

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Abstract. Let F be a tangent to the identity diffeomorphism in $(\mathbb{C}^2, 0)$ and X its infinitesimal generator. We prove that Camacho and Sad's formal invariant curves of X give summable formal power series, whose sums correspond to the parabolic curves found by Hakim for F and F^{-1}.

1. *Introduction*

Consider a tangent to the identity diffeomorphism F in $(\mathbb{C}^2, 0)$. A formal curve Γ in $(\mathbb{C}^2, 0)$ is given by $\Gamma = (g(x, y) = 0)$ with $g \in \mathbb{C}[[x, y]]$ irreducible and not a unit. A parameterization of Γ is a pair $\Gamma(t) = (\hat{\gamma}_1(t), \hat{\gamma}_2(t))$ of formal series $\hat{\gamma}_i(t) \in \mathbb{C}[[t]]$ such that $\hat{\gamma}_i(0) = 0$ and $g(\hat{\gamma}_1(t), \hat{\gamma}_2(t)) \equiv 0$. We say that Γ is invariant for F if $g \circ F$ belongs to the ideal $g\mathbb{C}[[x, y]]$. A parabolic curve for F is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^2$, where Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$ such that:

- φ is continuous at the origin and $\varphi(0) = 0$;
- invariance: $F(\varphi(\Delta)) \subset \varphi(\Delta)$;
- stability: $\varphi^{-1}(F^n(\varphi(x))) \to 0$ when $n \to \infty$ for all $x \in \Delta$. In particular, $F^n(\varphi(x)) \to 0$.

In this paper we prove the following result.

THEOREM. *If F has an isolated fixed point, there exists a parameterized formal invariant curve $\Gamma(t) = (\hat{\gamma}_1(t), \hat{\gamma}_2(t))$ of F with $\hat{\gamma}_1$, $\hat{\gamma}_2$ l-summable for some $l \geq 1$. Moreover, if v_1, v_2 are the l-sums of $\hat{\gamma}_1$, $\hat{\gamma}_2$ in a direction d, then*

$$t \mapsto (v_1(t), v_2(t))$$

is a parabolic curve for F or F^{-1}.

Camacho and Sad's theorem [**5**] guarantees the existence of formal invariant curves for the infinitesimal generator of F, that are automatically invariant for F. These curves are the formal invariant curves in the theorem. They lead after desingularization to a situation where Hakim's theorem [**11**] applies. The sums we obtain in the main theorem will coincide with Hakim's parabolic curves, up to reduction of singularities.

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The theorem ensures that Hakim's parabolic curves for a diffeomorphism can be recovered in a purely formal way, obtaining first Camacho and Sad's formal invariant curves for the infinitesimal generator and applying then the classical Borel–Laplace summation procedure to the formal series obtained.

Note that the theorem above is not a direct corollary of Hakim's result up to a desingularization morphism. Even in Hakim's situation we have to prove summability of the formal invariant curve. The main part of the paper is devoted to this proof. We proceed in two steps. First, we enlarge the domains of definition of Hakim's parabolic curves to reach the necessary wideness for summability. This enlargement will follow from a modified version of Hakim's proof, mainly based on some bounds of Loray [**14**]. The second step is to prove that the asymptotic expansions of the parabolic curves are of the corresponding Gevrey type. The main argument in this proof is based on the construction of quasi-solutions of an equation (see for example Canalis-Durand *et al* [**6**]).

In this paper we have only used elementary geometrical techniques, as a partial alternative to Écalle's theory in [**8**].

2. *The infinitesimal generator*

We denote by $\widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ the space

$$\widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0) = \{\hat{F}(x, y) = (x + F_1(x, y), y + F_2(x, y)) : F_1, F_2 \in \mathbb{C}[[x, y]]\}$$

of tangent to the identity formal diffeomorphisms in $(\mathbb{C}^2, 0)$, by $\mathrm{Diff}_1(\mathbb{C}^2, 0)$ the space of convergent elements of $\widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ and by $\hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ the space

$$\hat{\mathfrak{X}}_2(\mathbb{C}^2, 0) = \{X = A(x, y)\partial_x + B(x, y)\partial_y : A, B \in \mathbb{C}[[x, y]],\ \nu(A), \nu(B) \geq 2\}$$

of formal vector fields of order greater than or equal to two (here ν denotes the order of annulation of 0).

For any $X \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$, consider the map $\exp X : \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ given by

$$\exp X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g),$$

where $X^0(g) = g$ and $X^{j+1}(g) = X(X^j(g))$. It is known that the application

$$\begin{aligned} \mathrm{Exp} : \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0) &\to \widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0), \\ X &\mapsto \mathrm{Exp}\, X = (\exp X(x), \exp X(y)) \end{aligned}$$

is a bijection. The infinitesimal generator of $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ is the formal vector field X such that $F = \mathrm{Exp}\, X$.

Write $F(z) = (z + f_k(z) + f_{k+1}(z) + \cdots) \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$, where $z = (x, y)$ and f_j is a two-dimensional vector of homogeneous polynomials of degree j with $f_k \not\equiv 0$. A characteristic direction for F is a point $[v] \in \mathbb{P}^1_{\mathbb{C}}$ such that $f_k(v) = \lambda v$ for some $\lambda \in \mathbb{C}$; it is non-degenerate if $\lambda \neq 0$. The integer $\mathrm{ord}(F) := k \geq 2$ is the order of F at 0.

Note that the order of $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ coincides with the order of its infinitesimal generator $X \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ and that F has an isolated fixed point if and only if X has an isolated singularity. Moreover, the characteristic directions for F are exactly the lines of the tangent cone of X and a formal curve is invariant for F if and only if it is invariant for X.

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Definition 2.1. Consider $X \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ and take coordinates (x, y). We say that X is of Briot–Bouquet type with respect to (x, y) if

$$X = x^k[(\lambda + a(x, y))x\partial_x + (\beta x + \mu y + b(x, y))\partial_y],$$

where $\nu(a) \geq 1$, $\nu(b) \geq 2$, $\lambda \neq 0$ and $\mu/\lambda \notin \mathbb{Q}_{>0}$.

In these conditions, Briot and Bouquet's theorem [**2**] guarantees the existence of a unique formal invariant curve Γ for X (and hence for $\operatorname{Exp} X$) different from $(x = 0)$. Moreover, Γ is non-singular and transversal to $(x = 0)$. We say that $F \in \operatorname{Diff}_1(\mathbb{C}^2, 0)$ is of Briot–Bouquet type with respect to (x, y) if its infinitesimal generator is of Briot–Bouquet type and we call Γ the Briot–Bouquet curve of F.

Observe that if F is of Briot–Bouquet type with respect to (x, y), the characteristic direction transversal to $(x = 0)$ is non-degenerate.

Definition 2.2. Consider $F \in \operatorname{Diff}_1(\mathbb{C}^2, 0)$ with an isolated fixed point and let X be its infinitesimal generator. A Camacho–Sad curve for F is a formal curve whose strict transform by the reduction of singularities of X is a Briot–Bouquet curve.

The existence of Camacho–Sad curves for a given F is guaranteed by the formal version of Camacho and Sad's theorem [**5**] (see [**3, 17**]). Observe that in the dicritical case the transformed vector field at each generic (non-strictly singular) point on the dicritical component is of Briot–Bouquet type, so in this case there exist infinitely many Camacho–Sad curves.

3. *Existence of parabolic curves*

Given $k \geq 1$, $\eta \in (0, \pi/2)$ and $\delta > 0$, we denote by $\Delta^k_{\eta,\delta}$ the set

$$\Delta^k_{\eta,\delta} = \bigcup_{-\pi/2+\eta \leq \theta \leq \pi/2-\eta} \{x \in \mathbb{C} : |x^k - \delta e^{i\theta}| < \delta\}.$$

It has k connected components that are fitted in sectors of opening $2(\pi - \eta)/k$ bisected by the lines $\{x \in \mathbb{C} : x^k \in \mathbb{R}^+\}$. Observe that if Δ is a component of $\Delta^k_{\eta,\delta}$ and $\delta' < \delta$, then $\Delta \cap \Delta^k_{\eta,\delta'}$ is a component of $\Delta^k_{\eta,\delta'}$.

The dynamics of tangent to the identity diffeomorphisms in dimension one is described by Leau and Fatou's flower theorem [**9, 13**]. The following statement, that can be found in [**14**], provides a useful bound we will later need.

THEOREM 3.1. *Consider* $f(x) = x - x^{k+1} + O(x^{k+2}) \in \operatorname{Diff}_1(\mathbb{C}, 0)$. *Given* $\eta \in (0, \pi/2)$, *there exist constants* $\delta_0, C, C^* > 0$ *such that for every* $\delta \leq \delta_0$ *and every component* Δ *of* $\Delta^k_{\eta,\delta}$, *we have*

$$f(\Delta) \subset \Delta, \quad f^n(x) \sim 1/(kn)^{1/k} \quad \text{and} \quad |f^n(x)|^k \leq C|x|^k/(1 + n|x|^k)$$

for all $x \in \Delta$ *and* $n \in \mathbb{N}$. *Moreover,* $|\arg(f^n(x))^k| < \pi/2 - \eta$ *for all* $x \in \Delta$ *and* $n \geq C^*/(k|x|^k)$.

A generalization of Leau and Fatou's theorem to dimension two (valid in fact in arbitrary dimension) is given by the following result of Hakim [**11**] that can also be seen as a discrete analog of Briot and Bouquet's theorem.

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THEOREM 3.2. *Consider* $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$. *If* $[v]$ *is a non-degenerate characteristic direction for* F, *there exist* $\mathrm{ord}(F) - 1$ *disjoint parabolic curves* φ_i *for* F *such that* $[\varphi_i(x)] \to [v]$ *when* $x \to 0$.

As a consequence of this result, we obtain a discrete version of Camacho and Sad's theorem ([**1**], see also [**3**]).

4. *Parabolic curves in wide domains*

In this section, we will enlarge the domain of definition of Hakim's parabolic curves on [**11**] to reach the same opening as the attracting domains in Theorem 3.1.

Throughout this section, F will be of Briot–Bouquet type with respect to (x, y) and Γ will be the Briot–Bouquet curve of F, that we can parameterize as $\Gamma = (y = \sum_{n\geq 1} a_n x^n)$.

Denote $m = m(F; x, y) = \min\{n \in \mathbb{N} : a_n \neq 0\}$. We have

$$F(x, y) = (x + \lambda x^{k+1} + O(x^{k+1}y, x^{k+2}),\ y + \mu x^k y + O(x^k y^2, x^{k+1}y) + x^{k+m}\psi(x))$$

with $\psi(0) \neq 0$.

Definition 4.1. We say that (x, y) are normalized coordinates for F if

$$F(x, y) = (x - x^{k+1} + O(x^{k+1}y, x^{k+2}),\ y - \alpha x^k y + O(x^k y^2, x^{k+1}))$$

with $m(F; x, y) \geq 2$ and $m(F; x, y) > 1 + \mathrm{Re}\,\alpha$.

Observe that there always exist normalized coordinates for F. They can be obtained after a suitable polynomial change of coordinates of the form

$$(x, y) \mapsto \left(\delta x,\ y - \sum_{n\leq N} a_n x^n\right).$$

THEOREM 4.2. *Consider* $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ *of Briot–Bouquet type. Take normalized coordinates* (x, y) *for* F *and put* $m = m(F; x, y)$ *and* $k = \mathrm{ord}(F) - 1$. *Given* η *with* $0 < \eta < \pi/2$, *there exist* $\delta_0 > 0$ *and parabolic curves*

$$\varphi_\Delta(x) = (x, u_\Delta(x))$$

defined in each component Δ *of* Δ^k_{η,δ_0} *with the following properties:*

(1) $u_\Delta \in \mathcal{H}_\Delta = \{u \in \mathcal{O}(\Delta) : |u(x)| \leq |x|^{m-1},\ |u'(x)| \leq |x|^{m-2}\}$;

(2) *for every* $\delta \leq \delta_0$, *the restriction of* φ_Δ *to* $\Delta \cap \Delta^k_{\eta,\delta}$ *is a parabolic curve for* F;

(3) *uniqueness: if* $\delta \leq \delta_0$ *and* $\varphi(x) = (x, u(x))$ *is a parabolic curve for* F *defined in* $\Delta \cap \Delta^k_{\eta,\delta}$ *with* $u \in \mathcal{H}_{\Delta\cap\Delta^k_{\eta,\delta}}$, *then* φ *is the restriction of* φ_Δ *to* $\Delta \cap \Delta^k_{\eta,\delta}$.

Remark 4.3. The existence of parabolic curves is a particular case of Theorem 3.2, since every diffeomorphism of Briot–Bouquet type has a non-degenerate characteristic direction. The theorem will follow from Hakim's proof with a modification based on the first bound of Theorem 3.1.

Proof of Theorem 4.2. In the coordinates (x, y), we have $F = (F_1, F_2)$ with

$$F_1(x, y) = x - x^{k+1} + O(x^{k+1}y, x^{k+2}),$$
$$F_2(x, y) = y - \alpha x^k y + O(x^k y^2, x^{k+1}y) + x^{k+m}\psi(x)$$

with $\alpha \notin \mathbb{Q}_{>0}$, $m \geq 2$ and $m > 1 + \mathrm{Re}\,\alpha$.

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For any $\delta > 0$, we consider the space

$$\mathcal{B}_{\Delta^k_{\eta,\delta}} = \{u \in \mathcal{O}(\Delta^k_{\eta,\delta}) : u(x) = x^{m-1}u_0(x) \text{ with } \|u_0\|_\infty < \infty\},$$

which is a Banach space with the norm $\|u\| = \|u_0\|_\infty$ and the closed subspace

$$\mathcal{H}_{\Delta^k_{\eta,\delta}} = \{u \in \mathcal{B}_{\Delta^k_{\eta,\delta}} : |u(x)| \le |x|^{m-1},\ |u'(x)| \le |x|^{m-2}\}.$$

Given $u \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$, we denote $f_u(x) = F_1(x, u(x))$.

Arguing as in the proof of Theorem 3.1, that can be found in [**14**], there exist constants $\delta', C, C^* > 0$ such that for every $0 < \delta \le \delta'$, every component Δ of $\Delta^k_{\eta,\delta}$ and every $u \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$, we have

$$f_u(\Delta) \subset \Delta, \quad f_u^n(x) \sim 1/(kn)^{1/k} \quad \text{and} \quad |f_u^n(x)|^k \le C|x|^k/(1 + n|x|^k)$$

for $x \in \Delta$ and $n \in \mathbb{N}$. Moreover, $|\arg(f_u^n(x))^k| < \pi/2 - \eta$ for $x \in \Delta$ and $n \ge C^*/(k|x|^k)$.

We look for curves of the form $\varphi(x) = (x, u(x))$ with $u \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ that are invariant for F, that is, such that for all $x \in \Delta^k_{\eta,\delta}$,

$$u(F_1(x, u(x))) = F_2(x, u(x)). \tag{1}$$

Then, if δ is small enough, the restriction of φ to each component of $\Delta^k_{\eta,\delta}$ will be a parabolic curve for F.

If x and $F_1(x, y)$ belong to the same component of $\Delta^k_{\eta,\delta}$, we can define the function

$$H(x, y) = y - \frac{x^\alpha}{F_1(x, y)^\alpha} F_2(x, y) = O(x^k y^2, x^{k+1} y, x^{k+m}).$$

For $u \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ with δ small enough, let us consider Hakim's operator

$$Tu(x_0) = x_0^\alpha \sum_{n=0}^{\infty} x_n^{-\alpha} H(x_n, u(x_n)) = x_0^\alpha \sum_{n=0}^{\infty} \left(\frac{u(x_n)}{x_n^\alpha} - \frac{F_2(x_n, u(x_n))}{x_{n+1}^\alpha} \right),$$

where $x_n = f_u^n(x_0)$. The series is normally convergent, since $H(x_n, u(x_n)) = O(x_n^{k+m})$, $x_n = O(1/n^{1/k})$ and $m > \operatorname{Re}\alpha$. Moreover, u is a fixed point of T if and only if u satisfies the identity (1).

Let $u \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ with $\delta \le \delta'$. Since $|x_n|^k \le C|x_0|^k/(1 + n|x_0|^k)$, it follows that for any $s > k$ there exists a constant $C_s > 0$ such that

$$\sum_{n=0}^{\infty} |x_n|^s \le C_s |x_0|^{s-k}. \tag{2}$$

Arguing as in [**11**] we conclude, using the inequality (2), that there exists a constant $B > 0$ such that for all $u, v \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ with δ small we get

$$|Tu(x)| \le B|x|^m \quad \text{and} \quad |(Tu)'(x)| \le B|x|^{m-1},$$

so for $\delta > 0$ small enough T sends $\mathcal{H}_{\Delta^k_{\eta,\delta}}$ into itself.

Now we will prove that for δ small enough T is a contraction in $\mathcal{H}_{\Delta^k_{\eta,\delta}}$. To do so, we need the following lemma, that generalizes Hakim's corresponding statement to large domains.

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LEMMA 4.4. *If δ is small enough, there exists $M > 0$ such that for all $u, v \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ and all $x_0 = y_0 \in \Delta^k_{\eta,\delta}$ we have*

$$|y_n - x_n| \le M|x_0|^m \|v - u\|$$

for all $n \ge 0$, where $x_n = f_u^n(x_0)$ and $y_n = f_v^n(x_0)$.

Proof. For all $j \ge 0$,

$$y_{j+1} - x_{j+1} = (y_j - x_j)\left[1 - \frac{y_j^{k+1} - x_j^{k+1}}{y_j - x_j} + O(|z_j|^{k+1})\right] + (v(y_j) - u(x_j))O(|z_j|^{k+1}),$$

where $|z_j| = \max\{|x_j|, |y_j|\}$. Since

$$\begin{aligned} v(y_j) - u(x_j) &= v(y_j) - v(x_j) + v(x_j) - u(x_j) \\ &= (y_j - x_j)O(|z_j|^{m-2}) + \|v - u\|O(|z_j|^{m-1}) \end{aligned}$$

and $m \ge 2$, we have

$$y_{j+1} - x_{j+1} = (y_j - x_j)\left[1 - \frac{y_j^{k+1} - x_j^{k+1}}{y_j - x_j} + O(|z_j|^{k+1})\right] + \|v - u\|O(|z_j|^{k+m})$$

with the functions $O(|z_j|^{k+1})$ and $O(|z_j|^{k+m})$ bounded independently of u and v.

For $x_j, y_j \in W = \{w \in \Delta^k_{\eta,\delta} : |\arg(w^k)| < \pi/2 - \eta\}$ with δ small enough, the quantity inside brackets is bounded by 1. Using the bound for $|x_n|^k$ and $|y_n|^k$ obtained above, we get

$$\left|1 - \frac{y_j^{k+1} - x_j^{k+1}}{y_j - x_j} + O(|z_j|^{k+1})\right| \le 1 + A\frac{|x_0|^k}{1 + j|x_0|^k} \le 1 + A\frac{|x_0|^k}{1 + |x_0|^k}$$

for some constant $A > 0$ and for all $j \ge 1$, so we have

$$|y_j - x_j| \le |y_{j-1} - x_{j-1}|\left(1 + A\frac{|x_0|^k}{1 + |x_0|^k}\right) + K|z_{j-1}|^{k+m}\|v - u\| \quad \text{for } j \le p,$$

$$|y_j - x_j| \le |y_{j-1} - x_{j-1}| + K|z_{j-1}|^{k+m}\|v - u\| \quad \text{for } j \ge p + 1,$$

where $p \le C^*/(k|x_0|^k)$ is the lowest integer such that $x_p, y_p \in W$. Therefore,

$$|y_n - x_n| \le \left[A_1|z_0|^{k+m} + \cdots + A_{p-1}|z_{p-2}|^{k+m} + \sum_{j=p-1}^{n-1} |z_j|^{k+m}\right]K\|v - u\|,$$

where

$$A_j = \left(1 + A\frac{|x_0|^k}{1 + |x_0|^k}\right)^{p-j} \quad \text{for } 1 \le j \le p - 1.$$

Since

$$A_j \le \left(1 + A\frac{|x_0|^k}{1 + |x_0|^k}\right)^p$$

and

$$\left(1 + A\frac{|x_0|^k}{1 + |x_0|^k}\right)^{C^*/(k|x_0|^k)}$$

is bounded, we get

$$|y_n - x_n| \le K_1\|v - u\| \sum_{j=0}^{n-1} |z_j|^{k+m} \le M|x_0|^m\|v - u\|$$

from the inequality (2). □

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Again arguing as in [**11**], using the fact that $m > 1 + \operatorname{Re}\alpha$, we see that there exists $B' > 0$ such that for all $u, v \in \mathcal{H}_{\Delta^k_{\eta,\delta}}$ with δ small

$$|Tu(x) - Tv(x)| \leq B'|x|^m \|u - v\|,$$

so for $\delta > 0$ small enough T is a contraction in $\mathcal{H}_{\Delta^k_{\eta,\delta}}$. Hence, the Banach fixed point theorem ensures the existence of a unique fixed point u for T in $\mathcal{H}_{\Delta^k_{\eta,\delta}}$. We define u_Δ to be the restriction of u to Δ. This ends the proof of Theorem 4.2. □

The parabolic curves $(x, u_\Delta(x))$ for F given by Theorem 4.2 are called strong Hakim parabolic curves for F.

5. *Summability of Briot–Bouquet curves*

Take $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ of Briot–Bouquet type. Put $k = \mathrm{ord}(F) - 1$ and take normalized coordinates (x, y) for F. Let $\Gamma(x) = (x, \hat{\gamma}(x))$ be the Briot–Bouquet curve of F. In this section, we prove the following theorem.

THEOREM 5.1. *Let $\varphi(x) = (x, u(x))$ be a strong Hakim parabolic curve of F defined in a component Δ of Δ^k_{η,δ_0}. Then u admits $\hat{\gamma}$ as a $1/k$-Gevrey asymptotic expansion in Δ.*

Varying η between 0 and $\pi/2$ and applying Theorem 5.1 to F and to F^{-1}, we deduce that $\hat{\gamma}$ is k-summable in every direction except $\{x \in \mathbb{C} : x^k \in \pm i\mathbb{R}^+\}$ and we obtain the following consequence.

COROLLARY 5.2. *The formal series $\hat{\gamma}$ is k-summable, with at most $2k$ directions of non-summability. If u is the k-sum of $\hat{\gamma}$ in a direction, then $\varphi(x) = (x, u(x))$ is a strong Hakim parabolic curve for F or for F^{-1}.*

Fruchard and Zhang's theorem [**10**] ensures that if a function f is holomorphic and bounded in a sectorial domain S and has a Gevrey asymptotic expansion at the origin in a direction inside S, then this asymptotic expansion remains valid in the whole domain S. Therefore, Theorem 5.1 will follow from the following proposition.

PROPOSITION 5.3. *Let $\varphi(x) = (x, u(x))$ be a strong Hakim parabolic curve of F defined in the component Δ of Δ^k_{η,δ_0} bisected by $\mathbb{R}^+$. Then u admits $\hat{\gamma}$ as a $1/k$-Gevrey asymptotic expansion in the direction $\mathbb{R}^+$.*

The rest of this section is devoted to the proof of Proposition 5.3. To prove it, we use a standard method in summability theory: the construction of quasi-solutions of an equation and the computation of the difference between solutions and quasi-solutions. Given $0 < \beta \leq \pi/k$ and $\rho \in (0, +\infty]$, we denote by $V(0, \beta, \rho)$ the sector

$$V(0, \beta, \rho) = \{x \in \mathbb{C} : 0 < |x| < \rho, |\arg x| < \beta/2\}.$$

Since (x, y) are normalized coordinates for F, we have $F = (F_1, F_2)$ with

$$F_1(x, y) = x - x^{k+1} + O(x^{k+1}y, x^{k+2}),$$
$$F_2(x, y) = y - \alpha x^k y + O(x^k y^2, x^{k+1}y) + x^{k+m}\psi(x)$$

with $\alpha \notin \mathbb{Q}_{>0}$, $m \geq 2$ and $m > 1 + \operatorname{Re}\alpha$.

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LEMMA 5.4. *The formal series $\hat{\gamma}$ is $1/k$-Gevrey.*

Proof. In [**4**], it is proved that the infinitesimal generator of F is $1/k$-Gevrey. Then, according to the Gevrey version of Briot and Bouquet's theorem (the proof given in [**12**] can be easily adapted to the context of Gevrey vector fields), we obtain that $\hat{\gamma}$ is $1/k$-Gevrey (see also [**7**]). □

Quasi-invariant curves. Since $\hat{\gamma}$ is $1/k$-Gevrey, we know (see [**16**]) that there exists a function v holomorphic in the sector $V(0, \pi/k, \infty)$ that has $\hat{\gamma}$ as a $1/k$-Gevrey asymptotic expansion at the origin. For instance, we can take v to be the truncated Laplace transform of order k in the direction $\mathbb{R}^+$ of the Borel transform of order k of $\hat{\gamma}$.

LEMMA 5.5. *There exists $\varepsilon \leq \delta_0$ such that the difference*

$$R(x) = v(F_1(x, v(x))) - F_2(x, v(x))$$

is exponentially small of order k at the origin in $V = V(0, (\pi - \eta)/k, \varepsilon)$.

Proof. Since $v = O(x)$ in every proper subsector of $V(0, \pi/k, \infty)$, we get that for ε small enough $F_i(x, v(x))$ is well defined for $i = 1, 2$ and the sector $V = V(0, (\pi - \eta)/k, \varepsilon)$ is invariant for the map $x \mapsto F_1(x, v(x))$. Since $v \sim_{1/k} \hat{\gamma}$ in $V(0, \pi/k, \infty)$, we get

$$R(x) \sim_{1/k} \hat{\gamma}(F_1(x, \hat{\gamma}(x))) - F_2(x, \hat{\gamma}(x))$$

in V. Since Γ is an invariant curve for F, we conclude that $R \sim_{1/k} 0$ in V. □

Remark 5.6. The function $x \mapsto (x, v(x))$ is a *quasi-solution* (see [**6**]) of the equation satisfied by $\hat{\gamma}$, $\hat{\gamma}(F_1(x, \hat{\gamma}(x))) = F_2(x, \hat{\gamma}(x))$.

An equation for the correction $u - v$. Put $W_\delta = V(0, (\pi - 2\eta)/k, \delta) \subset V \cap \Delta^k_{\eta, \delta_0}$ for all $\delta \leq \varepsilon$. Denote $f_u(x) = F_1(x, u(x))$. We know that there exists δ small enough such that

$$f_u(W_\delta) \subset W_\delta, \quad |f_u^n(x)|^k \leq C|x|^k/(1 + n|x|^k) \quad \text{and} \quad f_u^n(x) \sim 1/(kn)^{1/k}$$

for all $x \in W_\delta$ and all $n \in \mathbb{N}$.

Consider the function $d = u - v$. We have

$$d(f_u(x)) = F_2(x, u(x)) - F_2(x, v(x)) - R(x) + v(F_1(x, v(x))) - v(F_1(x, u(x)))$$

for all $x \in W_\delta$. For all $x \in W_\delta$, we get

$$F_2(x, u(x)) - F_2(x, v(x)) = d(x)(1 - \alpha x^k + O(x^{k+1}))$$

and

$$v(F_1(x, v(x))) - v(F_1(x, u(x))) = d(x)O(x^{k+m}).$$

Then d satisfies the linear non-homogeneous f_u-difference equation

$$d(f_u(x)) = I(x)d(x) - R(x), \tag{3}$$

where $I(x) = 1 - \alpha x^k + O(x^{k+1})$ for all $x \in W_\delta$.

An exponentially small solution. Here we find a particular solution of equation (3) and we prove that it is exponentially small of order k at the origin in W_δ.

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Consider first the homogeneous equation

$$h(f_u(x)) = I(x)h(x) \tag{4}$$

in W_δ. For δ small enough, using the substitution

$$h(x) = x^\alpha e^{H(x)},$$

we can transform equation (4) into the standard f_u-difference equation

$$H(f_u(x)) - H(x) = J(x)$$

in W_δ with $J(x) = \log I(x) - \alpha \log(f_u(x)/x)$. We find a particular solution $h(x) = x^\alpha e^{H(x)}$ of equation (4) by putting

$$H(x) = -\sum_{n\geq 0} J(f_u^n(x)).$$

This series is normally convergent in W_δ, since $J(x) = O(x^{k+1})$ and $f_u^n(x) = O(1/n^{1/k})$.

Now, solutions of equation (3) are given by $s(x)h(x)$ with

$$s(f_u(x)) - s(x) = -\frac{R(x)}{h(f_u(x))},$$

so we obtain a particular solution $d_0(x) = s(x)h(x)$ of equation (3) by putting

$$s(x) = \sum_{n\geq 0} \frac{R(f_u^n(x))}{h(f_u^{n+1}(x))}.$$

Since R is exponentially small of order k at the origin in V, there exist constants $K_1, A_1, K_2, A > 0$ such that

$$|R(x)| \leq K_1 e^{-A_1/|x|^k} \leq K_2 |x|^m e^{-A/|x|^k}$$

for all $x \in W_\delta$. Then, since $f_u(W_\delta) \subset W_\delta$, we get

$$\left|\frac{R(f_u^n(x))}{h(f_u^{n+1}(x))}\right| \leq K_2 \frac{|f_u^n(x)|^m e^{-A/|f_u^n(x)|^k}}{|(f_u^{n+1}(x))^\alpha||e^{H(f_u^{n+1}(x))}|} \leq K e^{-AC^{-1}/|x|^k} e^{-An/C}$$

for all $x \in W_\delta$, so the series above converges normally in W_δ and there exist constants $K', B_1 > 0$ such that $|s(x)| \leq K' e^{-B_1/|x|^k}$ for all $x \in W_\delta$.

Then we obtain that for all $x \in W_\delta$,

$$|d_0(x)| = |x^\alpha e^{H(x)} s(x)| \leq M_1 |x|^{\operatorname{Re}\alpha} e^{-B_1/|x|^k} \leq M e^{-B/|x|^k},$$

so d_0 is exponentially small of order k at the origin in W_δ.

End of the proof. To finish the proof of Proposition 5.3, it suffices to show that $u - v = d_0$ in W_δ.

LEMMA 5.7. *There exists a unique solution $D \in \mathcal{O}(W_\delta)$ of equation (3) with $D = O(x^{m-1})$ in W_δ.*

Proof. Suppose that there exist two solutions $d_1, d_2 \in \mathcal{O}(W_\delta)$ of equation (3) such that $d_1, d_2 = O(x^{m-1})$. Since $d_2 - d_1$ satisfies equation (4), we get $d_2 - d_1 = hc$, where h is the solution of equation (4) constructed above and $c \in \mathcal{O}(W_\delta)$ satisfies

$$c(f_u(x)) = c(x).$$

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Since for each $x \in \Delta$ there exists $n \in \mathbb{N}$ such that $f_u^n(x) \in W_\delta$, we can extend c to Δ by putting $c(x) = c(f_u^n(x))$. By hypothesis, $|h(x)c(x)| \leq M_1|x|^{m-1}$ for some constant $M_1 > 0$, so $|c(x)| \leq M_2|x|^{m-1-\operatorname{Re}\alpha}|e^{H(x)}|$ for some $M_2 > 0$ and hence c is bounded and satisfies $\lim_{x\to 0} c(x) = 0$.

Let ϕ be the Fatou coordinate of f_u in Δ, that is, the biholomorphism $\phi : \Delta \to \mathbb{C}$ such that $\phi \circ f_u = \phi + 1$. Put $C = c \circ \phi^{-1}$. Since $C(z+1) = C(z)$ and C is defined in a right half-plane (see [**15**]), we can extend C to an entire function. Then, since C is bounded, it must be constant and then so is c. But, since $\lim_{x\to 0} c(x) = 0$, it follows that $c \equiv 0$. □

Then we get $D = d_0 = u - v$.

6. *Summability of Camacho–Sad curves*

Now we can prove summability of Camacho–Sad curves of a diffeomorphism.

THEOREM 6.1. *Consider $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ with an isolated fixed point and let Υ be a Camacho–Sad curve for F. Up to a change of order in the coordinates, there exists a Puiseux parameterization*

$$\Upsilon(t) = (t^n, \hat{\phi}(t)), \quad \hat{\phi}(t) \in \mathbb{C}[[t]]$$

of Υ such that $\hat{\phi}$ is l-summable for some $l \geq \mathrm{ord}(F) - 1$ and its sums define parabolic curves for F or for F^{-1}.

These sums coincide after desingularization with strong Hakim parabolic curves.

Proof. Let $\Gamma(z) = (z, \hat{\gamma}(z))$ be the Briot–Bouquet curve that corresponds to Υ after desingularization. Observe that the transformed vector field by the desingularization is the infinitesimal generator of the transformed diffeomorphism (see [**3**]), so by Corollary 5.2 $\hat{\gamma}$ is l-summable for some $l \geq \mathrm{ord}(F) - 1$ and its l-sums define strong Hakim parabolic curves. The desingularization morphism is locally of the form $(z, w) \mapsto (P_1(z, w), P_2(z, w))$, where $P_1(z, w)$, $P_2(z, w)$ are polynomials. Then $z \mapsto (P_1(z, \hat{\gamma}(z)), P_2(z, \hat{\gamma}(z)))$ is a parameterization of Υ and $P_i(z, \hat{\gamma}(z))$ is l-summable. Now it is enough to recall how to obtain Puiseux parameterizations for Υ. Up to a change of order in the coordinates, we have

$$n = \nu(P_1(z, \hat{\gamma}(z))) \leq \nu(P_2(z, \hat{\gamma}(z))).$$

We can write $P_1(z, \hat{\gamma}(z)) = z^n \hat{U}(z)$, where $\hat{U}(z) \in \mathbb{C}[[z]]$ is a unit. Since $\hat{U}(0) \neq 0$, the series $\hat{V} = \hat{U}^{1/n}$ is l-summable. As a consequence of the Lagrange inversion formula, the formal series $\hat{z}(t)$ such that $t = \hat{z}(t)\hat{V}(\hat{z}(t))$ is l-summable. Hence, we obtain a parameterization

$$t \mapsto (t^n, \hat{\phi}(t))$$

of Υ, where $\hat{\phi}(t) = P_2(\hat{z}(t), \hat{\gamma}(\hat{z}(t)))$ is l-summable. By construction, if v is the l-sum of $\hat{\phi}$ in a direction, then $t \mapsto (t^n, v(t))$ is a parabolic curve for F or for F^{-1}. □

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