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Topological dynamics of exponential maps on their escaping sets

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Abstract. For the family of exponential maps $E_\kappa(z) = \exp(z) + \kappa$, we prove an analog of Böttcher's theorem by showing that any two exponential maps E_{κ_1} and E_{κ_2} are conjugate on suitable subsets of their escaping sets, and this conjugacy is quasiconformal. Furthermore, we prove that any two attracting and parabolic exponential maps are conjugate on their sets of escaping points; in fact, we construct an analog of Douady's 'pinched disk model' for the Julia sets of these maps. On the other hand, we show that two exponential maps are generally *not* conjugate on their sets of escaping points. We also answer several questions about escaping endpoints of dynamic rays. In particular, we give a necessary and sufficient condition for the ray to be continuously differentiable in such a point, and show that escaping points can escape arbitrarily slowly. Furthermore, we show that the principle of topological renormalization is false for attracting exponential maps.

1. *Introduction*

If p is a polynomial of degree $d \geq 2$, then by Böttcher's theorem [**M**, Theorem 9.1], p is conjugate to $z \mapsto z^d$ in a neighborhood of ∞. In the case where no critical point of p is attracted to ∞ (or equivalently if the Julia set of p is connected), this conjugacy can be extended to a biholomorphic mapping between the complement of the unit disk and the basin of infinity of p. The images of radial rays under this map give rise to the foliation of this basin by *dynamic rays*, which have been used very successfully in the combinatorial study of polynomials [**DH**].

In the family of exponential maps $E_\kappa : z \mapsto \exp(z) + \kappa$, the point ∞ is no longer an attracting fixed point, but rather an essential singularity, and the set of escaping points

$$I(E_\kappa) := \{z \in \mathbb{C} : E_\kappa^n(z) \to \infty\}$$

has no interior [**EL**, §2] and is thus contained in the Julia set. Nevertheless, it was recently shown by Schleicher and Zimmer [**SZ1**] that $I(E_\kappa)$ is a union of curves to infinity that

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can be seen as an analog of dynamic rays of polynomials. However, this still leaves open many questions on the topology of $I(E_\kappa)$, and on the dynamics of E_κ thereon. For example, one can ask whether, as in the polynomial case, any two exponential maps with non-escaping singular values are conjugate on their sets of escaping points.

We show, by a simple argument, that this is not true in general (see §2). In fact, it is already false when one of the parameters has an attracting fixed point and the other is a postsingularly finite (or *Misiurewicz*) parameter; i.e. one for which the singular value κ is preperiodic. (For polynomials, Misiurewicz parameters are among the most easily understood.) The argument generalizes to a much larger class of parameters; see §8.

Despite these negative results, it is possible to make some statements about the topological dynamics on the set of escaping points in general. We show the following, which can be seen as an analog of Böttcher's theorem. (An *escaping parameter* is one for which the singular value escapes.)

THEOREM 1.1. (Conjugacy between exponential maps) *Let $\kappa_1, \kappa_2 \in \mathbb{C}$. Let $R > 0$ be large enough and consider the set*

$$A := \{z \in \mathbb{C} : |E_{\kappa_1}^n(z)| \geq R \text{ for all } n \geq 1\}.$$

Then there exists a quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(E_{\kappa_1}(z)) = E_{\kappa_2}(\varphi(z)) \quad \text{for all } z \in A \quad \text{and} \tag{1.1}$$

$$|E_{\kappa_1}^n(z) - E_{\kappa_2}^n(\varphi(z))| \to 0 \quad \text{for all } z \in I(E_{\kappa_1}) \cap A. \tag{1.2}$$

If neither κ_1 nor κ_2 are escaping parameters, then $\varphi|_{A \cap I(E_{\kappa_1})}$ extends to a bijection $\Phi : I(E_{\kappa_1}) \to I(E_{\kappa_2})$ satisfying (1.1) and (1.2) for all $z \in I(E_{\kappa_1})$.

Remark 1. In general, the extended map Φ will not be continuous.

Remark 2. The number R can be chosen of size $R = O(\max(|\kappa_1|, |\kappa_2|))$.

The proof of Theorem 1.1 is achieved by constructing an explicit model for the set of escaping points, and then constructing a conjugacy between an exponential map and this model on a suitable set A as in the theorem. In particular, this yields a simplified proof of the classification of escaping points given by Schleicher and Zimmer (see Corollary 4.5).

While exponential maps are generally not conjugate on their sets of escaping points, the situation is quite different for parameters with an attracting (or parabolic) periodic orbit, since such maps are expanding on their Julia sets.

THEOREM 1.2. (Topological Conjugacy) *Suppose that κ_1 and κ_2 are attracting (or parabolic) parameters. Then the map Φ from Theorem 1.1 is a conjugacy.*

In fact, we give an explicit topological model for the Julia set of such a parameter and the topological dynamics thereon, based on its combinatorics, as a quotient of our general 'straight brush' model. Such models have already been constructed for the case of an attracting fixed point in [**AO**] and for general periods in [**BDD**]. (These constructions depended on the specific parameter, and thus do not imply Theorem 1.2.)

On the other hand, we show a somewhat surprising result for attracting exponential maps. For polynomials, *renormalization* fuels much of the detailed study of parameter

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spaces, ever since its introduction by Douady and Hubbard. Although the concept does not generalize directly (due to the absence of compactness, there is no notion of 'exponential-like maps'), it was hoped that some form of renormalization exists also in the exponential family (see [**S1**, §VI.6] for a formulation of this question). In particular, it was thought that renormalization is *topologically* valid; i.e. that, by collapsing certain rays and the regions between them for an exponential map E_κ that is 'renormalizable' of period n, the projection of the map E_κ^n on this space will be conjugate to another exponential map. We show that this is false even for attracting exponential maps. More precisely, suppose that E_κ has an attracting periodic orbit of period $n > 1$, and let $E_{\kappa'}$ be an exponential map with an attracting fixed point of the same multiplier as the attracting cycle of E_κ. If U is an immediate attracting basin of E_κ, then it is known (compare the discussion in §10) that E_κ restricted to U is conformally conjugate to $E_{\kappa'}$ restricted to its Fatou set $F(E_{\kappa'})$.

THEOREM 1.3. (No topological renormalization) *This conformal conjugacy $U \to F(E_{\kappa'})$ does not extend continuously to ∂U.*

The classification of escaping points by Schleicher and Zimmer [**SZ1**] exposed a feature of dynamic rays that does not occur for polynomials: some dynamic rays have endpoints that also escape to ∞. For the purposes of their classification, Schleicher and Zimmer used topological arguments to reach these endpoints and do not provide much additional information about them. In particular, it is not *a priori* clear whether the escape speed of these endpoints is independent of the parameter. That this is the case follows from Theorem 1.1. As our model for escaping points is very explicit, we can also answer several other questions concerning these endpoints; in particular, we show that they can escape arbitrarily slowly, or in fact with any prescribed escape speed. (Escaping points that are not endpoints are known to always escape with iterated exponential speed.)

THEOREM 1.4. (Arbitrary escape speed) *Let $\kappa \in \mathbb{C}$. Suppose that r_n is a sequence of positive real numbers such that $r_n \to \infty$ and $r_{n+1} \leq \exp(r_n) + c$ for some $c > 0$. Then there is an escaping endpoint $z \in I(E_\kappa)$ and some $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $|\operatorname{Re}(E_\kappa^{n-1}(z)) - r_n| \leq 2 + 2\pi$.*

Furthermore, we give an explicit necessary and sufficient condition (independent of the parameter) under which a ray with escaping endpoint is differentiable in this endpoint (Theorem 6.2).

Finally, we also discuss the situation in parameter space. The set $\mathcal{I}$ of parameters for which the singular value lies on a dynamic ray can be described in terms of *parameter rays* [**FS**]; this result is extended to escaping endpoints in [**FRS**]. As in the dynamical plane, it is interesting to ask what the topology of this set is. In §11, we show that the bijection between our model space and the set of escaping parameters is a homeomorphism on large sets, and that there exists a sequence of 'Cantor Bouquets' in parameter space whose union covers $\mathcal{I}$.

1.1. *Organization of the article.* After a short exposition of an example of two exponential maps that are not conjugate on their escaping sets in §2, the fundamental construction of our model and the correspondence with the set of escaping points of

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an exponential map (including the proof of Theorem 1.1) are carried out in §3 and §4. These two sections form the backbone of the remainder of the article.

The remaining sections are mostly independent of each other. Section 5 contains a short discussion of the limiting behavior of dynamic rays, collecting results and techniques from [**SZ1**] and [**FRS**]. In §6 we answer the question which rays are differentiable in their escaping endpoints, and in §7 we prove some facts on the speed of escape (including Theorem 1.4), demonstrating that questions of this type can easily be answered using our model.

Section 8 discusses the uniqueness of our correspondence and some further results on escaping set rigidity of exponential maps. Sections 9 and 10 are devoted to the proofs of Theorems 1.2 and 1.3, respectively. Finally, §11 examines continuity properties of parameter rays and §12 discusses some questions that remain open.

1.2. *Remarks on notation.* The Julia, Fatou and escaping sets of an exponential map will be denoted $J(E_\kappa)$, $F(E_\kappa)$ and $I(E_\kappa)$, as usual. It is well known [**BR, EL**] that exponential maps have no wandering domains and at most one non-repelling cycle.

In particular, if $F(E_\kappa) \neq \emptyset$, then $F(E_\kappa)$ consists of the basin of attraction of some attracting or parabolic cycle, or of the iterated preimages of a Siegel disk. In this case, the map E_κ (and also the parameter κ) is called *attracting*, *parabolic* or *Siegel*, respectively. If the *postsingular set*

$$\mathcal{P}(E_\kappa) := \overline{\bigcup_{n\geq 0} E_\kappa^n(\kappa)}$$

is finite, then κ is called a *Misiurewicz* parameter; in this case, $J(E_\kappa) = \mathbb{C}$.

Throughout the article, we shall fix the function $F : [0, \infty) \to [0, \infty); t \mapsto \exp(t) - 1$ as a model for exponential growth. We shall routinely make use of the fact that, for all $t_2 \geq t_1 \geq 0$,

$$F(t_2) - F(t_1) = e^{t_1} F(t_2 - t_1) \geq F(t_2 - t_1). \tag{1.3}$$

Following [**SZ1**], a sequence $\underline{s} = s_1 s_2 s_3 \cdots \in \mathbb{Z}^{\mathbb{N}}$ of integers is called an *external address*; the shift map on external addresses is denoted by σ. We end any proof by the symbol ■; statements which are cited without proof are concluded by □.

2. *Two exponential maps not conjugate on their escaping sets*

In this section, we describe, as a motivation for our further discussions, an example of two exponential maps that are not conjugate on their sets of escaping points. The two maps we will discuss are fairly well-understood since the 1980s.

Set $\kappa := -2$ and $E := E_\kappa : z \mapsto \exp(z) - 2$. This map has an attracting fixed point between -2 and -1 that attracts the entire interval $(-\infty, 0)$. In particular, the singular value κ is contained in the Fatou set. It is well known [**DeG**, Proposition 3.3] that the Julia set of E is a 'Cantor Bouquet'; in particular, every connected component is an injective curve $\gamma : [0, \infty) \to \mathbb{C}$, where $\gamma(t)$ escapes for $t > 0$; i.e. $\gamma((0, \infty)) \subset I(E)$.

On the other hand, let $\tilde{\kappa} := \log(2\pi) + (\pi/2)i$, and denote

$$\tilde{E} := E_{\tilde{\kappa}} : z \mapsto \exp(z) + \log(2\pi) + \frac{\pi}{2} i.$$

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Note that $\tilde{E}(\tilde{\kappa}) = 2\pi i + \tilde{\kappa}$ is a fixed point. The Julia set of $\tilde{E}$ is the entire plane, and the dynamics of $\tilde{E}$ is less well-understood than that of E. Nonetheless, it has long been known that there exists an injective curve of escaping points landing at the fixed point $\tilde{E}(\tilde{\kappa})$ and tending to ∞ in the other direction. (For a proof, see [**BDG**, Theorem 3.9] or [**SZ1**, Proposition 6.11]; for arbitrary Misiurewicz parameters the same fact is proved in [**SZ2**, Theorem 4.3].) Pulling back, we obtain a curve $\tilde{\gamma} : [0, \infty) \to \mathbb{C}$ with $\tilde{\gamma}(0) = \tilde{\kappa}$, $\tilde{\gamma}((0, \infty)) \subset I(E_{\tilde{\kappa}})$ and $\tilde{\gamma}(t) \to \infty$ for $t \to \infty$.

In the following, we will denote the Julia set of E by J and that of $\tilde{E}$ by $\tilde{J}$; similarly for their sets of escaping points etc. We are now ready to prove that these two maps are not conjugate on their sets of escaping points.

PROPOSITION 2.1. (No conjugacy) *The maps $E|_I$ and $\tilde{E}|_{\tilde{I}}$ are not topologically conjugate.*

Proof. Assume, by contradiction, that there is a conjugacy between E and $\tilde{E}$ on their sets of escaping points, say $\Phi : I \to \tilde{I}$. Consider the curve γ corresponding to $\tilde{\gamma}$ under Φ, i.e. $\gamma := \Phi^{-1} \circ \tilde{\gamma} : (0, \infty) \to I$. We first claim that $\lim_{t\to\infty} \gamma(t) = \infty$. Suppose not; by the above description of the topology of J we would then have $\lim_{t\to\infty} \gamma(t) = z$ for some $z \in \mathbb{C}$. However, then $E(z)$ would be a fixed point of E, and all points on γ converge to $E(z)$ under iteration. This is a contradiction, as the map E has only repelling periodic points in its Julia set.

Again by the topology of J, γ has an endpoint $z_0 := \gamma(0) := \lim_{t\to 0} \gamma(t)$. Pick any point $w \in I$, say $w = 2$, and denote $\tilde{w} := \Phi(w)$. Choose any open neighborhood U of w such that

$$\Phi(U \cap I) \subset \{z \in \mathbb{C} : |z - \tilde{w}| < 1\}.$$

Now, since $w \in J$, we can find a point $z_1 \in U$ with $E^n(z_1) = z_0$ for some n. Pulling back γ along the corresponding branch of E^{-n}, we obtain a curve $\alpha : (0, \infty) \to I$ with $\lim_{t\to 0} \alpha(t) = z_1$. In particular, there exists $t_0 > 0$ with $\alpha(t) \in U$ for $t \le t_0$.

Now consider the curve $\tilde{\alpha} := \Phi \circ \alpha$. This curve satisfies

$$|\tilde{\alpha}(t) - \tilde{w}| < 1$$

for $t \le t_0$. On the other hand,

$$\lim_{t\to 0} \tilde{E}^n(\tilde{\alpha}(t)) = \lim_{t\to 0} \tilde{\gamma}(t) = \tilde{\kappa},$$

and thus $\lim_{t\to 0} |\tilde{\alpha}(t)| = \infty$. This is a contradiction. ∎

For further discussion and more general results on the (non-)existence of conjugacies on the sets of escaping points, see §8.

3. *A Model for the set of escaping points*

To motivate the definitions that follow, let us shortly review the structure of the Julia set for the map $E = E_{-2}$ of the previous section, as described, for example, in [**DeG**]. As the interval $(-\infty, -2)$ is contained in the Fatou set, the Julia set J is disjoint from

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the preimages of this interval, which are straight lines of the form $\{\operatorname{Im} z = (2k-1)\pi\}$. In other words, J is completely contained in the strips

$$S_k := \{z : \operatorname{Im} z \in ((2k-1)\pi, (2k+1)\pi)\}.$$

To any point $z \in J$ we can thus associate an integer sequence $\underline{s} = s_1 s_2 \dots$ such that $E_\kappa^{k-1}(z) \in S_{s_k}$ for all k. This sequence is called the *external address of* z. It turns out that the connected components of J are exactly the sets of the form

$$\{z \in \mathbb{C} : z \text{ has external address } \underline{s}\},$$

which are curves consisting of escaping points together with an endpoint that may or may not escape.

Let us now develop the promised model for the set of escaping points of an exponential map (with non-escaping singular orbit). Based on the description above, our model should consist of pairs $(\underline{s}, t) \in \mathbb{Z}^{\mathbb{N}} \times [0, \infty)$†. Note that the space of external addresses has a natural topological structure, namely that induced by the lexicographic order on external addresses (open sets are unions of open intervals). Thus $\mathbb{Z}^{\mathbb{N}} \times [0, \infty)$ is equipped with the product of this topology and the usual topology of the real numbers.

For a given point $(\underline{s}, t)$, the first entry s_1 of $\underline{s}$ should be thought of as the imaginary part, while t corresponds to the real part. We thus define $Z(\underline{s}, t) := t + 2\pi i s_1$ and abbreviate $|(\underline{s}, t)| := |Z(\underline{s}, t)|$. We shall also write T for the projection to the second component; i.e. $T(\underline{s}, t) = t$. In analogy to the potential-theoretic interpretation of dynamic rays of polynomials, we will sometimes refer to t as the 'potential' of the point $(\underline{s}, t)$.

We now define a model function that will naturally give rise to our model space. There is considerable freedom in the definition. To suit our needs, we have chosen here to use a function that allows very explicit calculations, given by

$$\mathcal{F}(\underline{s}, t) := (\sigma(\underline{s}), F(t) - 2\pi |s_2|).$$

The key feature of our model dynamics is that, as for exponential maps, the size of the image of a point is roughly the exponential of its real part. Indeed, if $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ and $t \geq 0$ with $T(\mathcal{F}(\underline{s}, t)) \geq 0$, then

$$\frac{1}{\sqrt{2}} F(t) \leq |\mathcal{F}(\underline{s}, t)| \leq F(t).$$

We now define

$$\overline{X} := \{(\underline{s}, t) : \forall n \geq 0, T(\mathcal{F}^n(\underline{s}, t)) \geq 0\} \text{ and}$$
$$X := \{(\underline{s}, t) \in \overline{X} : T(\mathcal{F}^n(\underline{s}, t)) \to \infty\}.$$

The space X will be our model of the set of escaping points; as we show in §9, $\mathcal{F}|_{\overline{X}}$ is conjugate to the attracting exponential map E considered in §2 on its Julia set. In particular, the set $\overline{X}$ is a 'straight brush' in the sense of [**AO**], which we show directly in the following observation.

† Note that not all external addresses can be realized by an exponential map; accordingly the same will be true of our model.

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OBSERVATION 3.1. (Comb structure of $\overline{X}$ and X) *For every external address $\underline{s}$, there exists a $t_{\underline{s}}$, $0 \le t_{\underline{s}} \le \infty$, such that*

$$\{t \ge 0 : (\underline{s}, t) \in \overline{X}\} = [t_{\underline{s}}, \infty);$$

this $t_{\underline{s}}$ depends lower semicontinuously on $\underline{s}$. Furthermore,

$$(t_{\underline{s}}, \infty) \subset X_{\underline{s}} := \{t \ge 0 : (\underline{s}, t) \in X\} \quad \textit{and} \tag{3.1}$$

$$\mathcal{F}(\underline{s}, t_{\underline{s}}) = (\sigma(\underline{s}), t_{\sigma(\underline{s})}). \tag{3.2}$$

Proof. Suppose that $(\underline{s}, t) \in \overline{X}$ and $t' = t + \delta$, $\delta > 0$. By the definition of $\mathcal{F}$, we have

$$T(\mathcal{F}(\underline{s}, t')) - T(\mathcal{F}(\underline{s}, t)) = F(t') - F(t) \ge F(t' - t) = F(\delta).$$

By induction,

$$T(\mathcal{F}^n(\underline{s}, t')) \ge T(\mathcal{F}^n(\underline{s}, t')) - T(\mathcal{F}^n(\underline{s}, t)) \ge F^n(\delta) \to \infty.$$

This proves the first claim as well as equation (3.1). Note also that (3.2) follows directly from the definitions. To prove semicontinuity, note that $\overline{X}$ is a closed set. Therefore, for any $R > 0$ the set $\{\underline{s} : t_{\underline{s}} \le R\} = \{\underline{s} : (\underline{s}, R) \in \overline{X}\}$ is closed. ∎

Following [**SZ1**], we will call an external address $\underline{s}$ *exponentially bounded* if $t_{\underline{s}} < \infty$. Furthermore, we will call such an address *fast* if $(\underline{s}, t_{\underline{s}}) \in X$; otherwise $\underline{s}$ is called *slow*. (It is not difficult to see that these definitions are equivalent to those given in [**SZ1**], compare Corollary 7.2.) We will also denote the space of all exponentially bounded external addresses by $\mathcal{S}_0$.

4. *Classification of escaping points*

In the following, we fix some arbitrary parameter $\kappa \in \mathbb{C}$. As before, we say that a point $z \in \mathbb{C}$ has external address $\underline{s}$ if

$$\operatorname{Im}(E_\kappa^{n-1}(z)) \in S_{s_n}$$

for all n, where $S_k = \{z : \operatorname{Im} z \in ((2k-1)\pi, (2k+1)\pi)\}$. Note that, for general κ, not all points $z \in I(E_\kappa)$ have an external address, as components of $I(E_\kappa)$ may cross the strip boundaries (see Figure 1). However, some forward iterate of z always has an external address. Indeed, $|E_\kappa^n(z) - \kappa| = \exp(\operatorname{Re} E_\kappa^{n-1}(z))$, so $\operatorname{Re} E_\kappa^{n-1}(z) \to \infty$. Thus, if n is large enough, the orbit of $E_\kappa^n(z)$ will be contained in the half plane $\{\operatorname{Re} z > \operatorname{Re} \kappa\}$. Images of points on the strip boundaries, on the other hand, lie in $\{\kappa - t : t > 0\}$, so the orbit of $E_\kappa^{n-1}(z)$ never intersects these boundaries and thus has an external address.

We now construct a conjugacy between $\mathcal{F}$ and E_κ (defined on a suitable subset of X). This is done by iterating forward in our model and then backwards in the dynamics of E_κ. To this end, we define the inverse branches L_k of E_κ by

$$L_k(w) := \operatorname{Log}(w - \kappa) + 2\pi i k,$$

where $\operatorname{Log} : \mathbb{C} \setminus (-\infty, 0] \to S_0$ is the principal branch of the logarithm. Thus $L_k(w)$ is defined and analytic whenever $w - \kappa \notin (-\infty, 0]$.

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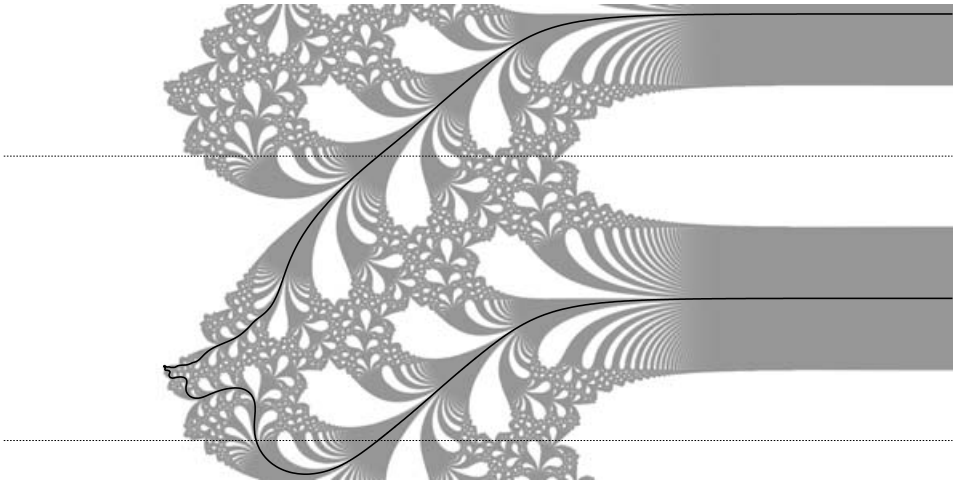


FIGURE 1. Dynamic rays may cross strip boundaries.

Define maps $\mathfrak{g}_k$ inductively by $\mathfrak{g}_0(\underline{s},t) := Z(\underline{s},t)$ and

$$\mathfrak{g}_{k+1}(\underline{s},t) := L_{s_1}(\mathfrak{g}_k(\mathcal{F}(\underline{s},t)))$$

(wherever this is defined).

Fix $K > 2\pi + 6$ such that $|\kappa| \le K$, and let us define

$$\begin{aligned} Q &:= Q(K) := \max\{\log(4(K+\pi+3)), \pi+2\} \quad \text{and} \\ Y &:= Y_Q := \{(\underline{s},t) \in \overline{X} : T(\mathcal{F}^n(\underline{s},t)) \ge Q \ \forall n\}. \end{aligned} \tag{4.1}$$

Note that Y contains the set $\{(\underline{s},t) : \underline{s} \in \mathcal{S}_0 \text{ and } t \ge t_{\underline{s}} + Q\}$. Note also that, for every $x = (\underline{s},t) \in X$, there exists some n such that $\mathcal{F}^n(x) \in Y$.

LEMMA 4.1. (Bound on $\mathfrak{g}_k$) *For all k, the map $\mathfrak{g}_k$ is defined on Y. For all $(\underline{s},t) \in Y$, we have $|\operatorname{Re}\mathfrak{g}_k(\underline{s},t) - t| < 2$, and in particular $|\mathfrak{g}_k(\underline{s},t) - Z(\underline{s},t)| < \pi + 2$.*

Proof. The idea of the proof, which proceeds by induction, is quite simple. By the induction hypothesis we know that $\mathfrak{g}_k(\mathcal{F}(\underline{s},t))$ and $Z(\mathcal{F}(\underline{s},t))$ are close, and by the definition of $\mathcal{F}$ the values $|Z(\mathcal{F}(\underline{s},t))|$ and $|E_\kappa(Z(\underline{s},t))|$ are essentially the same, namely $F(t)$ (up to a constant factor). Pulling back $\mathfrak{g}_k(\mathcal{F}(\underline{s},t))$ and $E_\kappa(Z(\underline{s},t))$ by the same branch of $E_\kappa^{-1} = \operatorname{Log}(z-\kappa)$, this constant factor translates to an additive constant for the real parts, as desired. The somewhat unpleasant calculations that follow flesh out this idea and fix the constants.

To begin the induction, note that the case $k = 0$ is trivial. Now fix $k \ge 0$ such that the claim is true for k; we will show that it holds also for $k+1$. Let $(\underline{s},t) \in Y$. By the induction hypothesis,

$$\begin{aligned} |\mathfrak{g}_k(\mathcal{F}(\underline{s},t)) - Z(\mathcal{F}(\underline{s},t))| &\le \pi + 2 \quad \text{and} \\ \operatorname{Re}(\mathfrak{g}_k(\mathcal{F}(\underline{s},t))) &\ge T(\mathcal{F}(\underline{s},t)) - 2 \ge Q - 2 \ge 0. \end{aligned}$$

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Furthermore, we have

$$F(t) = \exp(t) - 1 \geq \exp(Q) - 1 \geq 4(K + \pi + 2).$$

In particular,

$$|\mathfrak{g}_k(\mathcal{F}(\underline{s}, t))| \geq |Z(\mathcal{F}(\underline{s}, t))| - \pi - 2 \geq \frac{F(t)}{\sqrt{2}} - \pi - 2 > 2K.$$

Thus $\mathfrak{g}_k(\mathcal{F}(\underline{s}, t)) - \kappa \notin (-\infty, 0]$, so $\mathfrak{g}_{k+1}(\underline{s}, t)$ is defined. Furthermore, we can write

$$\mathfrak{g}_k(\mathcal{F}(\underline{s}, t)) - \kappa = Z(\mathcal{F}(\underline{s}, t)) + (\mathfrak{g}_k(\mathcal{F}(\underline{s}, t)) - Z(\mathcal{F}(\underline{s}, t)) - \kappa),$$

and, by the definition of Q,

$$|\mathfrak{g}_k(\mathcal{F}(\underline{s}, t)) - Z(\mathcal{F}(\underline{s}, t)) - \kappa| \leq \pi + 2 + K \leq \tfrac{1}{4}\exp(Q) - 1 \leq \tfrac{1}{4}\exp(t) - 1.$$

Therefore,

$$\begin{aligned}\operatorname{Re}(\mathfrak{g}_{k+1}(\underline{s}, t)) = \log|\mathfrak{g}_k(\mathcal{F}(\underline{s}, t)) - \kappa| &\geq \log\left(|Z(\mathcal{F}(\underline{s}, t))| + 1 - \frac{1}{4}\exp(t)\right) \\ &\geq \log\left(\frac{1}{\sqrt{2}}\exp(t) - \frac{1}{4}\exp(t)\right) > \log\left(\frac{\exp(t)}{4}\right) = t - \log 4.\end{aligned}$$

Analogously $\operatorname{Re}(\mathfrak{g}_{k+1}(\underline{s}, t)) \leq t + \log 4$, and thus $|\operatorname{Re}(\mathfrak{g}_{k+1}(\underline{s}, t)) - t| \leq \log 4 < 2$, as required. ■

With the estimate of Lemma 4.1, we can now construct the required conjugacy by a standard contraction argument.

THEOREM 4.2. (Convergence to a conjugacy) *On Y, the functions $\mathfrak{g}_k$ converge uniformly (in $(\underline{s}, t)$ and κ with $|\kappa| \leq K$) to a function $\mathfrak{g} : Y \to J(E_\kappa)$ such that $\mathfrak{g}(\underline{s}, t)$ has external address $\underline{s}$ for each $(\underline{s}, t) \in Y$. This function satisfies $\mathfrak{g} \circ \mathcal{F} = E_\kappa \circ \mathfrak{g}$ and*

$$|\mathfrak{g}(\underline{s}, t) - (\operatorname{Log}(Z(\mathcal{F}(\underline{s}, t))) + 2\pi i s_1)| \leq e^{-t} \cdot (2K + 2\pi + 4). \tag{4.2}$$

Furthermore, $\mathfrak{g}(\underline{s}, t) \in I(E_\kappa)$ if and only if $(\underline{s}, t) \in X$, $\mathfrak{g}$ is a homeomorphism between Y, and its image and $\mathfrak{g}(\underline{s}, t)$ depends holomorphically on κ for fixed $(\underline{s}, t) \in Y$.

Remark. Note that, for every $\underline{s} \in \mathcal{S}_0$, equation (4.2) implies that

$$\mathfrak{g}(\underline{s}, t) = t + 2\pi i s_1 + O(e^{-t}). \tag{4.3}$$

Proof. Recall from the previous proof that, for $(\underline{s}, t) \in Y$ and $n \geq 1$,

$$|\mathfrak{g}_k(\mathcal{F}^n(\underline{s}, t))| > 2K \quad (\geq |\kappa| + 2\pi + 6) \quad \text{and}$$
$$\operatorname{Re}(\mathfrak{g}_k(\mathcal{F}^n(\underline{s}, t))) > 0.$$

Furthermore, the distance between $\mathfrak{g}_k(\mathcal{F}(\underline{s}, t))$ and $\mathfrak{g}_{k+1}(\mathcal{F}(\underline{s}, t))$ is at most $2\pi + 4$. Thus we can connect these two points by a straight line within the set

$$\{z \in \mathbb{C} : |z - \kappa| \geq 2 \text{ and } z - \kappa \notin (-\infty, 0]\}.$$

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Since $L'_{s_1}(z) = 1/(z-\kappa)$,

$$|\mathfrak{g}_{k+1}(\underline{s},t) - \mathfrak{g}_{k+2}(\underline{s},t)| \le \tfrac{1}{2}|\mathfrak{g}_k(\mathcal{F}^{k+1}(\underline{s},t)) - \mathfrak{g}_{k+1}(\mathcal{F}^{k+1}(\underline{s},t))|.$$

It follows by induction that

$$|\mathfrak{g}_{k+1}(\underline{s},t) - \mathfrak{g}_{k+2}(\underline{s},t)| \le 2^{-(k+1)}|Z(\mathcal{F}^{k+1}(\underline{s},t)) - \mathfrak{g}_1(\mathcal{F}^{k+1}(\underline{s},t))| \le 2^{-(k+1)}(\pi+2),$$

so the $\mathfrak{g}_n$ converge uniformly on Y. By definition, $\mathfrak{g}(\underline{s},t) = L_{s_1}(\mathfrak{g}(\mathcal{F}(\underline{s},t)))$, and thus

$$E_\kappa \circ \mathfrak{g} = \mathfrak{g} \circ \mathcal{F}.$$

To prove the asymptotics (4.2), we first observe that

$$|Z(\mathcal{F}(\underline{s},t))| \ge F(t)/\sqrt{2} \ge 2\exp(t)/3$$

(which follows easily from the fact that $t \ge 3$), and thus

$$\left|\frac{\mathfrak{g}(\mathcal{F}(\underline{s},t)) - \kappa - Z(\mathcal{F}(\underline{s},t))}{Z(\mathcal{F}(\underline{s},t))}\right| \le \frac{3(\pi+2+K)}{2\exp(t)} \le \frac{3}{8}.$$

Since $|\mathrm{Log}(1+z)| \le 4|z|/3$ when $|z| \le 3/8$, it follows that

$$\begin{aligned}|\mathfrak{g}(\underline{s},t) - (\mathrm{Log}(Z(\mathcal{F}(\underline{s},t))) + 2\pi i s_1)| &= \left|\mathrm{Log}\left(1 + \frac{\mathfrak{g}(\mathcal{F}(\underline{s},t)) - \kappa - Z(\mathcal{F}(\underline{s},t))}{Z(\mathcal{F}(\underline{s},t))}\right)\right| \\ &\le \frac{4}{3}\frac{3(\pi+2+K)}{2\exp(t)} = e^{-t}\cdot(2\pi+4+2K).\end{aligned}$$

By Lemma 4.1, $\mathfrak{g}(\underline{s},t)$ escapes under iteration of E_κ if and only if $(\underline{s},t)$ escapes under iteration of $\mathcal{F}$. Clearly the point $\mathfrak{g}(\underline{s},t)$ has the correct external address (note that $\arg(\mathfrak{g}_{k-1}(\underline{s},t))$ is bounded away from $\pm\pi$, so that the values $\mathfrak{g}_k(\underline{s},t)$ cannot converge to the strip boundaries). In particular, $\mathfrak{g}(\underline{s},t) \ne \mathfrak{g}(\underline{s}',t')$ whenever $\underline{s} \ne \underline{s}'$, because the points have different external addresses. On the other hand, the $\mathcal{F}$-orbits of $(\underline{s},t)$ and $(\underline{s},t')$, $t \ne t'$ will eventually be arbitrarily far apart, and therefore the same holds for $\mathfrak{g}(\underline{s},t)$ and $\mathfrak{g}(\underline{s},t')$ under E_κ. This proves injectivity.

The function $\mathfrak{g}$ is continuous as uniform limit of continuous functions; for the same reason, $\mathfrak{g}(\underline{s},t)$ is analytic in κ. To prove that the inverse $\mathfrak{g}^{-1}$ is continuous, note that we can compactify both Y and $\mathfrak{g}(Y)$ by adding a point at infinity. The extended map $\mathfrak{g}$ is still continuous, and the inverse of a continuous bijective map on a compact space is continuous. ∎

Remark. The asymptotic description of $\mathfrak{g}(\underline{s},t)$ in terms of

$$\begin{aligned}\mathrm{Log}(Z(\mathcal{F}(\underline{s},t))) + 2\pi i s_1 = \log\sqrt{(F(t) - 2\pi|s_2|)^2 + (2\pi s_2)^2} \\ + i\arg((F(t) - 2\pi|s_2|) + 2\pi i s_2) + 2\pi i s_1\end{aligned}$$

is somewhat awkward. Had we used, instead of $\mathcal{F}$, the map

$$\mathcal{F}'(\underline{s},t) := (\sigma(\underline{s}), \sqrt{F(t)^2 - (2\pi s_2)^2}),$$

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then the whole construction would have carried through analogously (with somewhat improved constants). For the map $\mathfrak{g}' : Y \to J(E_\kappa)$ that we obtain this way, we would correspondingly have the following asymptotics:

$$|\operatorname{Re}(\mathfrak{g}'(\underline{s}, t)) - t| < e^{-t} \cdot (2K + 2\pi + 4).$$

In this article, we will never use the asymptotics in any other form than equation (4.3), whereas we shall rather often make direct calculations. This is why we have opted to use the function $\mathcal{F}$ rather than $\mathcal{F}'$.

In order to extend $\mathfrak{g}$ to a bijection $\mathfrak{g} : X \to I(E_\kappa)$, the main remaining problem is to decide when a point is contained in $\mathfrak{g}(Y)$. The following is a counterpart to Theorem 4.2.

THEOREM 4.3. (Points in the image of $\mathfrak{g}$) *Suppose that $z \in \mathbb{C}$ spends its entire orbit in the halfplane $\{w \in \mathbb{C} : \operatorname{Re} w \geq Q + 1\}$. Then z has an external address $\underline{s}$, and there exists $t \geq t_{\underline{s}}$ such that $(\underline{s}, t) \in Y$ and $z = \mathfrak{g}(\underline{s}, t)$.*

Proof. First note that, for $n \geq 1$,

$$|E_\kappa^n(z) - \kappa| = \exp(\operatorname{Re}(E_\kappa^{n-1}(z))) \geq \exp(Q + 1) > 2K$$

and $\operatorname{Re}(E_\kappa^n(z)) > 0$, so $E_\kappa^n(z) - \kappa \notin (-\infty, 0)$. Therefore, no iterate of z lies on the strip boundaries, and thus z has an external address $\underline{s}$.

Consider the sequence t_k of potentials uniquely defined by

$$\mathcal{F}^k(\underline{s}, t_k) = (\sigma^k(\underline{s}), \operatorname{Re}(E_\kappa^k(z))).$$

We claim that (similarly to Lemma 4.1), for $j \leq k$,

$$|T(\mathcal{F}^j(\underline{s}, t_k)) - \operatorname{Re}(E_\kappa^j(z))| \left(= \left| \log \frac{T(\mathcal{F}^{j+1}(\underline{s}, t_k)) + 2\pi s_{j+2} + 1}{|E_\kappa^{j+1}(z) - \kappa|} \right| \right) \leq 1. \tag{4.4}$$

The idea is again to prove this by induction: since $E_\kappa^{j+1}(z)$ belongs to the strip $S_{s_{j+2}}$ and $T(\mathcal{F}^{j+1}(\underline{s}, t_k))$ and $\operatorname{Re}(E_\kappa^{j+1}(z))$ are close by the induction hypothesis, the ratio in the logarithm is bounded. We omit the precise calculations here.

Let t be any limit point of the sequence t_k. Then $|T(\mathcal{F}^j(\underline{s}, t)) - \operatorname{Re}(E_\kappa^j(z))| \leq 1$ by equation (4.4); in particular, $(\underline{s}, t) \in Y$.

Since $\mathfrak{g}(\underline{s}, t)$ also has external address $\underline{s}$, it now follows that the distance between $E_\kappa^j(\mathfrak{g}(\underline{s}, t))$ and $E_\kappa^j(z)$ is bounded for all j. By the same contraction argument as in the proof of Theorem 4.2, they are equal. ∎

We can now prove the existence of a global correspondence between X and $I(E_\kappa)$.

COROLLARY 4.4. (Global correspondence) *Suppose that $\kappa \notin I(E_\kappa)$. Then $\mathfrak{g}|_{Y \cap X}$ extends to a bijective function*

$$\mathfrak{g} : X \to I(E_\kappa)$$

which satisfies $\mathfrak{g}(\mathcal{F}(\underline{s}, t)) = E_\kappa(\mathfrak{g}(\underline{s}, t))$. The function $\mathfrak{g}$ is a homeomorphism on every $\mathcal{F}^{-k}(Y \cap X)$ and for every $\underline{s} \in \mathcal{S}_0$ the function $t \mapsto \mathfrak{g}(\underline{s}, t)$ is continuous ('$\mathfrak{g}$ is continuous along rays').

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If $\kappa \in I(E_\kappa)$, then $\mathfrak{g}$ has a similar extension as follows. There exists $(\underline{s}^0, t^0) \in X$ with $\mathfrak{g}(\underline{s}^0, t^0) = \kappa$, and $\mathfrak{g}$ is defined for all $(\underline{s}, t) \in X$ except those with $\mathcal{F}^n(\underline{s}, t) = (\underline{s}^0, t')$ for some $n \geq 1$ and $t' \leq t^0$. For every k, $\mathfrak{g}$ is continuous on the intersection of $\mathcal{F}^{-k}(Y \cap X)$ with its domain of definition and it is continuous along rays.

Whenever $x_0 \in \mathcal{F}^{-n}(Y)$ such that $\mathfrak{g}(x_0)$ is defined, there exists a neighborhood U of x_0 in $\mathcal{F}^{-n}(Y)$ such that $\mathfrak{g}|_U$ is defined and continuous (as a function of $x = (\underline{s}, t)$ and κ) for all parameters with $|\kappa| \leq K$.

Proof. We will consider only the case of $\kappa \notin I(E_\kappa)$; the other statements follow similarly.

It is sufficient to show, by induction, that $\mathfrak{g}$ extends to a homeomorphism

$$\mathfrak{g} : \mathcal{F}^{-k}(Y \cap X) \to E_\kappa^{-k}(\mathfrak{g}(Y \cap X))$$

for every $k \geq 1$. Indeed, the sets of definition clearly exhaust all of X, while the range exhausts $I(E_\kappa)$ by Theorem 4.3. Continuity along rays also follows because every $(\underline{s}, t) \in X$ has a neighborhood on the ray that is completely contained in the same $\mathcal{F}^{-k}(Y \cap X)$.

So let us suppose that $\mathfrak{g}$ has been extended to $\mathcal{F}^{-k}(Y \cap X)$. First note that we can extend $\mathfrak{g}$ to $\mathcal{F}^{-(k+1)}(Y \cap X)$ in such a way that the extension is continuous along rays. Indeed, for every $\underline{s}$, we can choose a branch L of E_κ^{-1} on the ray such that $L(\mathfrak{g}(\mathcal{F}(\underline{s}, t))) = \mathfrak{g}(\underline{s}, t)$ whenever $(\underline{s}, t) \in \mathcal{F}^{-k}(Y \cap X)$. This extension is also continuous in both variables because the branch L varies continuously. ∎

As a direct consequence of Corollary 4.4, we obtain the classification theorem from [**SZ1**].

COROLLARY 4.5. (Classification of escaping points [**SZ1**, Corollary 6.9]) *Let E_κ be an exponential map. For every escaping point $z \in I(E_\kappa)$, exactly one of the following holds:*

- *there exists a unique $x \in X$ such that $z = \mathfrak{g}(\underline{s}, t)$; or*
- *the singular value κ escapes; there exist $\underline{s}$ and $t_0 > t_{\underline{s}}$ with $\kappa = \mathfrak{g}(\underline{s}, t_0)$, and there is $n \geq 1$ such that $E_\kappa^n(z) = \mathfrak{g}(\underline{s}, t)$ with $t_{\underline{s}} \leq t \leq t_0$.* ∎

Proof of Theorem 1.1. Set $K := \max(|\kappa_1|, |\kappa_2|, 2\pi + 6) + 1$, and define $R := Q(K) + 1$. For every $\kappa \in \mathbb{C}$ with $|\kappa| < K$, let $\mathfrak{g}^\kappa : Y_{Q(K)} \to \mathbb{C}$ denote the map from Theorem 4.2.

If A is as in the statement of the theorem, then, by Theorem 4.3, $A \subset \mathfrak{g}^{\kappa_1}(Y_{Q(K)})$. Thus the map

$$\varphi^\kappa : A \to \mathbb{C}, z \mapsto \mathfrak{g}^\kappa((\mathfrak{g}^{\kappa_1})^{-1}(z))$$

conjugates E_{κ_1} on A to E_κ on $\varphi^\kappa(A)$. Since φ^κ is a holomorphic motion of A (compare [**MSS**]), φ^{κ_2} extends to a quasiconformal homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$.

If κ_1 and κ_2 are non-escaping parameters, then $\varphi^{\kappa_2}|_{A \cap I(E_{\kappa_1})}$ extends to the bijection $\Phi := \mathfrak{g}^{\kappa_2} \circ (\mathfrak{g}^{\kappa_1})^{-1}$ from Corollary 4.4 (which is continuous along rays, but not continuous in general). ∎

When considering individual rays, it is often cumbersome to take into account the starting potential $t_{\underline{s}}$. For convenience, we make the following definition.

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Definition 4.6. (Dynamic rays) Let E_κ be an exponential map and let $\underline{s} \in \mathcal{S}_0$. We define a curve $g_{\underline{s}}$—the *dynamic ray at address* $\underline{s}$—by

$$g_{\underline{s}}(t) = \mathfrak{g}(\underline{s}, t + t_{\underline{s}}).$$

If $g_{\underline{s}}$ is not defined for all $t > 0$ (i.e. if there exists $t_0 > t_{\underline{s}}$ such that $\mathfrak{g}(\mathcal{F}^n(\underline{s}, t_0)) = \kappa$), then we call $g_{\underline{s}}$ a 'broken ray'. We say that an unbroken ray $g_{\underline{s}}$ *lands* at a point z_0 if $\lim_{t\to 0} g_{\underline{s}}(t) = z_0$. Similarly, we say that $g_{\underline{s}}(t)$ *has an escaping endpoint* if $g_{\underline{s}}(0)$ is defined and escaping; i.e. if $(\underline{s}, t_{\underline{s}}) \in X$.

LEMMA 4.7. (Convergence of rays) *Let E_κ be an exponential map. Suppose that $\underline{s}^n$ is a sequence of external addresses converging to a sequence $\underline{s}^0 \in \mathcal{S}_0$ such that also $t_{\underline{s}^n} \to t_{\underline{s}^0}$, and let $t_0 > 0$ such that $g_{\underline{s}^0}(t)$ is defined for all $t \geq t_0$. Then*

$$g_{\underline{s}^n}|_{[t_0,\infty)} \to g_{\underline{s}^0}|_{[t_0,\infty)}$$

uniformly.

Proof. Let $Q = Q(|\kappa|)$ as before. There exists k such that $\{(\underline{s}, t) : t - t_{\underline{s}} \geq t_0\} \subset \mathcal{F}^{-k}(Y_Q)$. The claim then follows from Corollary 4.4. ■

Remark. In the case where $g_{\underline{s}^0}$ is broken, we can say the following (with the same proof). Suppose that $\underline{s}^n > \underline{s}^0$ (or $< \underline{s}^0$) for all n. Then, under the assumptions of Lemma 4.7 the rays $g_{\underline{s}^n}$ converge locally uniformly (in the spherical metric on the Riemann Sphere $\hat{\mathbb{C}}$) to a curve $\widetilde{g_{\underline{s}^0}} : (0, \infty) \to \hat{\mathbb{C}}$. This curve has $\widetilde{g_{\underline{s}^0}}(t) = \infty$ if and only if $\mathfrak{g}(\mathcal{F}^n(\underline{s}^0, t + t_{\underline{s}^0})) = \kappa$ for some $n \geq 1$ and coincides with $g_{\underline{s}^0}$ where the latter is defined. If the ray that contains κ is periodic, then ∞ is assumed infinitely many times on this curve. In this case, the curve accumulates everywhere on itself. (See Figure 2.) A theorem of Curry [**C**] can then be used to show that the accumulation set of $\widetilde{g_{\underline{s}^0}}$ in $\mathbb{C}$ can be compactified to an indecomposable continuum. This was previously done for $\kappa \in (-1, \infty)$ in [**De**] and for certain other parameters in [**M-R**].

In the remainder of the article, we sometimes write $\mathfrak{g}^\kappa$ or $g_{\underline{s}}^\kappa$ for the objects constructed previously when the parameter is not fixed in the context.

There is an interesting corollary of Theorem 4.2. Note that Y_Q contains many points of $\overline{X} \setminus X$; in particular endpoints of periodic addresses. Which of these addresses lie in Y_Q depends on Q (and thus on the parameter); however, we can use this fact to give an elementary bound on those parameters for which we know that these rays cross sector boundaries or are not defined. This result is used in [**R2**] and [**RS1**] to bound *parameter rays* (see §11) and *wakes of hyperbolic components*.

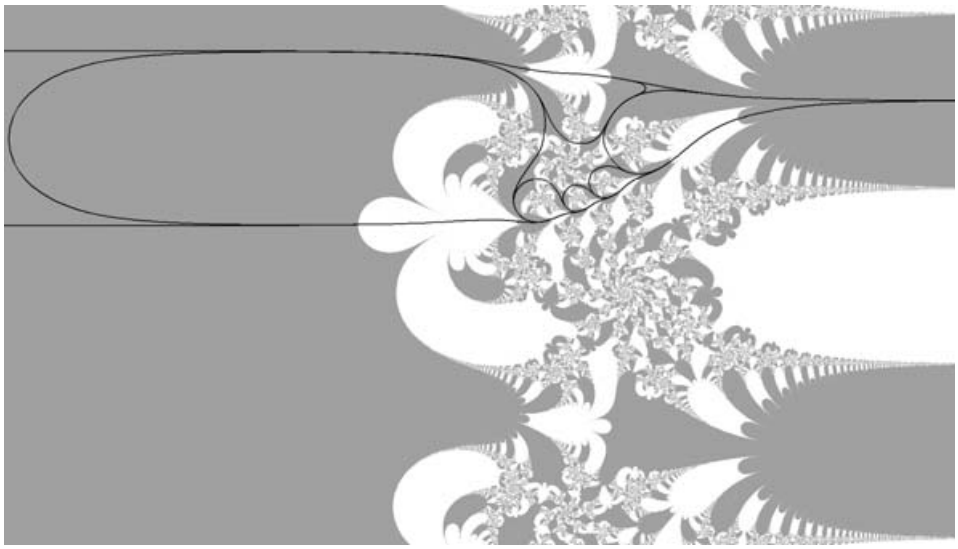
COROLLARY 4.8. (Bound on parameter rays) *Let $\kappa \in \mathbb{C}$, $Q := Q(|\kappa|)$ as in equation (4.1) and suppose that $(\underline{s}, t) \in \overline{X} \setminus Y_Q$. If the number $t_0 := \inf_{j\geq 0} T(\mathcal{F}^j(\underline{s}, t))$ satisfies $t_0 \geq \pi + 2$, then $|\kappa| > \frac{1}{5}\exp(t_0)$.*

In particular, suppose that $\underline{s}$ is periodic of period n and

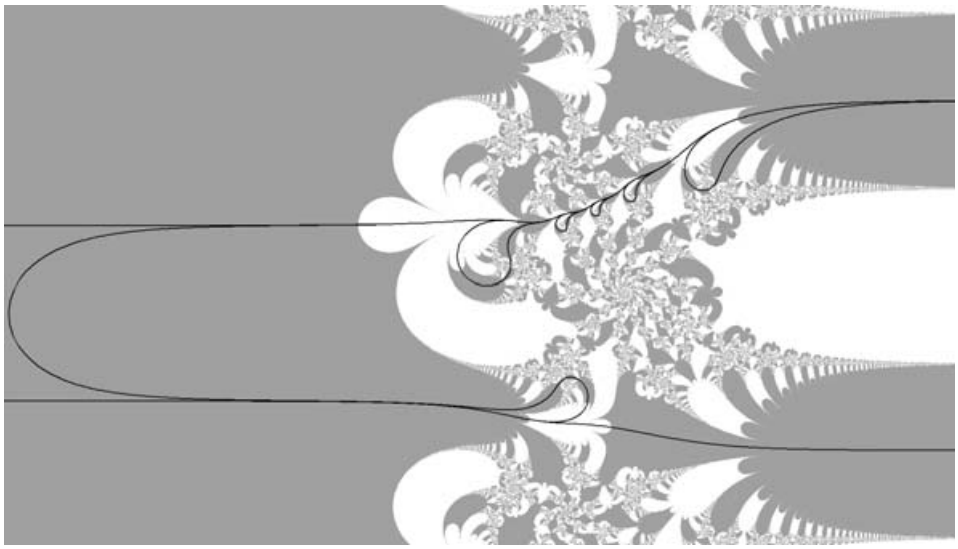
$$M := \max |s_k| \geq F^n(\pi + 2)/2\pi.$$

If $\kappa \in \mathbb{C}$ with $\kappa \in g_{\underline{s}}^\kappa$, then $|\kappa| \geq \frac{1}{5}F^{-(n-1)}(2\pi M)$.

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(a)



(b)

FIGURE 2. An illustration of the remark about broken rays after Lemma 4.7. Here $\kappa \in g_{\underline{s}}$, where $\underline{s}$ is the periodic address $\underline{s} = \overline{01}$. (a) A beginning piece of the curve $\widetilde{g_{\overline{10}}}$ obtained when approximating $g_{\overline{10}}$ from above. $\widetilde{g_{\overline{10}}}$ first traverses the ray $g_{\overline{10}}$ (the bottom curve in the picture), from right to left, then the upper curve (a preimage of the ray piece connecting $E_\kappa(\kappa)$ to $-\infty$) from left to right, followed by further preimages of this piece. (b) The analogous situation when approximating $g_{\overline{10}}$ from below.

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Similarly, suppose that $\underline{s}^1, \underline{s}^2 \in \mathcal{S}_0$ *are external addresses for which there is* $n \in \mathbb{N}$ *and* $M \geq F^n(\pi + 2)/2\pi$ *such that*

$$\max_{k+1 \leq m \leq k+n} |\underline{s}^j_m| \geq M$$

for all $k \geq 0$ *and* $j \in \{1, 2\}$. *If* κ *is a parameter such that* $g^{\kappa}_{\underline{s}^1}$ *and* $g^{\kappa}_{\underline{s}^2}$ *land together, then* $|\kappa| \geq \frac{1}{5} F^{-(n-1)}(2\pi M)$.

Proof. The first claim is an immediate consequence of the definition of Q and Y_Q. Note that, if $\underline{s}$ is an address such that $\kappa \in g_{\sigma(\underline{s})}$ or such that $g_{\underline{s}}$ lands at a point that does not have external address $\underline{s}$, then $(\underline{s}, t_{\underline{s}}) \notin Y_Q$.

Furthermore, among all addresses $\underline{s}$ one of whose entries $s_2, \ldots, s_{n+1}$ is of size at least M, the value of $t_{\underline{s}}$ is minimized by the address $\underline{s} = 00 \ldots 0M\bar{0}$ (where the first block of 0s consists of n entries). For this $\underline{s}$, we have $t_{\underline{s}} = F^{-n}(2\pi M)$. Thus the second and third statements follow from the first. ∎

5. *Limiting behavior of dynamic rays*

For completeness, this section collects some results from [**SZ1**] and [**FRS**] on the limiting behavior of dynamic rays. First we state and prove two lemmas that were implicitly contained in the proof of [**SZ1**, Corollary 6.9] and imply that a dynamic ray at a slow external address cannot land at an escaping point. From this, we deduce, as first outlined in [**FRS**], that every dynamic ray is a path-connected component of $I(E_\kappa)$.

LEMMA 5.1. (Limit set of ray) *Let* $g : (0, \infty) \to I(E_\kappa)$ *be an unbroken dynamic ray, and let*

$$L := \bigcap_{t>0} \overline{g((0, t))}$$

denote the limit set of g. *If there is* $(\underline{s}, t_0)$ *such that* $g_{\underline{s}}(t_0) \in L$, *then* $g_{\underline{s}}(t) \in L$ *whenever* $t \leq t_0$ *is such that* $g_{\underline{s}}(t)$ *is defined.*

Remark. If the ray $g_{\underline{s}}$ is broken, we could replace $g_{\underline{s}}$ in the last statement by the curve $\tilde{g}_{\underline{s}}$ from the remark after Lemma 4.7.

Proof. Let us define addresses $\underline{s}^{n\pm}$ by

$$s^{n\pm}_k = \begin{cases} s_k \pm 1, & k = n, \\ s_k, & \text{otherwise.} \end{cases}$$

One sees easily that $t_{\underline{s}^{n\pm}} \to t_{\underline{s}}$ for $n \to \infty$ (see Lemma 7.1). Now pick some t, $0 < t < t^0$. Then, by Lemma 4.7,

$$g_{\underline{s}^{n\pm}}([t, \infty)) \to g_{\underline{s}}([t, \infty))$$

uniformly. Therefore, any curve that does not intersect the $g_{\underline{s}^{n\pm}}$ and accumulates at $g_{\underline{s}}(t_0)$ must also accumulate at $g_s(t)$. ∎

LEMMA 5.2. (Addresses of rays landing together) *Suppose that for some* $\underline{s}, \underline{s}' \in \mathcal{S}_0$ *the dynamic rays* $g_{\underline{s}}$ *and* $g_{\underline{s}'}$ *land at the same point. Then* $|s_k - s'_k| \leq 1$ *for all* k.

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Proof. Assume, by contradiction, that $|s_k - s_k'| > 1$ for some k; by passing to a forward iterate if necessary we can assume that $k = 1$. Let S denote the union of $g_{\underline{s}}$, $g_{\underline{s}'}$ and their common landing point z_0, which is a Jordan arc tending to ∞ in both directions. Note that E_κ is injective on S. Indeed, E_κ is injective on every ray, and it is injective on $g_{\underline{s}} \cup g_{\underline{s}'}$ unless $\underline{s}$ and $\underline{s}'$ differ only in their first entries. However, in that case $g_{\underline{s}'}$ would be a translate of $g_{\underline{s}}$ by a multiple of $2\pi i$, which means that $g_{\underline{s}}$ and $g_{\underline{s}'}$ cannot land together. Finally, E_κ is injective on S as otherwise z_0 would lie on a dynamic ray, which contradicts Lemma 5.1.

On the other hand, if $|s_1 - s_1'| > 1$, then the two ends of S tend to ∞ with a difference of more than 2π in their imaginary parts, which implies that $S \cap (S + 2\pi i) \neq \emptyset$. Thus E_κ is not injective on S, a contradiction. ■

COROLLARY 5.3. (No landing at escaping points) *Suppose that $\underline{s} \in \mathcal{S}_0$ is a slow external address (i.e. $(\underline{s}, t_{\underline{s}}) \notin X$). Then $g_{\underline{s}}$ does not land at an escaping point.*

Proof. By Lemma 5.1, $g_{\underline{s}}$ could only land at the escaping endpoint $g_{\underline{s}'}(t_{\underline{s}'})$ for some fast address $\underline{s}'$. By Lemma 5.2, $|s_k - s_k'| \leq 1$ for all k. It easily follows that $\underline{s}$ would also have to be a fast address, which contradicts our assumption. (See Corollary 7.2.) ■

To infer that the path-connected components of $I(E_\kappa)$ are given by dynamic rays, we require the following topological fact.

PROPOSITION 5.4. (Path components [**FRS**, Proposition 4.2]) *Let I be a Hausdorff topological space. Let Γ be a partition of I into path-connected subsets such that no union of two different elements of I is path-connected.*

Suppose that I can be written as a countable union of closed subsets I_k such that:

(a) *$I_k \subset I_{k+1}$;*
(b) *every path-connected component of I_k is contained in some element of Γ;*
(c) *every element of Γ contains at most one path-connected component of I_k; and*
(d) *if $c \subset I$ is a simple closed curve that is completely contained in some element of Γ, then either $c \subset I_k$ or $c \cap I_k = \emptyset$.*

Then Γ is the set of path-connected components of I. □

COROLLARY 5.5. (Path components of $I(E_\kappa)$ [**FRS**, Corollary 4.3]) *Let $\kappa \in \mathbb{C}$. Then every path connected component of $I(E_\kappa)$ is:*

(a) *a dynamic ray;*
(b) *a dynamic ray together with its escaping endpoint; or*
(c) *(if $\kappa \in I(E_\kappa)$) an iterated preimage component of the dynamic ray containing the singular value.*

Proof. Set $I := I(E_\kappa)$, and let Γ be the set of curves of types (a)–(c). By Corollary 5.3, no union of two different elements of Γ is path-connected, and no element of Γ contains a simple closed curve.

We now set $I_0 := \mathfrak{g}(Y_Q \cap X)$ and $I_j := E_\kappa^{-j}(I_0)$. It is easy to see that these sets satisfy the hypotheses of Proposition 5.4, and the claim follows. ■

Little else is known about the possible limiting behavior of dynamic rays. For polynomials, dynamic rays cannot accumulate at escaping points. This is not the case for exponential maps: in [**DJ**] it was shown that, for $\kappa \in (-1, \infty)$, there exists a ray that

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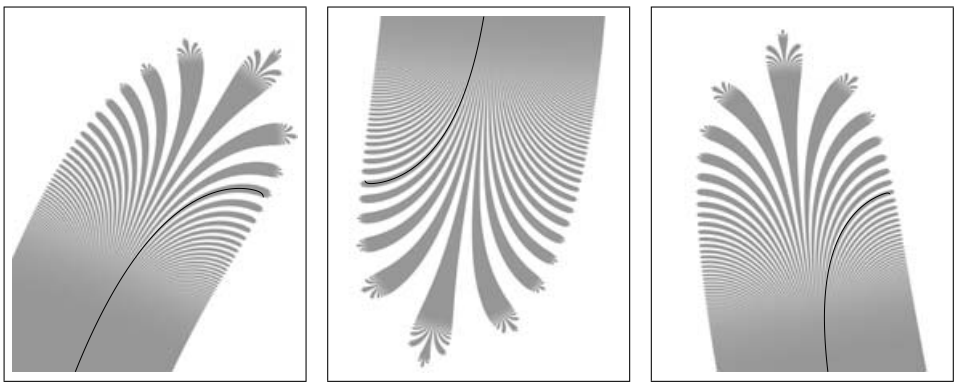


FIGURE 3. These pictures are intended to illustrate Theorem 6.2. Shown is the ray at address 01234... for the parameter $\kappa = -2$ in three successive magnifications. The images demonstrate that the ray indeed spirals around its escaping endpoint.

accumulates on itself. This was also shown for a larger class of exponential maps in [**R3**]; see also [**R1**, Section 3.8]. On the other hand, it is now known that every (pre-)periodic dynamic ray of an exponential map lands [**R2**].

6. *Differentiability of rays*

Viana [**V**] proved (using a different parametrization) that the rays $g_{\underline{s}}$ are C^∞. His arguments also apply to the parametrization of the curves given by our construction. (Compare also the proof of Theorem 6.2 below.)

THEOREM 6.1. (Rays are differentiable [**V**]) *Let $\underline{s} \in \mathcal{S}_0$. Then $g_{\underline{s}} : (0, \infty) \to \mathbb{C}$ is a C^∞ function.* □

A proof of the differentiability of rays can also be found in [**FS**], where this was carried out to obtain specific estimates on the first and second derivatives. However, previously there was no information about which rays with escaping endpoints are also differentiable in these endpoints, and whether this may depend on the parameter. Using the results of §4, we can answer this question (see Figure 3).

THEOREM 6.2. (Differentiability of rays in endpoints) *Let $\underline{s} \in \mathcal{S}_0$ be a fast external address, and let κ be a parameter for which the ray $g_{\underline{s}}$ is unbroken. Then the curve $g_{\underline{s}}([0, \infty))$ is continuously differentiable in $g_{\underline{s}}(0)$ if and only if the series*

$$\sum_{j=0}^{\infty} \frac{2\pi s_{j+1}}{T(\mathcal{F}^j(\underline{s}, t_{\underline{s}}))} \tag{6.1}$$

converges.

Remark. By the formulation 'the curve is continuously differentiable in $g_{\underline{s}}(0)$' we mean that it is continuously differentiable under a suitable parametrization (e.g. by arclength), *not* that the function $g_{\underline{s}}$ itself is necessarily differentiable in 0. If the convergence of the sum is absolute, then one can show that the function $g_{\underline{s}}$ itself is differentiable in 0.

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Proof. Let Q be as in equation (4.1). Then all $\mathfrak{g}_k$ are defined on the set Y_Q and converge uniformly to the function $\mathfrak{g}$ there; it is clearly sufficient to prove the theorem for addresses for which $(\underline{s}, t_{\underline{s}}) \in Y_Q \cap X$.

From the definition of the functions $\mathfrak{g}_k$, we can compute their t-derivatives in any point $(\underline{s}, t) \in Y_Q$ by repeatedly applying the chain rule:

$$\begin{aligned}\frac{\partial \mathfrak{g}_k}{\partial t}(\underline{s}, t) &= \frac{1}{\mathfrak{g}_{k-1}(\mathcal{F}(\underline{s}, t)) - \kappa} \cdot \frac{\partial \mathfrak{g}_{k-1}}{\partial t}(\mathcal{F}(\underline{s}, t)) \cdot \exp(t) = \cdots \\ &= \prod_{j=1}^{k} \frac{\exp(T(\mathcal{F}^{j-1}(\underline{s}, t)))}{\mathfrak{g}_{k-j}(\mathcal{F}^j(\underline{s}, t)) - \kappa} \\ &= \left(\prod_{j=1}^{k} \frac{\exp(T(\mathcal{F}^{j-1}(\underline{s}, t)))}{\mathfrak{g}(\mathcal{F}^j(\underline{s}, t)) - \kappa}\right) \cdot \prod_{j=1}^{k} \left(1 + \frac{\mathfrak{g}(\mathcal{F}^j(\underline{s}, t)) - \mathfrak{g}_{k-j}(\mathcal{F}^j(\underline{s}, t))}{\mathfrak{g}_{k-j}(\mathcal{F}^j(\underline{s}, t)) - \kappa}\right).\end{aligned}$$

Recall from the proof of Theorem 4.2 that

$$|\mathfrak{g}(\mathcal{F}^j(\underline{s}, t)) - \mathfrak{g}_{k-j}(\mathcal{F}^j(\underline{s}, t))| \leq 2^{-(k-j)} \cdot (2\pi + 4),$$

so the second product converges uniformly for $t \geq t_{\underline{s}}$. It is not difficult to see that the first product converges locally uniformly (and is non-zero) for $t > t_{\underline{s}}$ (see, e.g., [**FS**]). Note that this proves that the ray without the endpoint is C^1.

The ray is continuously differentiable in its endpoint if and only if $\arg((\partial \mathfrak{g}/\partial t)(\underline{s}, t))$ has a limit as $t \to t_{\underline{s}}$. The above argument shows that this is equivalent to the question of whether the function

$$\Theta(t) := \sum_{j=1}^{\infty} \arg(\mathfrak{g}(\mathcal{F}^j(\underline{s}, t)) - \kappa)$$

has a limit for $t \to t_{\underline{s}}$. Let us set $\arg(k, t) := \arg(\mathfrak{g}(\mathcal{F}^k(\underline{s}, t)) - \kappa) \in (-\pi, \pi)$ for all $k \geq 1$ and $t \geq t_{\underline{s}}$.

CLAIM. The limit $\lim_{t \to t_{\underline{s}}} \Theta(t)$ exists if and only if the series $\Theta(t_{\underline{s}})$ is convergent.

Proof. Choose some number $m \geq 3$ (to be fixed below) and define, for n large enough, $t_n > t_{\underline{s}}$ to be the unique number for which

$$T(\mathcal{F}^n(\underline{s}, t_n)) = T(\mathcal{F}^n(\underline{s}, t_{\underline{s}})) + \log m.$$

We will prove the claim by comparing the summands of $\Theta(t_n)$ with those of $\Theta(t_{\underline{s}})$. Note that $|\mathfrak{g}(\mathcal{F}^n(\underline{s}, t_n)) - \mathfrak{g}(\mathcal{F}^n(\underline{s}, t_{\underline{s}}))| \leq K := 2\pi + 4 + \log m$. It follows again by contraction that, for $k \leq n$,

$$|\mathfrak{g}(\mathcal{F}^k(\underline{s}, t_n)) - \mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}}))| \leq 2^{-(n-k)} \cdot K.$$

Thus

$$\left|\sum_{k-1}^{n} \arg(k, t_{\underline{s}}) - \sum_{k=1}^{n} \arg(k, t_n)\right| \leq \pi K \sum_{k=1}^{n} \frac{2^{-(n-k)}}{|\mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}})) - \kappa|},$$

which is easily seen to converge to 0 as $n \to \infty$.

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Also observe that, for $k \geq n+1$,

$$\begin{aligned} T(\mathcal{F}^k(\underline{s}, t_n)) &\geq F^{k-n-1}(T(\mathcal{F}^{n+1}(\underline{s}, t_n)) - T(\mathcal{F}^{n+1}(\underline{s}, t_{\underline{s}}))) \\ &= F^{k-n-1}((m-1)\cdot \exp(T(\mathcal{F}^n(\underline{s}, t_{\underline{s}})))) \\ &\geq F^{k-n-1}((m-1)\cdot F(T(\mathcal{F}^n(\underline{s}, t_{\underline{s}})))) \end{aligned}$$

and

$$2\pi s_{k+1} \leq F^{k-n-1}(F(T(\mathcal{F}^n(\underline{s}, t_{\underline{s}})))).$$

It easily follows that

$$\sum_{k=n+2}^{\infty} |\arg(k, t_n)| \to 0$$

as $n \to \infty$. Similarly, for large enough n, the value $|\arg(n+1, t_n)|$ is no larger than $2/(m-1)+\varepsilon$. If $\arg(n+1, t_{\underline{s}})$ tends to 0 as $n \to \infty$, then $\arg(n+1, t_n)$ also does.

Now let us first consider the case that $\arg(n, t_{\underline{s}})$ does not converge to 0 (and thus the sum $\Theta(t_{\underline{s}})$ is divergent). So let $\delta > 0$ and let n_k be a subsequence for which $\arg(n_k, t_{\underline{s}}) \geq \delta$. If m was chosen to be $1 + (5/\delta)$, then it follows from the above considerations that

$$|\Theta(t_{n_k-1}) - \Theta(t_{n_k})| \geq \delta - \frac{4}{m-1} + o(1) > \frac{\delta}{5} + o(1)$$

(as $k \to \infty$). In particular, the sequence $\Theta(t_n)$ does not have a limit for $n \to \infty$. This proves the claim in this case.

So we can now suppose that $\arg \mathfrak{g}(\mathcal{F}^j(\underline{s}, t_{\underline{s}})) \to 0$. Then, by our observations,

$$\left|\Theta(t_n) - \sum_{k=1}^{n} \arg(k, t_{\underline{s}})\right| \to 0.$$

Thus in particular the sequence $\Theta(t_n)$ has a limit if and only if the sum $\Theta(t_{\underline{s}})$ is convergent. It remains to show that this implies that Θ has a limit as $t \to t_{\underline{s}}$. However, it is easy to show that

$$\sup_{t \in [t_n, t_{n+1}]} |\Theta(t_n) - \Theta(t)| \to 0$$

as $n \to \infty$. Indeed, by the above observations,

$$\sum_{k=1}^{n-1} |\arg(k, t) - \arg(k, t_n)|$$

is small, as is

$$\sum_{k=n+2}^{\infty} |\arg(\mathfrak{g}(\mathcal{F}^k(\underline{s}, t)) - \kappa)|.$$

The two entries that remain to be dealt with tend to 0 because $\arg \mathfrak{g}(\mathcal{F}^j(\underline{s}, t_{\underline{s}}))$ does. This proves the claim in the second case.

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To conclude the proof, we need to show that the convergence of the sum $\Theta(t_{\underline{s}})$ is equivalent to the convergence of the sum (6.1) in the statement of the theorem. It is clear that the terms of the sum (6.1) converge to 0 if and only if those of $\Theta(t_{\underline{s}})$ do. So we can suppose that $\arg \mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}})) \to 0$. It is easy to see that then there exists $x > 0$ such that $|\mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}}))| \geq F^k(x)$ for all large enough k (see Corollary 7.2). Because of this and since $|\mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}})) - Z(\mathcal{F}^k(\underline{s}, t_{\underline{s}}))|$ is bounded by $2 + \pi$, we have

$$\left| \sum_k^{k_0} \arg(\mathfrak{g}(\mathcal{F}^k(\underline{s}, t_{\underline{s}})) - \kappa) - \sum_k^{k_0} \arg Z(\mathcal{F}^k(\underline{s}, t_{\underline{s}})) \right| \leq \pi \cdot (2 + \pi + |\kappa|) \cdot \sum_k^{k_0} \frac{1}{F^k(x)}.$$

The last sum is clearly absolutely convergent, and so the convergence of the sum $\Theta(t_{\underline{s}})$ and that of

$$\sum_k^{\infty} \arg Z(\mathcal{F}^k(\underline{s}, t_{\underline{s}}))$$

are equivalent. Similarly, one sees that the convergence of this last sum and the sum (6.1) are equivalent. ■

7. *Speed of escape*

Our construction provides information on the speed of escaping endpoints of dynamic rays, about which previously little was known. To deal with such questions, it is often useful to introduce a quantity $t^*_{\underline{s}}$ that is closely related to $t_{\underline{s}}$, but can be computed more easily.

DEFINITION AND LEMMA 7.1. (Growth of escaping endpoints) *Let $\underline{s}$ be an external address and define*

$$t^*_{\underline{s}} := \sup_{k \geq 1} F^{-k}(2\pi |s_{k+1}|).$$

Then $\underline{s}$ is exponentially bounded if and only if $t^*_{\underline{s}} < \infty$, in which case $t^*_{\underline{s}} \leq t_{\underline{s}} \leq t^*_{\underline{s}} + 1$.

Proof. Suppose first that $\underline{s}$ is exponentially bounded; i.e. $t_{\underline{s}} < \infty$. By definition of $t_{\underline{s}}$, we have $F(t_{\underline{s}}) \geq 2\pi |s_2|$ for all external addresses $\underline{s}$. Since $t_{\sigma(\underline{s})} \leq F(t_{\underline{s}})$, it follows inductively that $F^k(t_{\underline{s}}) \geq 2\pi |s_{k+1}|$. This proves that $t^*_{\underline{s}} \leq t_{\underline{s}} < \infty$.

Now suppose that $t^*_{\underline{s}} < \infty$. Then

$$T(\mathcal{F}(\underline{s}, t^*_{\underline{s}} + 1)) = F(t^*_{\underline{s}} + 1) - 2\pi |s_2| \geq 2F(t^*_{\underline{s}}) + 1 - 2\pi |s_2| \geq F(t^*_{\underline{s}}) + 1 \geq t^*_{\sigma(\underline{s})} + 1$$

(where we used the facts that $F(t + 1) \geq 2F(t) + 1$ and $2\pi |s_2| \leq F(t^*_{\underline{s}})$). It follows by induction that $(\underline{s}, t^*_{\underline{s}} + 1) \in \overline{X}$; i.e. $t_{\underline{s}} \leq t^*_{\underline{s}} + 1 < \infty$. ■

As a first application, we recover the characterization of exponentially bounded, slow and fast addresses given in [**SZ1**]. Also, we obtain a new description of addresses with *positive minimal potential* in the sense of [**SZ1**]. These are addresses $\underline{s} \in \mathcal{S}_0$ for which there exists $x > 0$ with $2\pi |s_n| \geq F^{n-1}(x)$ for infinitely many n.

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COROLLARY 7.2. (Properties of external addresses) *Let $\underline{s}$ be an external address. Then:*

- *$\underline{s}$ is exponentially bounded if and only if there exists $x \geq 0$ with $2\pi|s_k| \leq F^{k-1}(x)$ for all $k \geq 1$;*
- *$\underline{s}$ is slow if and only if there exist $x \geq 0$ and infinitely many $n \geq 0$ which satisfy $2\pi|s_{n+k}| \leq F^{k-1}(x)$ for all $k \geq 1$; and*
- *$\underline{s}$ has positive minimal potential if and only if there exists $x > 0$ with the property that $T(\mathcal{F}^j(\underline{s}, t_{\underline{s}})) = t_{\sigma^j(\underline{s})} > F^j(x)$ for all j.*

Proof. By the previous lemma, $\underline{s}$ is exponentially bounded if and only if $t^*_{\underline{s}} < \infty$, which is clearly equivalent to the stated condition. The other claims follow in a similar fashion. ■

We now prove Theorem 1.4, restated here for convenience.

THEOREM 1.4. (Arbitrary escape speed) *Let $\kappa \in \mathbb{C}$. Suppose that r_n is a sequence of positive real numbers such that $r_n \to \infty$ and $r_{n+1} \leq \exp(r_n) + c$ for some $c > 0$. Then there is an escaping endpoint $z \in I(E_\kappa)$ and some $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $|\operatorname{Re}(E_\kappa^{n-1}(z)) - r_n| \leq 2 + 2\pi$.*

Proof. By changing the first few entries of (r_n), if necessary, we may assume without loss of generality that $F(r_n + 1) > r_{n+1} + 1$. We define an external address $\underline{s}$ by $s_{n+1} := \lfloor F(r_n)/2\pi \rfloor$. Then

$$t_{\sigma^{n-1}(\underline{s})} \geq F^{-1}(2\pi|s_{n+1}|) \geq r_n - 2\pi$$

for all n. Furthermore

$$F^{-k}(2\pi|s_{n+k}|) \leq F^{-(k-1)}(r_{n+k-1}) \leq r_n + 1$$

for $k \geq 1$. By Lemma 7.1, we thus have

$$r_n - 2\pi \leq t_{\sigma^{n-1}(\underline{s})} \leq t^*_{\sigma^{n-1}(\underline{s})} + 1 \leq r_n + 2$$

for all n. The claim now follows by Lemma 4.1 and Theorem 4.2: we pick n_0 sufficiently large so that $\mathcal{F}^{n_0}(\underline{s}, t_{\underline{s}}) \in Y_Q$ and $z_{n_0} := \mathfrak{g}(\mathcal{F}^{n_0}(\underline{s}, t_{\underline{s}})) \neq \kappa$ and choose z to be any element of $E_\kappa^{-n_0}(z_{n_0})$. ■

Frequently, one considers escaping points that eventually escape in a sector (i.e. $\operatorname{Im} E_\kappa^n(z) \leq C \operatorname{Re} E_\kappa^n(z)$) or, more generally, a parabola (i.e. $\operatorname{Im} E_\kappa^n(z) \leq C(\operatorname{Re} E_\kappa^n(z))^K$). For example, McMullen [**McM**] showed that the set of escaping points satisfying a sector condition has Hausdorff dimension two. On the other hand, Karpinska [**Ka**] proved that the Hausdorff dimension of the set of points satisfying a parabola condition of exponent $K < 1$ is at most $1 + K$. Also, results of Hemke for a certain class of meromorphic functions imply in the exponential case that, if the singular orbit of E_κ satisfies a parabola condition, then the orbit of almost every point accumulates precisely on the postsingular set [**He**, Corollary 6.1]. As a final illustration of the use of our model in determining escape speed, we derive a combinatorial condition for this type of behavior. It is easy to see that we need only consider the case of escaping endpoints, since any other escaping point satisfies the parabola condition for all $K > 0$ (compare [**SZ1**, Proposition 4.5]).

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THEOREM 7.3. (Endpoints that satisfy a parabola condition) *Let $\underline{s} \in \mathcal{S}_0$ be a fast external address, and define*

$$b := \limsup_{n\to\infty} F^{-(n-1)}(2\pi|s_n|).$$

Set $t_n := T(\mathcal{F}^{n-1}(\underline{s}, t_{\underline{s}})) = t_{\sigma^{n-1}(\underline{s})}$. Then for every $K > 0$, the following two conditions are equivalent:

(a) *there are $C_1 > 0$ and $n_0 \in \mathbb{N}$ such that $2\pi|s_n| < C_1 \cdot t_n^K$ for $n \geq n_0$;*

(b) *there are $C_2 > 0$ and n_0 such that $2\pi|s_n| < C_2(F^{n-1}(b))^K$ for $n \geq n_0$.*

Remark. b is the *minimal potential* as defined in [**SZ1**].

Proof. Set $t_n^* := t^*_{\sigma^{n-1}(\underline{s})}$. Note that b is the limit of the decreasing sequence $F^{-(n-1)}(t_n^*)$. By Lemma 7.1, (a) is equivalent to the same statement with t_n replaced by t_n^*. In particular, (b) implies (a).

To prove the other direction, suppose that $2\pi|s_n| < C_3 \cdot (t_n^*)^K$ for $n \geq n_0$. Since $t_n^* \to \infty$, for sufficiently large n we will get

$$2\pi|s_n| < C_3 \cdot (t_n^*)^K < \left(\frac{t_n^*}{K+1} + 1\right)^{K+1} - 1,$$

and similarly

$$t_n^* < \left(\frac{t_n^*}{K+1} + 1\right)^{K+1} - 1.$$

It follows that

$$\begin{aligned} t_n^* &= F^{-1}(\max(t_{n+1}^*, 2\pi|s_{n+1}|)) \\ &< F^{-1}\left(\left(\frac{t_{n+1}^*}{K+1} + 1\right)^{K+1} - 1\right) = (K+1)F^{-1}\left(\frac{t_{n+1}^*}{K+1}\right). \end{aligned}$$

In other words, we have $t_n^*/(K+1) < F^{-1}(t_{n+1}^*/(K+1))$, which implies by induction that $t_n^* \leq (K+1) \cdot (F^{n-1}(b))^K$. In particular, $2\pi|s_n| < C_3 \cdot (K+1)^K \cdot (F^{n-1}(b))^K$. ■

8. *Canonical correspondence and escaping set rigidity*

The bijection $\mathfrak{g} : X \to I(E_\kappa)$ constructed in §4—while having certain continuity properties—is, in general, quite far from being a conjugacy. The question presents itself whether it is possible to construct a different map which is a conjugacy, or at least is continuous on a larger set. We will now show that this is not the case.

The underlying reason for this is the rigidity presented by the $2\pi i$-periodic structure of the dynamical plane: since the real and imaginary directions interact under an exponential maps, this rigid structure means that the escape speed of two points that correspond to each other under a conjugacy cannot dramatically differ. This idea is quite similar to that used by Douady and Goldberg [**DoG**] showing that for $\kappa_1, \kappa_2 \in (-1, \infty)$, the maps E_{κ_1} and E_{κ_2} are not conjugate on their Julia sets (and we obtain a generalization of their result in Theorem 8.5 below).

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THEOREM 8.1. (No non-trivial self-conjugacies) *Let $Q > 0$ and suppose that $f : Y_Q \cap X \to X$ is a continuous map with $f \circ \mathcal{F} = \mathcal{F} \circ f$ and*

$$f(\underline{r}, t) \in \{\underline{r}\} \times [0, \infty) \tag{8.1}$$

for all $\underline{r}$ and t. Then f is the identity.

More precisely, let $(\underline{s}, t_0) \in Y_Q \cap X$. Suppose that a function

$$f : Y_Q \cap \bigcup_{j \geq 0} \mathcal{F}^{-j}(\mathcal{F}(\underline{s}, t_0)) \to X$$

satisfies $f \circ \mathcal{F} = \mathcal{F} \circ f$ and equation (8.1). If $f(\underline{s}, t_0) \neq (\underline{s}, t_0)$, then f is not continuous in $(\underline{s}, t_0)$.

Proof. The first statement follows immediately from the second. So let $x = (\underline{s}, t_0)$ and f be as in the second part of the theorem, and suppose that $f(\underline{s}, t_0) = (\underline{s}, t_0')$ with $t_0 \neq t_0'$. If n is large enough, we can find $m(n) \in \mathbb{N}$ such that

$$T(\mathcal{F}^n(x)) \leq \log(2\pi m(n) + t_0 + 1) < T(\mathcal{F}^n(x)) + 1.$$

Let us define $y_n := (m(n)\, s_2 s_3 s_4 \ldots t_0)$. Now pull back the points y_n along the orbit of x. More precisely, let z_n be the uniquely defined point with address $s_1 s_2 \ldots s_n m(n) s_2 s_3 \ldots$ such that $\mathcal{F}^n(z_n) = y_n$.

By the choice of $m(n)$,

$$T(\mathcal{F}^{n-1}(x)) \leq T(\mathcal{F}^{n-1}(z_n)) < T(\mathcal{F}^{n-1}(x)) + 1,$$

and thus

$$T(\mathcal{F}^j(x)) \leq T(\mathcal{F}^j(z_n)) < T(\mathcal{F}^j(x)) + F^{-(n-j-1)}(1)$$

for every $j < n$. In particular, $z_n \in Y_Q$ and $z_n \to x$.

On the other hand, consider the image points $f(z_n)$. Since their nth images under $\mathcal{F}$ are given by $\mathcal{F}^n(f(z_n)) = f(y_n) = (m(n) s_2 s_3 \ldots t_0')$, we see that

$$|T(f(z_n)) - T(z_n)| < F^{-n}(|t_0' - t_0|) \to 0.$$

Thus $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} z_n = x \neq f(x) = f(\lim_{n\to\infty} z_n)$, and f is not continuous in x. ∎

COROLLARY 8.2. (No other conjugacies) *Suppose that E_{κ_1} and E_{κ_2} are exponential maps with non-escaping singular values that are conjugate on their sets of escaping points by a conjugacy f that sends each dynamic ray of E_{κ_1} to the corresponding dynamic ray of E_{κ_2}. Then $f = \mathfrak{g}^{\kappa_2} \circ (\mathfrak{g}^{\kappa_1})^{-1}$.*

Proof. Define $\Phi : I(E_{\kappa_1}) \to I(E_{\kappa_1})$ by $\Phi := f^{-1} \circ \mathfrak{g}^{\kappa_2} \circ (\mathfrak{g}^{\kappa_1})^{-1}$, and abbreviate $\mathfrak{g} := \mathfrak{g}^{\kappa_1}$. If Q is large enough, then Φ is continuous on $\mathfrak{g}^{\kappa_1}(Y_Q)$. Suppose that $z = \mathfrak{g}(\underline{s}, t_0) \in I(E_{\kappa_1})$ such that $\Phi(z) \neq z$. We may assume (by possibly exchanging κ_1 and κ_2 and passing to a forward image of z if necessary) that $z \in Y_Q$ and $T(\mathfrak{g}^{-1}(\Phi(z))) > t_0$. Then the function

$$f : \bigcup_{j \geq 0} \mathcal{F}^{-j}(\mathcal{F}(\underline{s}, t_0)) \cap Y_Q \to Y_Q, (\underline{r}, t) \mapsto \mathfrak{g}^{-1}(\Phi(\mathfrak{g}(\underline{r}, t)))$$

is continuous, contradicting Theorem 8.1. ∎

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The condition of every dynamic ray being sent to the corresponding ray in the limit dynamics should be satisfied by every 'reasonable' conjugacy (up to a possible relabeling of the combinatorics). A natural condition placed upon a topological conjugacy between two functions on an open set is that it preserves orientation. In our setting of conjugacies on the sets of escaping points, we replace this condition by a notion of 'order-preserving' conjugacies.

The collection of dynamic rays is endowed with a natural vertical order: of any two dynamic rays, one is *above* the other. More precisely, define $\mathcal{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}$. If $g_{\underline{s}}$ is a dynamic ray and R is large enough, then the set $\mathcal{H}_R \setminus g_{\underline{s}}([1, \infty))$ has exactly two unbounded components, one above and one below $g_{\underline{s}}$, and any other dynamic ray must tend to ∞ within one of these. It follows immediately from the construction of dynamic rays that this order coincides precisely with the lexicographic order of their external addresses.

We now call a continuous function from some subset of X to X *order-preserving* if it induces an order-preserving map on external addresses. Similarly, if $\kappa_1, \kappa_2 \in \mathbb{C}$ and $f : I(E_{\kappa_1}) \to I(E_{\kappa_2})$ is continuous, then we call f order-preserving if it preserves the vertical (and thus the lexicographic) order of dynamic rays. Any orientation-preserving conjugacy of two exponential maps induces an order-preserving map on their sets of escaping points.

There are only a few order-preserving self-conjugacies of the shift map on external addresses. The simple proof of the following fact is left to the reader.

LEMMA 8.3. (Self-conjugacies of the shift) *Let f be an order-preserving homeomorphism of the space of external addresses such that $f \circ \sigma = \sigma \circ f$. Then there exists $j \in \mathbb{Z}$ such that*

$$f(s_1 s_2 s_3 \dots) = (s_1 + j)(s_2 + j)(s_3 + j) \dots$$

for all external addresses $\underline{s} = s_1 s_2 s_3 \dots$. □

Using Corollary 8.2, we can now describe all possible order-preserving conjugacies between the escaping dynamics of two exponential maps.

COROLLARY 8.4. (Order-preserving conjugacies) *Suppose that f is an order-preserving conjugacy between two maps E_{κ_1} and E_{κ_2} (with non-escaping singular orbit) on their escaping sets. Then there exists a parameter $\kappa_2' = \kappa_2 + 2\pi i k$ (with $k \in \mathbb{Z}$) such that*

$$\mathfrak{g}^{\kappa_2'} \circ (\mathfrak{g}^{\kappa_1})^{-1} : I(E_{\kappa_1}) \to I(E_{\kappa_2'})$$

is a conjugacy.

Proof. The map f induces an order-preserving self-conjugacy of the shift. By Lemma 8.3 this map consists of shifting all labels by some number k. Let $\kappa_2' := \kappa_2 - 2\pi i k$. The maps E_{κ_2} and $E_{\kappa_2 - 2\pi i k}$ are conjugate by the map $z \mapsto z - 2\pi i k$, and the induced self-conjugacy of the shift consists of shifting all labels by $-k$. Thus the map $f' : z \mapsto f(z) - 2\pi i k$ is a conjugacy between E_{κ_1} and E_{κ_2}, and preserves dynamic rays. The claim follows by Corollary 8.2. ■

The next theorem is a generalization of the previously mentioned result of Douady and Goldberg [**DoG**].

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THEOREM 8.5. (No conjugacy for escaping parameters) *Let* $\underline{s} \in \mathcal{S}_0$ *and let* $(\underline{s}, t_1)$, $(\underline{s}, t_2) \in X$ *with* $t_1 \neq t_2$. *Suppose that* κ_1 *and* κ_2 *are parameters such that* $\mathfrak{g}^{\kappa_1}(\underline{s}, t_1) = \kappa_1$ *and* $\mathfrak{g}^{\kappa_2}(\underline{s}, t_2) = \kappa_2$. *Then* E_{κ_1} *and* E_{κ_2} *are not conjugate on* $\mathbb{C}$.

Proof. By contradiction, let $f : \mathbb{C} \to \mathbb{C}$ be a conjugacy between E_{κ_1} and E_{κ_2}. For some $Q > 0$, the map

$$\alpha : Y_Q \to X; (\underline{s}, t) \mapsto (\mathfrak{g}^{\kappa_2})^{-1}(f(\mathfrak{g}^{\kappa_1}(\underline{s}, t)))$$

is defined. This map is either order-preserving or order-reversing, depending on whether f is orientation-preserving or -reversing. Also, $\alpha \neq \mathrm{id}$ because $\alpha(\mathcal{F}^n(\underline{s}, t_1)) = \mathcal{F}^n(\underline{s}, t_2) \neq \mathcal{F}^n(\underline{s}, t_1)$.

Suppose first that α is order-preserving. Since f must map $g^{\kappa_1}_{\sigma^n(\underline{s})}$ to $g^{\kappa_2}_{\sigma^n(\underline{s})}$, it follows by Lemma 8.3 that α preserves external addresses (i.e. satisfies equation (8.1)). As in the proof of Corollary 8.2, this contradicts Theorem 8.1.

Now suppose that α is order-reversing. Since f maps postsingular rays of E_{κ_1} to those of E_{κ_2}, this is possible only when $\underline{s}$ is periodic of period 1. We may assume without loss of generality that $\underline{s} = \overline{0}$, in which case we can replace α in the above argument by the order-preserving map $\tilde{\alpha}(s_1 s_2 \dots) := \alpha((-s_1)(-s_2)\dots)$. ■

We are now in a position to extend Proposition 2.1 to a larger class of examples.

THEOREM 8.6. (Rigidity for parameters with different combinatorics) *Suppose that* $\underline{s} \in \mathcal{S}_0$ *and* $\tilde{\kappa}$ *is a non-escaping parameter for which the singular value is contained in the limit set of* $g^{\tilde{\kappa}}_{\underline{s}}$. *Suppose furthermore that* κ *is another non-escaping parameter such that the limit set of* $g^{\kappa}_{\underline{s}}$ *does not contain the singular value and is bounded. Then* $\mathfrak{g}^{\tilde{\kappa}} \circ (\mathfrak{g}^{\kappa})^{-1}$ *is not continuous.*

Proof. Suppose by contradiction that $\mathfrak{g} := \mathfrak{g}^{\tilde{\kappa}} \circ (\mathfrak{g}^{\kappa})^{-1}$ is continuous. As in the proof of Proposition 2.1, pick an arbitrary point $w \in I(E_\kappa)$ and a neighborhood U of w whose image under $\mathfrak{g}$ is bounded.

Let A denote the accumulation set of $g^{\kappa}_{\underline{s}}$. We shall show that there is an iterated preimage of A which is contained in U. The conclusion then follows in the same way as in Proposition 2.1.

First note that every component of the preimage of A is compact. Indeed, otherwise there is no continuous branch of E_κ^{-1} on A, which means that A separates κ from ∞. However, this is impossible: if $\kappa \in J(E_\kappa)$ then there must be escaping points close to κ, which are connected to ∞ by a dynamic ray, and if $\kappa \in F(E_\kappa)$ then it is easy to see that there is a curve in the Fatou set that connects κ to ∞ (see [**S2**] or §9). All preimages of A are translates of each other; let K denote the diameter of any of these preimages. Choose n sufficiently large and let $V \subset U$ be a small neighborhood of w that is mapped biholomorphically to $\mathbb{D}_{2\pi+1}(E^n_\kappa(w))$ by E^n_κ. (The existence of such a V is easily shown using a pullback argument.) Choose among the preimages of A one, call it A_0, which satisfies

$$|E^{n+1}_\kappa(w) - \kappa| \leq |z - \kappa| \leq |E^{n+1}_\kappa(w) - \kappa| + 2\pi + K$$

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and for all $z \in A_0$. If n was chosen large enough, then $|E_\kappa^{n+1}(w) - \kappa| > 2\pi + K$ and thus

$$\log|z-\kappa| - \log|E_\kappa^{n+1}(w)-\kappa| \le \log\left(1 + \frac{2\pi+K}{|E_\kappa^{n+1}(w)-\kappa|}\right) \le \frac{2\pi+K}{|E_\kappa^{n+1}(w)-\kappa|} < 1.$$

Thus if we take the pullback A_1 of A_0 by the same branch of E_κ^{-1} that carries $E_\kappa^{n+1}(w)$ to $E_\kappa^n(w)$, then $A_1 \subset \mathbb{D}_{2\pi+1}(E_\kappa^n(w))$. We can then further pull back A_1 to $V \subset U$, which concludes the proof. ■

The preceding result, together with theorems on the combinatorial rigidity of escaping and Misiurewicz parameters [**FS, FRS, HSS**], easily implies the following statement on the rigidity of the escaping dynamics of such parameters.

COROLLARY 8.7. (Rigidity for escaping and Misiurewicz parameters) *Suppose that $\kappa_1 \neq \kappa_2$ are attracting, parabolic, Misiurewicz or escaping parameters, at least one of which is not attracting or parabolic. Suppose that $\operatorname{Im}\kappa_1, \operatorname{Im}\kappa_2 \in (-\pi, \pi]$. Then E_{κ_1} and E_{κ_2} are not conjugate on their sets of escaping points by an order-preserving conjugacy.*

Proof. Clearly an escaping parameter cannot be conjugate to a non-escaping parameter. So let us first suppose that κ_1 and κ_2 are escaping parameters, and that the singular values lie on the rays at external addresses $\underline{s}^1$ and $\underline{s}^2$. Then both addresses have first entry 0 by [**FRS**, Corollary 1]. Since the conjugacy must map the singular value of E_{κ_1} to that of E_{κ_2}, it follows by Lemma 8.3 that $\underline{s}^1 = \underline{s}^2$. As in the proof of Theorem 8.5, their potentials are equal as well. However, this contradicts the fact that for every $x \in X$ there exists only one parameter κ with $\mathfrak{g}^\kappa(x) = \kappa$ [**FRS**, Theorem 2].

Now suppose that both κ_1 and κ_2 are Misiurewicz. Assume that the preperiod of κ_1 is smaller or equal to that of κ_2. By [**SZ2**, Theorem 4.3], there exists a preperiodic address $\underline{s}$ such that $g_{\underline{s}}^{\kappa_1}$ lands at κ_1. By [**SZ2**, Theorem 3.2], all periodic rays of E^{κ_2} land at periodic points. As the preperiod of κ_2 is greater or equal to that of κ_1, this implies that $g_{\underline{s}}^{\kappa_2}$ lands at a preperiodic point. By the results of [**HSS**], Misiurewicz parameters with given combinatorics are unique, so this landing point is $\neq \kappa_2$. Thus we can apply Theorem 8.6. The same argument works (without reference to [**HSS**]) if κ_2 is parabolic or attracting. ■

It seems reasonable to conjecture that the escaping dynamics of exponential maps whose singular value lies in the Julia set is always rigid. This conjecture would imply density of hyperbolicity: a non-hyperbolic stable parameter would be (quasiconformally) conjugate to all nearby parameters, and in particular the maps would be conjugate on their sets of escaping points.

CONJECTURE 8.8. (Escaping set rigidity) *Suppose that κ_1 is a parameter with $\kappa_1 \in J(E_{\kappa_1})$, and let $\kappa_2 \notin \{\kappa_1 + 2\pi i k\}$. Then there exists no order-preserving conjugacy*

$$f : I(E_{\kappa_1}) \to I(E_{\kappa_2})$$

between E_{κ_1} and E_{κ_2}.

9. *Topology of the Julia set for attracting and parabolic parameters*

We will now completely describe the Julia sets of attracting and parabolic exponential maps (and the dynamics thereon) as a quotient of our model $\overline{X}$. In particular, any two attracting

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exponential maps are conjugate on their sets of escaping points. We will give the complete construction for attracting parameters, and remark on the parabolic case later.

So let $\kappa \in \mathbb{C}$ such that E_κ has an attracting cycle $a_0 \mapsto a_1 \mapsto \cdots \mapsto a_n = a_0$ and corresponding Fatou components $A_0 \mapsto A_1 \mapsto \cdots \mapsto A_n = A_0$. This cycle of immediate attracting basins must contain the singular value [**B**, Theorem 7]; let us choose our labeling in such a way that $\kappa \in A_1$. By the Koenigs linearization theorem [**M**, Theorem 8.2], we can find open Jordan neighborhoods V_j of a_j such that $\kappa \in V_1$, $E_\kappa(V_j) \Subset V_{j+1}$† and $E_\kappa(V_n) \Subset V_1$. For $j = 0, \ldots, n-1$, let U_j denote the component of $E_\kappa^{-1}(V_{j+1})$ containing a_j. Since $\kappa \in U_1$, the component U_0 contains a left halfplane; the other U_j are bounded Jordan domains. We consider the set

$$W := \mathbb{C} \setminus \left(\bigcup_{i=0}^{n-1} \overline{U_i} \right).$$

Then $\overline{E_\kappa^{-1}(W)} \subset W$, and $E_\kappa : E_\kappa^{-1}(W) \to W$ is a covering map.

From now on let us suppose that $n \geq 2$. The minor modifications necessary in the case $n = 1$ are straightforward and are left to the reader.

We can connect κ and $E^n(\kappa)$ by a curve in U_1 (e.g. by a straight line in linearizing coordinates). Pulling this curve back under E_κ^n, we obtain a curve $\gamma \subset A_1$ that connects κ to ∞. Define $V := \mathbb{C} \setminus \gamma$. The set $E_\kappa^{-1}(V)$ then consists of countably many strips bounded by two preimages of γ. Let us label these strips as R_k in such a way that $t + (2k+1)\pi i \in R_k$ for large enough t, and let

$$\tilde{L}_k : V \to R_k$$

denote the corresponding branch of E_κ^{-1}. Observe that these differ from the branches L_k considered in §4. Note that $\tilde{L}_k$ is well-defined everywhere on the Julia set.

If $z \in J(E_\kappa)$, we can associate to z an *itinerary* $\mathrm{itin}(z) := \mathrm{u}_1\mathrm{u}_2\mathrm{u}_3 \ldots$ such that

$$E_\kappa^{n-1}(z) \in R_{\mathrm{u}_n}$$

for all $n \geq 1$. If $\underline{s} \in \mathcal{S}_0$, then all points in $g_{\underline{s}}$ clearly have the same itinerary, which is also denoted by $\mathrm{itin}(\underline{s})$. (This itinerary can be defined in a purely combinatorial way by associating an 'intermediate external address' to the curve γ; see [**S2, SZ2**] or, for a general approach, [**R1**, §3.7].)

Choose some $A > 0$ and $B < 0$ such that the map $\mathfrak{g} = \mathfrak{g}^\kappa : X \to I(E_\kappa)$ satisfies $|\operatorname{Re} \mathfrak{g}(\underline{s}, t) - t| < 2$ on Y_A (as in Theorem 4.2) and such that

$$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Re} z > A - 2\} \subset W \subset \{z \in \mathbb{C} : \operatorname{Re} z > B\}.$$

Now let us define functions $H_k : \overline{X} \to J(E_\kappa)$ by

$$H_0(\underline{s}, t) := \mathfrak{g}(\underline{s}, t + A) \quad \text{and} \quad H_{k+1}(\underline{s}, t) := \tilde{L}_{\mathrm{u}_1}(H_k(\mathcal{F}(\underline{s}, t))),$$

where u_1 is the first entry of $\mathrm{itin}(\underline{s})$. Note that $H_k(\underline{s}, t)$ always lies on the dynamic ray $g_{\underline{s}}$.

† The notation $U \Subset V$ means, as usual, that $\overline{U}$ is a compact set contained in V.

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THEOREM 9.1. (Conjugacy for attracting parameters) *In the hyperbolic metric of W, the functions H_k converge uniformly to a continuous, surjective function $H : \overline{X} \to J(E_\kappa)$ with $H \circ \mathcal{F} = E_\kappa \circ H$. Furthermore, $H|_X : X \to I(E_\kappa)$ is a conjugacy.*

Remark. By Corollary 8.2, $H|_X$ must be equal to $\mathfrak{g}$. In particular, Theorem 9.1 implies Theorem 1.2.

Proof. Let us denote the hyperbolic metric of any hyperbolic domain $U \subset \mathbb{C}$ by $ds = \rho_U|dz|$. Since $E_\kappa : E_\kappa^{-1}(W) \to W$ is a covering map, it expands the hyperbolic metric of W. In fact, there exists $K > 1$ such that $\|DE_\kappa(z)\|_{\text{hyp}} \geq K$ for all $z \in E_\kappa^{-1}(W)$.

To prove this, set $W' := E_\kappa^{-1}(W)$, and let

$$\tilde{W} := \{z \in W : z + 2\pi ik \in W \text{ for all } k \in \mathbb{Z}\}.$$

(Thus $\tilde{W}$ is obtained from W by removing all translates of the sets $\overline{U_i}$.)

Then $W' \subset \tilde{W} \subset W$. As $\rho_{\tilde{W}} \geq \rho_W$ by monotonicity of the hyperbolic metric, it is sufficient to show that, for every $z \in W'$,

$$q(z) := \frac{\rho_{W'}(z)}{\rho_{\tilde{W}}(z)} \geq K > 1. \tag{9.1}$$

Recall that U_0 contains a left halfplane, so that, for some $R_0 > 0$, the set $\mathbb{C} \setminus W'$ contains the curves $\{R + (2k+1)\pi i : R \geq R_0\}$. By standard estimates on the hyperbolic metric, it follows that $\rho_{W'}(z)$ is bounded from below as $\operatorname{Re} z \to +\infty$. On the other hand, $\tilde{W}$ contains the right halfplane $\mathcal{H}$, so $\rho_{\tilde{W}}(z) \to 0$, and thus $q(z) \to \infty$, as $\operatorname{Re} z \to +\infty$. Since $\overline{W'} \subset \tilde{W}$ and $\tilde{W}$ is bounded to the left, it follows that

$$K := \inf_{\substack{z \in W': \\ |\operatorname{Im} z| \leq \pi}} q(z) > 1.$$

The expression $q(z)$ is $2\pi i$-periodic, so equation (9.1) follows.

For an arbitrary $(\underline{s}, t) \in X$, consider the two points $z_1 := E_\kappa(H_0(\underline{s}, t)) = \mathfrak{g}(\mathcal{F}(\underline{s}, t + A))$ and $z_2 := E_\kappa(H_1(\underline{s}, t)) = H_0(\mathcal{F}(\underline{s}, t))$. Both points have real parts greater than $A - 2$ and thus can be connected by a straight line g_0 in W. Note that g_0 is homotopic (in W) to the piece of the ray $g_{\underline{s}}$ between z_1 and z_2, as this piece is also contained in the halfplane $\mathcal{H}$. Thus we can pull back g_0 and obtain a curve g_1 between $H_0(\underline{s}, t)$ and $H_1(\underline{s}, t)$. We claim that the (euclidean) length of g_1 is uniformly bounded (independent of $\underline{s}$ and t).

To prove this claim, recall that

$$\operatorname{Re}(z_1) \leq T(\mathcal{F}(\underline{s}, t + A)) + 2 = \exp(t + A) - 2\pi|s_2| + 1$$

and

$$\operatorname{Re}(z_2) \geq T(\mathcal{F}(\underline{s}, t)) - 2 = \exp(t) - 2\pi|s_2| - 3.$$

It follows that the euclidean length of g_0 satisfies

$$\ell(g_0) \leq \exp(t + A) - \exp(t) + 4 + 2\pi = O(\exp(t)).$$

Because all points of g_0 have an absolute value of at least $|z_2| - 2\pi \geq \exp(t)/\sqrt{2} - 2 - 3\pi$, we see that

$$\ell(g_1) \leq \frac{1}{|z_2| - 2\pi} \cdot \ell(g_0) = O(1).$$

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Since $\rho_W(z) \to 0$ as $\operatorname{Re} z \to \infty$, the function ρ_W is uniformly bounded on W'. Thus the hyperbolic length of g_1 in the hyperbolic metric of W is also bounded by some constant C. Now, taking pullbacks inductively, we see that the hyperbolic distance between $H_k(\underline{s}, t)$ and $H_{k+1}(\underline{s}, t)$ is bounded by C/K^k. Thus the H_k converge uniformly. The functional equation $H \circ \mathcal{F} = E_\kappa \circ H$ is satisfied by construction.

To show surjectivity of H, it is sufficient to see that $H(\overline{X})$ is dense in J (note that, because the hyperbolic distance between $H(x)$ and $H_0(x)$ is uniformly bounded, H is again continuous as a map $\overline{X} \cup \{\infty\} \to J \cup \{\infty\}$). However, density of the image is trivial because $E_\kappa^{-1}(H(\overline{X})) \subset H(\overline{X})$, and backward orbits of any point (except κ) accumulate on the entire Julia set. Injectivity of H on X follows by the same argument as before. ■

The following immediate corollary was previously proved in [**BDD**] (with somewhat different notation).

PROPOSITION 9.2. (Dynamic rays landing at a common point) *Let κ be an attracting parameter. Then every non-escaping point in $J(E_\kappa)$ is the landing point of at least one dynamic ray. Two dynamic rays land at the same point if and only if they have identical itineraries.*

Proof. The only thing we still need to check is that, whenever $\operatorname{itin}(\underline{s}) = \operatorname{itin}(\underline{s}')$, then $H(\underline{s}, t_{\underline{s}}) = H(\underline{s}', t_{\underline{s}'})$. However, the strips R_k have height 2π, so the nth entries of $\underline{s}$ and $\underline{s}'$, for any n, differ by at most 1. It follows easily (for example by Lemma 7.1) that $|t_{\sigma^n(\underline{s})} - t_{\sigma^n(\underline{s}')}|$ is bounded independently of n. Thus the distance between the points $H_0(\sigma^n(\underline{s}), t_{\sigma^n(\underline{s})})$ and $H_0(\sigma^n(\underline{s}'), t_{\sigma^n(\underline{s}')})$ is uniformly bounded as well. The claim now follows by the contraction argument from the previous proof. ■

Note that the proof of Proposition 9.2 for periodic addresses is much easier, compare [**SZ2**, Proposition 4.5].

Another direct consequence of Theorem 9.1 is the following result, which describes an abstract model of the Julia set of an attracting exponential map, in analogy to the 'Pinched Disk Model' for polynomials [**Do**].

COROLLARY 9.3. ('Pinched Cantor bouquet') *Let κ be an attracting parameter. Form the quotient $\tilde{X}$ of X by identifying all points $(\underline{s}, t_{\underline{s}})$ and $(\underline{s}', t_{\underline{s}'})$ for which* $\operatorname{itin}(\underline{s}) = \operatorname{itin}(\underline{s}')$. *Then $\mathcal{F}$ projects to a map $\mathcal{F} : \tilde{X} \to \tilde{X}$ that is conjugate to $E_\kappa : J(E_\kappa) \to J(E_\kappa)$.* ■

Remark. All the preceding theorems remain true for parabolic parameters. The issue is to find a replacement for the strict hyperbolic contraction used in the proof of Theorem 9.1. This issue is the same as that which appears in the proof of local connectivity for quadratic polynomials with a parabolic orbit—see [**DH**, Exposé 10] or [**LY**]—and can be dealt with in a similar manner. Those arguments, however, are somewhat technical and hardly very enlightening; it seems to us that there is little to be gained by their detailed adoption to the exponential case. Furthermore, in recent work by Haissinsky [**Ha**], parabolic rational

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maps were constructed from hyperbolic maps by using Guy David's transquasiconformal surgery. In particular, the resulting parabolic map is topologically conjugate to the hyperbolic function it originated from. Such methods should generalize to the space of exponential maps and thus yield a natural proof of the conjugacy of parabolic exponential maps to attracting exponential maps with the same combinatorics (i.e. with the same *intermediate external address* [**S2**]). In view of these facts, we have decided against a presentation of rigorous proofs of the above theorems in the parabolic case.

10. *Invalidity of renormalization*

Suppose that E_κ is any attracting exponential map of period $n > 1$ and let μ be the multiplier of its attracting orbit. As in the previous section, label the cycle of immediate basins, $A_0 \mapsto A_1 \mapsto \cdots \mapsto A_n = A_0$ in such a way that A_0 contains a left half plane. $E_\kappa^n|_{A_0}$ is conformally conjugate to $E_{\kappa_0}|_{F(E_{\kappa_0})}$ where κ_0 is such that E_{κ_0} has an attracting fixed point with multiplier μ (in fact, $\kappa_0 = \log\mu - \mu$). This can be proved either by constructing the conjugacy directly using the linearizing coordinates of E_κ and E_{κ_0}, or by conjugating these maps to a normal form as in [**S1**, §III.4] or [**DeG**]. Let

$$\Psi : A_0 \to F(E_{\kappa_0})$$

be this conjugacy, and note that $\Psi(z + 2\pi i) = \Psi(z) + 2\pi i$. We will now prove Theorem 1.3, i.e. that this map does not extend continuously to $\overline{A_0}$. The reason, as in the argument from Theorem 8.1, is that the $2\pi i$-periodic structure of the dynamical plane must be preserved under a conjugacy, which makes it impossible to conjugate E_{κ_0} to the much faster growing function E_κ^n. Combinatorially speaking, this means that there are dynamic rays on ∂A_0 that would be mapped to points with an external address that is not exponentially bounded, which is clearly impossible (compare the combinatorial tuning formula in [**RS2**]). However, our proof does not use these combinatorial notions and is, in fact, completely elementary.

THEOREM 10.1. (No topological renormalization) *Let E_κ and Ψ be as above. Then Ψ does not have a continuous extension to ∂A_0.*

Proof. Assume, by contradiction, that Ψ does extend continuously to ∂A_0. The idea of the proof is the following: orbits of points in $\overline{A_0}$ under E_κ^n (with bounded imaginary parts) grow essentially as iterates of F^n. So for any large enough K and k, we can find a point z_0 with real part around K whose imaginary part under E_κ^{kn} is about $F^{kn}(K)$. The orbit of $\Psi(z_0)$, on the other hand, can grow only like F under iteration of E_{κ_0}, which leads to a contradiction because $\Psi(E_\kappa^{kn}(z_0)) = E_{\kappa_0}^k(\Psi(z_0))$ must also have imaginary part roughly $F^{kn}(K)$. In the following, we fix the details of the proof.

Cut the plane into the strips R_k from the previous section; recall that these strips have bounded imaginary parts and the strip boundaries lie in A_0. We may assume that the R_k are numbered so that R_0 contains 0. Let

$$R := R_0 \cup \bigcup_{1 \le j \le n-1} R_{\mathrm{u}_j},$$

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where R_{u_j} is the strip containing A_j. Then the orbit of any point $z \in \overline{A_0}$ lies in $\overline{A_0} \cup R$. Define $A := \max_{z \in \overline{R}} |\mathrm{Im}(z)|$ and $B := \max_{t \in [-3\pi, -\pi]} |\mathrm{Im}\, \Psi(ti)|$. Note that both quantities are at least π.

Choose $K > B$ large enough such that, whenever $|E_\kappa(z)| \geq K$, then

$$|E_\kappa(z)| - A - 1 \geq F(\mathrm{Re}(z) - 1) \quad \text{and} \tag{10.1}$$

$$|E_\kappa(z)| + A + 1 \leq F(\mathrm{Re}(z) + 1). \tag{10.2}$$

Let

$$M := \max\{|\Psi(z)| : z \in \overline{A_0 \cap R_0} \text{ and } \mathrm{Re}(z) \in [K - 1, K + 1]\};$$

by enlarging M, if necessary, we can also assume that $\exp(t) + |\kappa_0| + 1 \leq F(t + 1)$ for all $t \geq M$. Note that this implies $|E^j_{\kappa_0}(z)| < F^j(M + 1)$ for all j and all z with $\mathrm{Re}(z) \leq M$. Finally, choose k so large that $F^k(K - B) > M + 1$.

Pick any point $z_1 \in \overline{A_0}$ with $\mathrm{Re}(z_1) = 0$ and

$$\mathrm{Im}(z_1) \in [F^{kn}(K) - \pi, F^{kn}(K) + \pi].$$

By repeatedly pulling back the point z_1 under E_κ^{-n}, we obtain a point $z_0 \in \overline{R_0 \cap A_0}$ with $E^{kn}_\kappa(z_0) = z_1$ and $E^{jn}_\kappa(z_0) \in \overline{R_0}$ for $j < k$. By (10.1), we see that

$$\mathrm{Re}(E^{kn-1}_\kappa(z_0)) - 1 \leq F^{-1}(|z_1| - A - 1) \leq F^{kn-1}(K)$$

and, similarly, by (10.2), $\mathrm{Re}(E^{kn-1}_\kappa(z_0)) + 1 \geq F^{kn-1}(K)$. In particular,

$$\mathrm{Re}(E^{kn-1}_\kappa(z_0)) \geq K.$$

Repeating this argument inductively, it follows that $\mathrm{Re}(z_0) \in [K - 1, K + 1]$.

As $\Psi(z + 2\pi i) = \Psi(z) + 2\pi i$, we can estimate that $\mathrm{Im}(\Psi(z_1)) \geq F^{kn}(K) - B$. On the other hand, $\mathrm{Re}(\Psi(z_0)) \leq M$, and thus

$$|\Psi(z_1)| = |E^k_{\kappa_0}(\Psi(z_0))| < F^k(M + 1) < F^{2k}(K - B) \leq F^{kn}(K) - B.$$

This is a contradiction. ∎

11. *Parameter space*

In [**FS**], it was shown that the parameters for which the singular value lies on a dynamic ray (but is not the endpoint of a ray) are organized in *parameter rays*.

PROPOSITION 11.1. (Classification of parameters on rays [**FS**]) *For every $\underline{s} \in \mathcal{S}_0$ and every $t > t_{\underline{s}}$, there exists a unique parameter $\kappa = \mathcal{G}(\underline{s}, t)$ such that $\mathfrak{g}^\kappa(\underline{s}, t) = \kappa$. For any fixed $\underline{s}$, the value $\mathcal{G}(\underline{s}, t)$ depends continuously on t and satisfies* $\mathrm{Im}\, \mathcal{G}(\underline{s}, t) \to 2\pi i s_1$ *as* $t \to \infty$.

The curves $G_{\underline{s}} : (0, \infty) \to \mathbb{C}; t \mapsto \mathcal{G}(\underline{s}, t + t_{\underline{s}})$, are called *parameter rays*. As for dynamic rays, we say that $G_{\underline{s}}$ *lands* at a parameter $\kappa \in \hat{\mathbb{C}}$ if $\lim_{t \to 0} G_{\underline{s}}(t) = \kappa$. If $\kappa \neq \infty$, we set $G_{\underline{s}}(0) := \mathcal{G}(\underline{s}, t_{\underline{s}}) := \kappa$ in this case. (Similarly, in the following we will write $\mathfrak{g}^\kappa(\underline{s}, t_{\underline{s}}) = z$ if the dynamic ray $g^\kappa_{\underline{s}}$ lands at $z \in \mathbb{C}$.)

In this section, we investigate continuity properties of the map $\mathcal{G}$, using the results of §4. The means to transfer this dynamical information into the parameter plane is provided by the following result. (Recall the definition of $Q(K)$ and Y_Q from equation (4.1)).

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LEMMA 11.2. (Continuity of $\mathcal{G}$) *Let $\kappa_0 \in \mathbb{C}$. Suppose that there are $n \geq 0$, $Q_1 > Q(|\kappa_0|)$ and $x_0 \in \mathcal{F}^{-n}(Y_{Q_1})$ such that $\mathfrak{g}^{\kappa_0}(x_0) = \kappa_0$. Then $\kappa_0 = \mathcal{G}(x_0)$; furthermore, there is a neighborhood V of x_0 in $\mathcal{F}^{-n}(Y_{Q_1})$ such that the map $\mathcal{G} : V \to \mathbb{C}$ is defined and a homeomorphism onto its image.*

Proof. Pick a neighborhood U of κ_0 with $Q(|\kappa|) < Q_1$ for all $\kappa \in U$. If V_1 is a small neighborhood of x_0 in $\mathcal{F}^{-n}(Y_{Q_1})$ and U was chosen small enough, then $\mathfrak{g}^{\kappa}(x)$ is defined for all $\kappa \in U$ and $x \in V_1$, and jointly continuous in κ and x.

By Hurwitz's theorem, there is a compact neighborhood $V \subset V_1$ of x_0 and a function $V \to U; x \mapsto \kappa(x)$ such that $\mathfrak{g}^{\kappa(x)}(x) = \kappa(x)$. Furthermore, this map can be chosen in such a way that $\kappa(x) \to \kappa_0$ as $x \to x_0$.

Suppose that $x_0 = (\underline{s}, t_0)$. Then there is t_1 such that $(\underline{s}, t) \in V$ for all $t \in (t_0, t_1)$. By Proposition 11.1, we have $\kappa(\underline{s}, t) = \mathcal{G}(\underline{s}, t)$, and hence $\kappa_0 = \lim_{t \to t_0} \kappa(\underline{s}, t) = \lim_{t \to t_0} \mathcal{G}(\underline{s}, t)$. Thus $\kappa_0 = \mathcal{G}(\underline{s}, t_0)$, as required.

For every $x \in V$, the parameter $\kappa(x)$ also satisfies the hypotheses of the theorem. Thus, by what we have just shown, $\kappa(x) = \mathcal{G}(x)$ for all $x \in V$. So $\mathcal{G}$ is defined on V and continuous in x_0. Continuity in any other point of V follows by replacing κ_0 by $\mathcal{G}(x)$. Since $\mathcal{G}$ is clearly injective, and V was chosen to be compact, $\mathcal{G}|_V$ is a homeomorphism onto its image. ∎

COROLLARY 11.3. (Continuity away from endpoints) *For every $\varepsilon > 0$, the map $\mathcal{G}$ is a homeomorphism when restricted to the set*

$$Z_\varepsilon := \{(\underline{s}, t) : t \geq t_{\underline{s}} + \varepsilon\}.$$

Proof. Let $x_0 \in Z_\varepsilon$ and $\kappa_0 := \mathcal{G}(x_0)$. Choose n so large that $F^n(\varepsilon) \geq Q(|\kappa|) + 1$. Then $\mathcal{F}^n(Z_\varepsilon) \subset Y_{Q(|\kappa|)+1}$, and continuity of $\mathcal{G}|_{Z_\varepsilon}$ in x_0 follows from the previous lemma.

To prove that $\mathcal{G}|_{Z_\varepsilon}$ is a homeomorphism onto its image, it remains to show that $\mathcal{G}(x_n) \to \infty$ as $x_n = (\underline{s}^n, t_n) \to \infty$ in Z_ε. If $s_1^n \to \infty$, then $\operatorname{Im} \mathcal{G}(x) \to \infty$ since parameter rays cannot intersect the lines $\{r + (2k+1)\pi i : r \in \mathbb{R}, k \in \mathbb{Z}\}$, which consist entirely of attracting and parabolic parameters. On the other hand, if $(\underline{s}, t) \in Z_\varepsilon$, then

$$T(\mathcal{F}(\underline{s}, t)) \geq F(t) - F(t - \varepsilon) \geq (1 - \exp(-\varepsilon))F(t).$$

In particular, if t is large enough, then $T(\mathcal{F}(\underline{s}, t)) \geq t$. By Corollary 4.8, this implies that $|\mathcal{G}(\underline{s}, t)| \geq t/5$ when $(\underline{s}, t) \in Z_\varepsilon$ with sufficiently large t. ∎

In particular, we obtain the following analog of Lemma 5.1.

COROLLARY 11.4. (Limit set of parameter rays) *Let $\underline{s}^0 \in \mathcal{S}_0$ and denote the limit set of $G_{\underline{s}^0}$ by L. If there exist some $\underline{s} \in \mathcal{S}_0$ and $t > 0$ with $G_{\underline{s}}(t) \in L$, then $G_{\underline{s}}((0, t]) \subset L$.*

Proof. Analogous to Lemma 5.1. ∎

Recall from the introduction that the set $\mathcal{I}$ consists of all escaping parameters.

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THEOREM 11.5. (Cantor bouquets in parameter space) *There exists a sequence of closed subsets $\overline{X}_k \subset \overline{X}$ with the following properties:*

(a) $\overline{X}_k \subset \overline{X}_{k+1}$;

(b) *every connected component of $\overline{X}_k$ is of the form $\{\underline{s}\} \times [t, \infty)$ for some $t \geq t_{\underline{s}}$;*

(c) *the function $\mathcal{G}$ is defined on $\overline{X}_k$ and is a homeomorphism onto its image $\mathcal{J}_k := \mathcal{G}(\overline{X}_k)$;*

(d) $\mathcal{I} \subset \bigcup_k \mathcal{J}_k$.

Proof. (Compare [**FRS**, Proof of Theorem 5.4].) Let $\overline{X}'_n$ denote the set of all $x \in \overline{X}$ for which $\kappa := \mathcal{G}(x)$ is defined and $x \in \mathcal{F}^{-n}(Y_{Q(|\kappa|)+1})$. Then by Lemma 11.2, $\mathcal{G} : \overline{X}'_n \to \mathbb{C}$ is defined, injective and a local homeomorphism. Furthermore, for any sequence (x_n) in $\overline{X}'_n$, clearly $\mathcal{G}(x_n) \to \infty$ if and only if $x_n \to \infty$.

It follows from the continuity of $\mathfrak{g}^\kappa(x)$ in κ and x that $\overline{X}'_n$ is closed. Thus $\mathcal{G}|_{\overline{X}_n}$ is a homeomorphism onto its image; let $\overline{X}_n$ be the union of all unbounded connected components of $\overline{X}'_n$. Then $\overline{X}_n$ satisfies (b) and (c).

It remains to establish (d). Let $\underline{s} \in \mathcal{S}_0$. We need to show that, for every escaping parameter of the form $\kappa_0 = \mathcal{G}(\underline{s}, t_0)$, there is some n such that $A := \underline{s} \times [t_0, \infty) \subset \overline{X}'_n$.

By [**FS**, Theorem 3.2], there exists $t_1 > t_{\underline{s}}$ with $\{\underline{s}\} \times [t_1, \infty) \subset \overline{X}_0$. Let us set $K := \max_{t \in [t_0, t_1]} |\mathcal{G}(\underline{s}, t)|$; we can then $n \geq 0$ such that $\mathcal{F}^n(A) \subset Y_{K+2}$. Then $(\{\underline{s}\} \times [t_0, t_1]) \subset \overline{X}_n$. We thus have $A \subset \overline{X}'_n$, as required. ■

COROLLARY 11.6. [**FRS**, Theorem 5.4] *Every path-connected component of $\mathcal{I}$ consists either of a single parameter ray or of a single parameter ray at a fast address together with an escaping landing point.*

Proof. By Corollary 11.4, no parameter ray can land at a point which is on another parameter ray. Furthermore, by [**RS2**, Theorem A.3], no other parameter ray can accumulate at the landing point of a fast parameter ray. The claim now follows by applying Proposition 5.4, where $I := \mathcal{I}$ and $I_n := \mathcal{J}_n \cap \mathcal{I}$ with $\mathcal{J}_n$ from the previous theorem. ■

12. *Further questions*

One of the most intriguing questions arising from our results is whether Theorem 1.1 generalizes to other classes of entire functions, such as the class $\mathcal{B}$ of functions whose set of singular values is bounded. Little can be said about the escaping sets of such functions without further restrictions. Although for many $f \in \mathcal{B}$, the escaping set is arranged in dynamic rays, there also exist such functions whose Julia sets contain no curves to ∞ at all [$\mathbf{R}^3\mathbf{S}$]. In view of these facts it is perhaps surprising that an analog of Theorem 1.1 *does* hold in full generality for class $\mathcal{B}$ [**R4**]: if $f, g \in \mathcal{B}$ are quasiconformally equivalent in the sense of [**EL**], then for sufficiently large $R > 0$ there is a conjugacy between f and g defined on the set $\{z : |f^n(z)| \geq R \text{ for all } n \geq 0\}$. Furthermore, this conjugacy extends to a quasiconformal map of the plane.

In our arguments of escaping set rigidity for Misiurewicz parameters, we used the fact that there is an asymptotic value in the Julia set that interferes with the topology of the escaping set. The same fact is used in proving the existence of non-landing rays for exponential Misiurewicz parameters (see [**R3**]). Schleicher [**S3**] has shown that *all*

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dynamic rays of Misiurewicz members of the cosine family $z \mapsto a\exp(z) + b\exp(-z)$ land. It is therefore an interesting question whether rigidity of escaping dynamics remains for functions without asymptotic values.

QUESTION 12.1. (Escaping dynamics in the cosine family) *Are there two distinct Misiurewicz parameters in the cosine family that are topologically conjugate on their sets of escaping points?*

Let us now return to exponential dynamics. As mentioned previously, little is known of the accumulation behavior of dynamic rays in general. In the case of quadratic polynomials, it is still unknown whether a dynamic ray can accumulate on the entire Julia set (although it is known [**Ki**] that this could happen only for Siegel or Cremer parameters). In the exponential family, this possibility becomes even more disconcerting:

QUESTION 12.2. (Rays accumulating on the plane) *Can the accumulation set of a dynamic ray be the entire complex plane?*

We can ask a stronger question:

QUESTION 12.3. (Accumulating rays) *If $\underline{s}^n$ is a sequence of addresses with $|\underline{s}^n_1| \to \infty$, is it true that $z_n \to \infty$ whenever $z_n \in g_{\underline{s}^n}$ for all n?*

More generally, is this true whenever $(\underline{s}^n)$ converges to an address that is not exponentially bounded?

Let us now depart from questions concerning single rays and consider the escaping sets of exponential maps in their entirety. We have already formulated the conjecture that two exponential maps whose singular value lies in the Julia set are never conjugate on their escaping sets by an order-preserving conjugacy. We can ask whether the map is already determined by the topology of this set. We say that a homeomorphism between $I(E_{\kappa_1})$ and $I(E_{\kappa_2})$ is *natural* if it preserves the addresses of dynamic rays.

QUESTION 12.4. (Natural homeomorphisms) *If $I(E_{\kappa_1})$ and $I(E_{\kappa_2})$ are naturally homeomorphic, are E_{κ_1} and E_{κ_2} conjugate on their sets of escaping points?*

The answer to this question can be seen to be 'yes' when κ_1 and κ_2 are Misiurewicz-parameters, using the construction of non-landing dynamic rays [**R3**]. With some more care one can also do this when κ_1 and κ_2 are escaping parameters lying on different parameter rays. The first interesting case in which to investigate thus seems to be that of two parameters on the same parameter ray; for example $\kappa_1, \kappa_2 \in (-1, \infty)$.

We have described the escaping dynamics completely only in the case of attracting and parabolic dynamics. As we have seen, the situation becomes much more complicated when the singular value moves into the Julia set. Nevertheless, Misiurewicz parameters are uniquely determined by their combinatorics. One would thus hope that their topological dynamics can also be completely understood in terms of their combinatorics, which again might be a starting point to understand also more complicated types of exponential dynamics.

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QUESTION 12.5. (Topological dynamics of Misiurewicz maps) *Let κ be a Misiurewicz parameter. Can one construct a model for the topological dynamics of $E_\kappa|_{I(E_\kappa)}$ in terms of* addr(κ)*?*

Our deliberations in §11 quite naturally lead to the question of further continuity properties of the map $\mathcal{G}$.

QUESTION 12.6. (Pinched cantor bouquet) *Is the map $\mathcal{G} : X \to \mathcal{I}$ a homeomorphism? Does $\mathcal{G}$ extend to a continuous (and surjective) map from $\overline{X}$ to the exponential bifurcation locus?*

A positive answer to the second question would imply density of hyperbolicity.

Finally, we have seen that the notion of renormalization fails topologically even for attracting parameters. However, there seem to be similarity features in the parameter space of exponential maps. So it is natural to ask whether some—different—notion of renormalization might exist in the exponential family. Let us formulate this (vaguely) in a special case. Let W be any hyperbolic component in exponential parameter space, and let W' be the unique hyperbolic component of period 1. Then the multiplier maps $\mu : W \to \mathbb{D}^*$ and $\mu' : W' \to \mathbb{D}^*$ are universal covering maps [**EL**], so we can continue some branch of $\mu'^{-1} \circ \mu$ to obtain a biholomorphic map $\mathcal{R} : W \to W'$. By the results of [**S1, RS1**], this map extends to a homeomorphism $\mathcal{R} : \overline{W} \to \overline{W'}$.

QUESTION 12.7. (Renormalization) *Is there some analytic way to construct the parameter $\mathcal{R}(\kappa)$ from the dynamics of κ in such a way that dynamical features such as linearizability etc. are preserved?*

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