Axes for symmetric convex curves

DAVID L. FARNSWORTH

1. *Introduction*

Curves are given in polar coordinates (r, θ) by equations of the form $r = f(\theta)$, where $f(\theta) > 0$ for all θ. Consider curves which are *symmetric about the origin* O, so that, $f(\theta + \pi) = f(\theta)$ for all θ. For such a curve, its *interior* is the set $\{(r, \theta) : 0 \leqslant r \leqslant f(\theta)\}$. Further, assume that the curve is convex. Recall that a closed curve is *convex* if a line segment between any two of its points has no points exterior to the curve [1], [2, pp. 198-203]. We call these curves *M-curves*, because the curves are fundamental objects in Minkowski geometry, where they are called *Minkowski circles* or simply *circles* [3, 4]. That application is briefly discussed in the Section 4 but is not required for our purposes.

Examples of M-curves are displayed in Figures 1 to 6. In order to express these curves as functions in rectangular coordinates, we need axes.

It is shown that all M-curves possess at least two axes, which are defined below, so that the curves can be expressed as functions in rectangular coordinates. Our attention is directed especially to M-curves that are piecewise linear.

A *line of support* is a line that intersects the M-curve and has none of its points interior to the curve [2, p. 41, p. 209], [3, p. 34]. Because the M-curve is convex, there is at least one line of support at each of its points [1, p. 626], [2, p. 41]. For sufficiently smooth portions of M-curves, the lines of support are tangent lines. If the curve has a linear portion, a line of support at each point in the linear segment is the line containing the segment. For some vertices, such as $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ in Figure 3, there is a pencil of concurrent supporting lines.

An *axis* is a line through the origin that is perpendicular to a line of support at the intersection of the line through the origin and the M-curve.

An axis can be used as the horizontal coordinate axis by rotating the M-curve or equivalently the coordinate system about the origin, so that the curve in the new first and second quadrants can be described by a function in Cartesian coordinates. The curve will be given in the upper half-plane by a formula of the form $y = g(x)$ and in the lower half-plane by $y = -g(-x)$.

Finding axes for the Euclidean circle in Figure 1 is easy, since all lines through the origin are axes. However, we would like to know whether all M-curves have axes and to have techniques for finding them. In Section 2, Theorem 1 says that for any M-curve there are always at least two lines through the origin that can serve as axes.

In Section 3, our sole focus is the properties of piecewise-linear M-curves. Figures 3, 4, 5 contain examples of such curves. Their linear portions are called *sides*, and the intersections of adjacent sides are called *vertices*. Theorem 2 gives an upper bound on the number of axes. Theorems

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3 and 4 together say that at least one axis must contain a vertex and at least one axis must intersect a side in a point that is not a vertex. For the curve in Figure 3, the current x-axis and y-axis are the first type of axes, and the lines $y = x$ and $y = -x$ are the second type of axes. Theorem 5 says that a piecewise-linear M-curve exists for any given number of axes.

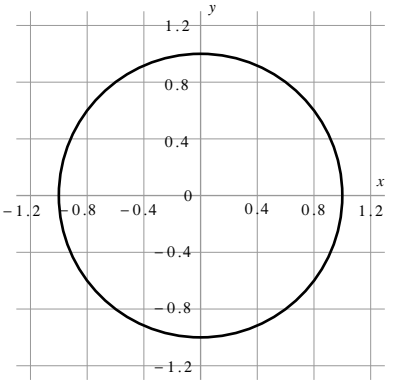


FIGURE 1: The usual Euclidean unit circle $x^2 + y^2 = 1$

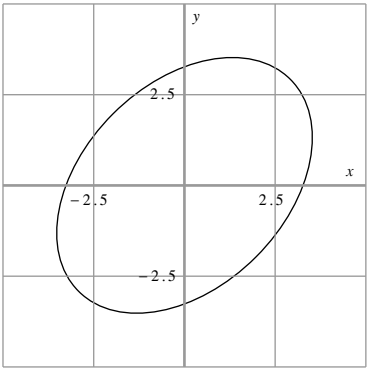


FIGURE 2: The ellipse $13x^2 + 13y^2 - 10xy - 36 = 0$

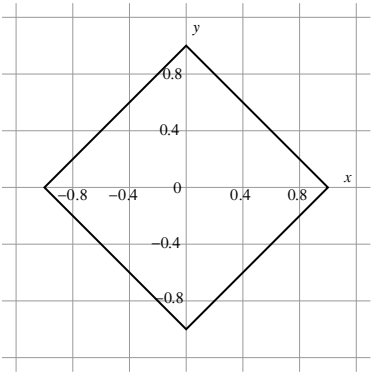


FIGURE 3: The piecewise-linear M-curve $|x| + |y| = 1$

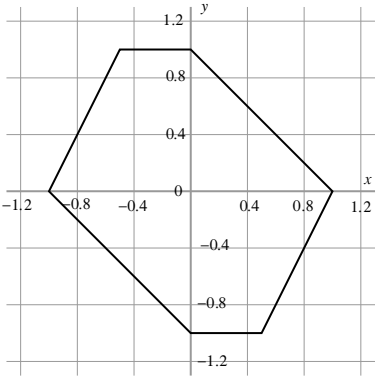


FIGURE 4: A piecewise-linear M-curve

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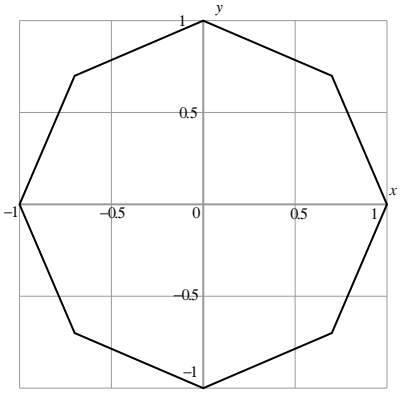


FIGURE 5: A regular octagon

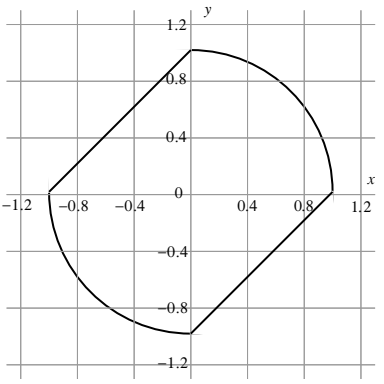


FIGURE 6: An M-curve that contains linear and nonlinear portions

2. *Existence of axes*

Theorem 1: There are at least two axes for any M-curve.

Proof: Define the portion C of the M-curve in the first and second quadrants in polar coordinates by $r = f(\theta)$. Since f is a real-valued, continuous, bounded function that is defined on the compact set $[0°,\ 180°]$, it attains its greatest and least values on that set [5, p. 73], [6, pp. 55-56], [7, pp. 275-276].

By rotating if necessary, first assume that C is at its greatest distance from the origin O at the point with Cartesian coordinates $(a,\ 0)$, $a > 0$. Then, C must lie entirely inside the disc $x^2 + y^2 \leqslant a^2$, so that the line $x = a$ is a line of support and $y = 0$ is an axis.

Secondly, assume C is at its least distance from O at $(a,\ 0)$, $a > 0$. We claim that once again $x = a$ is a line of support. There are points of C or its interior in the closed half-plane $x \leqslant a$; we claim there are no such points in the half-plane $x > a$. If there were such a point $P(b, c)$, then $b > a$, $c \neq 0$, and one of the tangents from P to the circle $x^2 + y^2 = a^2$ meets the circle at a point Q and crosses the x-axis at a point $(d,\ 0)$ between P and Q with $d > a$. So, this point is not on C or in its interior, and this contradicts the convexity condition. Thus $x = a$ is a line of support, and $y = 0$ is an axis.

2.1. *Axes for a smooth M-curve*

For sufficiently smooth M-curves, the search for axes can be straightforward. In rectangular coordinates, evaluate $y' = \dfrac{dy}{dx}$ at the intersection of the curve and $y = ax$, and solve $y' = -\dfrac{1}{a}$ for a. If necessary, separately consider special cases, such as $x = 0$. For the curve in Figure 2, implicitly differentiating its equation $13x^2 + 13y^2 - 10xy - 36 = 0$ yields $13x + 13yy' - 5y - 5xy' = 0$. Then

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$$y' = \frac{5y - 13x}{13y - 5x} = \frac{5(ax) - 13x}{13(ax) - 5x} = -\frac{1}{a}$$

gives $a = \pm 1$. Thus, $y = \pm x$ are the only axes. Alternatively, in polar coordinates, the curve is $r^2(13 - 5\sin 2\theta) - 36 = 0$. Differentiate with respect to θ to obtain $2rr'(13 - 5\sin 2\theta) - 10r^2\cos 2\theta = 0$. Putting $r' = 0$ yields $\cos 2\theta = 0$ and $\theta = \pm\frac{\pi}{4}$.

2.2. *Axes for a piecewise-linear M-curve*

For an example of axes for a piecewise-linear M-curve, consider the curve in Figure 4, which is reproduced in Figure 7 with its five axes as dashed lines. An efficient way to find axes for piecewise-linear M-curves is for each ray from the origin to a vertex, determine whether a support at the vertex is perpendicular to the ray, and for each side, determine whether any line perpendicular to it contains the origin.

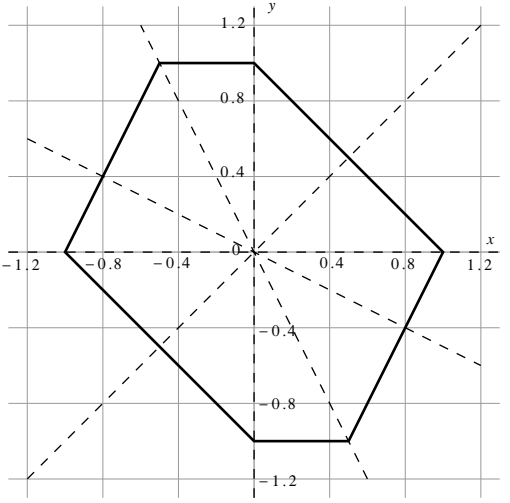


FIGURE 7: The axes for the M-curve in Figure 4 are the five dashed lines

2.3. *Axes for a mixed linear and nonlinear M-curve*

The M-curve in Figure 6 is composed of portions of $x^2 + y^2 = 1$ in the first and third quadrants and line segments with slope -1 in the second and fourth quadrants. Its axes are $x = 0$, $y = mx$ for any $m \geqslant 0$, and $y = -x$.

3. *Properties of axes for piecewise-linear M-curves*

Axes of a piecewise-linear M-curve either do or do not contain a vertex. These two types of axis may alternate, as in Figure 3, or not, as in Figure 7.

The number of axes can be even or odd. There are four axes for the M-curve in Figure 3 and five axes for the M-curve in Figure 7.

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Since by Theorem 1 there exist at least two axes, there is at least one direction to choose for the second, non-horizontal axis. Unlike the usual rectangular coordinates in Euclidean geometry, there may be no way to select two axes that are perpendicular. The M-curves in Figures 3, 4, 5 contain pairs of axes that are perpendicular. However, the M-curve in Figure 8, whose four axes are shown as dashed lines, has no pair of perpendicular axes.

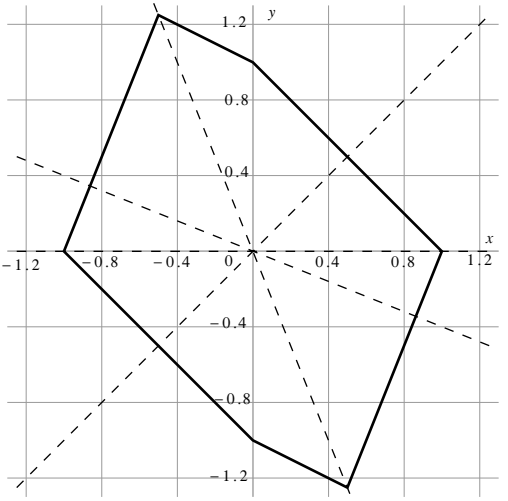


FIGURE 8: An M-curve with no pair of perpendicular axes

Theorem 2: A piecewise-linear M-curve with n vertices has at most n axes.

Proof: Each axis either meets two of the n vertices, or if not, then it meets the interiors of two sides, of which there are also n. So, the number of axes cannot be more than $\frac{1}{2}n + \frac{1}{2}n = n$.

As examples, the piecewise-linear M-curves in Figures 3 and 5 have the maximum number of axes for their number of vertices. The curves in Figures 4 and 7 have six vertices, but only five axes.

Theorem 3: For any piecewise-linear M-curve, at least one axis intersects the curve in the interior of a side.

Proof: Let the vertex Q be chosen such that no other vertex is closer to the origin O than Q, and let P and R be the vertices adjacent to Q. By convexity, at least one of $\angle PQO$ and $\angle RQO$ must be acute, say $\angle PQO < \frac{1}{2}\pi$. Since $|OP| \geqslant |OQ|$ we see that $\angle OPQ < \frac{1}{2}\pi$ too. It follows that the perpendicular line from O to PQ meets PQ between P and Q, and this perpendicular line is the required axis.

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Theorem 4: For any piecewise-linear M-curve, at least one axis intersects the curve in a vertex.

Proof: By rotating if necessary, assume that the point $(a,\ 0)$, $a > 0$, is a vertex of the curve, and that no other vertex is further from the origin. Then the other vertices, and hence the entire curve and its interior, must be in the closed half-plane $x \leqslant a$. So, $x = a$ is a line of support and $y = 0$ is an axis, and it contains the vertex $(a,\ 0)$.

Theorem 5: Piecewise-linear M-curves exist with exactly k axes for every $k \geqslant 2$.

Proof: For even $k \geqslant 4$, the regular polygons with k vertices have k axes, which pass through the vertices and the midpoints of the sides. The M-curve in Figure 9 has $k = 2$ axes, which are the x- and y-axes and are dashed. For odd $k \geqslant 3$, use the previously selected M-curve with $k - 1$ vertices and alter it with two bump-outs, which replace a side and its symmetrically placed side, each with two sides, such that the axis to the midpoint of the former side is replaced by two axes – one to the new vertex and another to one of the two new sides – while all other axes remain. Figure 10 contains a regular hexagon with six axes as dashed lines and an M-curve with seven axes as dashed lines, created from it with bump-outs.

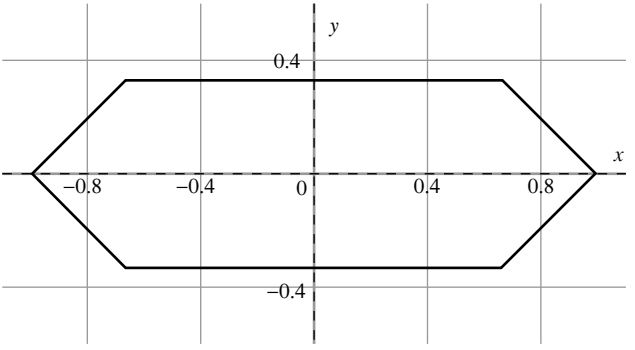


FIGURE 9: Piecewise-linear M-curve with two axes

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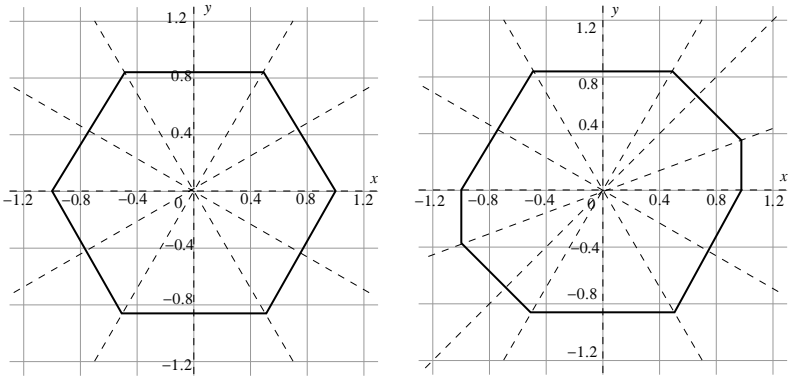


FIGURE 10: Creating bump-outs in order to replace an axis with two axes

4. *Discussion*

The M-curves are called Minkowski circles because each one is the unit circle of a Minkowski geometry. The Euclidean unit circle in Figure 1 defines the distance from the origin to (x, y) as $\sqrt{x^2 + y^2}$. The M-curve in Figure 3 is the unit circle of the Minkowski geometry called taxicab geometry, where the distance from the origin to (x, y) is $|x| + |y|$ [8]. For each Minkowski geometry, the radial distance from the origin to (x, y) is the Euclidean distance from the origin to (x, y) divided by the Euclidean distance from the origin to the point of intersection of the ray to (x, y) and the M-curve that is its unit circle. Thus distance is defined as a multiple of radial distance to the geometry's unit circle. The unit circle defines the distance function and vice versa. For more on Minkowski geometry, see [3, 4, 9, 10]. For ideas for classroom use and undergraduate research and for applications, see [4, pp. 17-18].

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DAVID L. FARNSWORTH
School of Mathematical Sciences, 84 Lomb Memorial Drive, Rochester Institute of Technology, Rochester, NY 14623 USA
e-mail: *DLFSMA@rit.edu*

The answers to the *Nemo* page from November on infinity were:

1.	GK Chesterton	The Man who was Thursday	
2.	Arthur Koestler	Darkness at Noon	
3.	John Donne	Love's Growth	
4.	Jonathan Swift	On Poetry	
5.	Robert Browning	Two in the Campagna	
6.	EM Forster	The Longest Journey	Part I, Chapter 3

Congratulations to Ian Anderson and Henry Ricardo on tracking all of these down. This month the theme is logical axioms and inferences. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 31st May 2018.

1. Poincaré concluded that the axioms of geometry are conventions, our choice among all possible conventions is guided by experimental facts, but it remains free and is limited only by the necessity of avoiding all contradiction. Thus it is that the postulates can remain rigorously true even though the experimental laws that have determined their adoption are only approximative. The axioms of geometry, in other words, are merely disguised definitions.

2. Men may dream in demonstrations, and cut out any illusory world in the shape of axioms, definitions, and propositions, with the final exclusion of fact signed Q.E.D.

Continued on page 62.

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RORY ALLEN

Department of Psychology, Goldsmiths, University of London, London SE14 6NW

Nemo, continued from page 30.

3. "For a complete logical argument", Arthur began with admirable solemnity, "we need two prim Misses—"

   "Of course!" she interrupted. "I remember that word now. And they produce—"

   "A Delusion," said Arthur.

   "Ye–es?" she said dubiously. "I don't seem to remember that so well. But what is the whole argument called?"

   "A Sillygism."

4. Euclidean axioms made holiday with the formulae of elementary statics. Arithmetic held a wild saturnalian kermesse; algebra cut capers. The music came to an end in an orgy of mathematical merry-making.

5. Syllogism: major premise, a definite and limited amount of bird-meat; minor premise, the only food the mutineers now have is bird-meat; conclusion, destroy the available food and the mutineers will be compelled to come back to duty.

6. "I believed nothing. I simply trusted you, as you asked me."

   "Quod erat demonstrandum!" cried Rowland. "I think you know Latin."

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