*Ergod. Th. & Dynam. Sys.* (2001), **21**, 1867–1881
DOI: 10.1017/S0143385701001900
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*Printed in the United Kingdom*

# Toeplitz flows, ordered Bratteli diagrams and strong orbit equivalence

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(*Received* 28 *April* 2000 *and accepted in revised form* 5 *February* 2001)

*Abstract*. Let $(X, T)$ be a Cantor minimal system associated with a properly ordered Bratteli diagram with the equal path number property. In this paper we will show that there exists a Toeplitz flow $(Y, S)$ such that $(X, T)$ and $(Y, S)$ are strongly orbit equivalent. This is an affirmative answer to the open problem in R. Gjerde and Ø. Johansen (Bratteli–Vershik models for Cantor minimal systems: applications to Toeplitz flows. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1687–1710).

## 1. *Introduction*

Cantor minimal systems, which are topological dynamical systems acting homeomorphically and minimally on the Cantor set by a single transformation, include a variety of dynamical systems that have very important and interesting properties. In fact, Cantor minimal systems are obtained by many types of constructions such as substitutions, interval exchange transformations, Toeplitz flows and so on. Moreover, Cantor minimal systems have connections with other fields of mathematics, that is $C^*$-algebras and (ordered) $K$-theory. In [**GPS**], T. Giordano, I. Putnam and C. Skau succeeded in exhibiting such connections by proving the following result.

Let $(X_i, \varphi_i)$ be a Cantor minimal system $(i = 1, 2)$. Then the following three statements are equivalent:

(a) $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are strongly orbit equivalent;
(b) associated $C^*$-crossed products $C(X_1) \rtimes_{\varphi_1} \mathbb{Z}$ and $C(X_2) \rtimes_{\varphi_2} \mathbb{Z}$ are isomorphic; and
(c) associated dimension groups $K_0(X_1, \varphi_1)$ and $K_0(X_2, \varphi_2)$ are order isomorphic by a map preserving the distinguished order units.

A definition for strong orbit equivalence is given as follows.

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https://doi.org/10.1017/S014333857010019900 Published online by Cambridge University Press

*Definition.* Two dynamical systems $(X_1, T_1)$ and $(X_2, T_2)$ are *orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ which sends $T_1$ orbits onto $T_2$ orbits. If there is an orbit equivalence between $(X_1, T_1)$ and $(X_2, T_2)$, then there exist functions $n : X_1 \to \mathbb{Z}$, $m : X_1 \to \mathbb{Z}$ such that for $x \in X_1$, $F \circ T_1 x = T_2^{n(x)} \circ Fx$ and $F \circ T_1^{m(x)} x = T_2 \circ Fx$, which are called *orbit cocycles associated with* $F$. $(X_1, T_1)$ and $(X_2, T_2)$ are *strongly orbit equivalent* if the two systems are orbit equivalent and each of the orbit cocycles, which are associated with the orbit equivalence, has at most one point of discontinuity.

The concept of 'strong orbit equivalence' plays a very important role in those connections and gives a new classification of dynamical systems (see [**GJ2, Or, Su1, Su2, Su3**]).

Here let us consider minimal subshifts. It is easy to see that the family of expansive Cantor minimal systems coincides with the family of minimal subshifts up to conjugacy. Concerning the strong orbit equivalence of a Cantor minimal system, we can raise the following question: within any strong orbit equivalence class, is there a Cantor minimal system which is conjugate to a subshift? In [**Su3**] we obtained the following result: given a Cantor minimal system $(X, T)$, there exists a minimal subshift strongly orbit equivalent to $(X, T)$. Furthermore, we can choose $(X, T)$ to be of any finite entropy.

One example of minimal subshifts is Toeplitz flows, which can be defined as follows.

Let $\Sigma$ be a finite alphabet. We say a non-periodic sequence $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ is *a Toeplitz sequence* if for any $m \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $\eta_m = \eta_{m+kn}$ for all $k \in \mathbb{Z}$. A *Toeplitz flow* is a minimal subshift $(X_\eta, T_\eta)$, where $\eta$ is Toeplitz sequence, $X_\eta$ is the closure of the orbit of $\eta$ under the shift, and $T_\eta$ is the restriction of the shift to $X_\eta$. (For details, see [**Wi**] and [**GJ1**].)

Toeplitz flows were first introduced by Jacobs and Keane [**JK**]. Williams extended the basic results in [**JK**] (see [**Wi**]). There are many interesting results concerning Toeplitz flows obtained by Downarowicz, Lacroix, Gjerde and Johansen, especially concerning their invariant measures and their maximal equicontinuous factors [**Do, DL, GJ1**]. Concerning the relationship between Toeplitz flows and Bratteli–Vershik systems (Cantor minimal systems) associated with properly ordered Bratteli diagram, Gjerde and Johansen succeeded in obtaining the following theorem.

THEOREM 1.1. [**GJ1**, Theorem 8] *The family of expansive Bratteli–Vershik systems associated with Bratteli diagrams with the equal path number property coincides with the family of Toeplitz flows up to conjugacy.*

Concerning the strong orbit equivalence of Toeplitz flows, they raised an open problem in [**GJ1**] which can be stated as follows: 'Let $(V, E)$ be a simple Bratteli diagram with the equal path number property. Does there exist a Toeplitz flow $(Y, \psi)$ such that $K^0(Y, \psi) \cong K_0(V, E)$ as ordered groups with distinguished order units?' We obtained an affirmative answer, Theorem 1.2, to this problem, which is the main theorem in this paper.

THEOREM 1.2. *The family of Bratteli–Vershik systems associated with properly ordered Bratteli diagrams with the equal path number property coincides with the family of Toeplitz flows up to strong orbit equivalence.*

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This theorem tells us that Bratteli diagrams with the equal path number property 'present' all the $K^0$-groups that arise from Toeplitz flows.

In §2, we introduce some definitions and results concerning Cantor minimal systems. The crucial tool in obtaining our results is a model theorem of Cantor minimal systems [**HPS**]. We define a properly ordered Bratteli diagram and introduce the relation between Cantor minimal systems and properly ordered Bratteli diagram (Bratteli–Vershik systems). Next we define the dimension group associated with Cantor minimal systems. From [**GPS**], we know that there is a relationship between a strong orbit equivalence class for Cantor minimal systems and an equivalence relation on Bratteli diagrams associated with the systems (Theorem 2.1).

In §3, we give some modification lemmas for the construction of Bratteli diagrams. In each modification lemma, we modify diagrams preserving the equivalence relation on Bratteli diagrams and the equal path number property. In general Bratteli diagrams, it is very important to control the number of vertices $|V_n|$ at the $n$th level of the Bratteli diagram within the equivalence relation on diagrams so that the number does not increase 'too rapidly' with respect to $n \in \mathbb{N}$. We show that we can do this in Lemma 3.2 which plays an important role in this paper. Moreover, we need to modify a Bratteli diagram in order to realize an expansive Bratteli–Vershik system with the equal path number property. Indeed, we define distinct orderings and we have to control both the number of vertices $|V_n|$ and that of edges $|E_n|$ so that the modified Bratteli diagram $\tilde{\mathcal{B}}$ has a distinct ordering for all levels. In §4, we show the main theorem (Theorem 1.2). The point which we have to prove is that the constructed Bratteli–Vershik system $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is expansive. Since $\mathcal{B}$ will have a distinct ordering for all levels, it is not hard to prove it.

## 2. *Definition of the Bratteli diagram the Bratteli–Vershik system and dimension group for Cantor minimal systems*

2.1. *Bratteli diagram.* A *Bratteli diagram* consists of a vertex set $V$ and an edge set $E$ satisfying the following conditions. We decompose $V$ into finite sets $V_k$ $(k \geq 0)$ where each $V_k$ is non-empty and disjoint from each other and $|V_0| = 1$. The index $k$ of $V_k$ implies that a vertex in $V_k$ lies on the $k$th level. Similarly, we decompose $E$ into finite sets $E_k$ $(k \geq 1)$ where each $E_k$ is non-empty and disjoint from each other. The relation between $V$ and $E$ is given by the maps $r, s : E \to V$ so that $r(E_k) \subset V_k$ and $s(E_k) \subset V_{k-1}$ $(k \geq 1)$. We call $r$, $s$ a *range map* and a *source map* respectively. We also assume that for all $v \in V$, $s^{-1}(v) \neq \emptyset$ and for all $v \in V \setminus V_0$, $r^{-1}(v) \neq \emptyset$.

For $k \geq 1$ , we write $V_k = \{v_i^{(k)} \mid 1 \leq i \leq |V_k|\}$. Let $m_{i,j}^{(k)}$ be the number of edges which connect $v_i^{(k)} \in V_k$ and $v_j^{(k-1)} \in V_{k-1}$. We call the matrix $[m_{i,j}^{(k)}]$ , whose size is $|V_k| \times |V_{k-1}|$, the *incidence matrix of the kth level*. We write $M^{(k)} = [m_{i,j}^{(k)}]$. Define $M_i^{(k)} = [m_{i,1}^{(k)}, m_{i,2}^{(k)}, \ldots, m_{i,|V_{k-1}|}^{(k)}]$. We call $M_i^{(k)}$ the *incidence vector of* $v_i^{(k)} \in V_k$.

We say that $(V, E)$ is *simple* if for all $k \geq 0$, there exists an $n > k$ such that for every pair of vertices $v \in V_k$, $v' \in V_n$, there is a path from $v$ to $v'$ or, to put it differently, if the product of incidence matrices $M^{(n)} M^{(n-1)} \cdots M^{(k+2)} M^{(k+1)}$ consists of all positive entries.

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For a given Bratteli diagram $(V, E)$ and $k, n \in \mathbb{N}$ with $k < n$, let $\mathcal{P}(V_k, V_n)$ be the set of all paths from $V_k$ to $V_n$; i.e. $\mathcal{P}(V_k, V_n) = \{(e_{k+1}, e_{k+2}, \dots, e_n) \mid e_i \in E_i \text{ for } i \in (k, n] \text{ and } r(e_i) = s(e_{i+1}) \text{ for } i \in (k, n)\}$. Define $r : \mathcal{P}(V_k, V_n) \to V_n$, and $s : \mathcal{P}(V_k, V_n) \to V_k$ by $r(e_{k+1}, e_{k+2}, \dots, e_n) = r(e_n)$, $s(e_{k+1}, e_{k+2}, \dots, e_n) = s(e_{k+1})$. We simply write $\mathcal{P}(V_k)$ instead of $\mathcal{P}(V_0, V_k)$. For $v \in V_k$, let $\mathcal{P}(v)$ denote the set of all paths from $V_0$ to $v$.

Given a Bratteli diagram $\mathcal{B} = (V, E)$ and sequence $c_0 = 0 < c_1 < c_2 < c_3 < \cdots$ in $\mathbb{Z}^+$, we say that a Bratteli diagram $(V', E')$ is a *telescoping* (or *contraction*) of $\mathcal{B}$ to $\{c_k\}_{k=0}^{\infty}$ if $V'_n = V_{c_n}$ and $E'_n$ consists of $\mathcal{P}(V_{c_{n-1}}, V_{c_n})$, and source and range maps are as before. In particular, we define $\mathcal{B}_{\text{odd}}$ as a telescoping $\mathcal{B}$ to odd levels $(0, 1, 3, \dots)$ and define $\mathcal{B}_{\text{even}}$ as telescoping $\mathcal{B}$ to even levels $(0, 2, 4, \dots)$.

2.2. *Ordered Bratteli diagram.* An *ordered Bratteli diagram* is a Bratteli diagram $(V, E)$ together with a partial order $\geq$ on $E$. (We write $(V, E, \geq)$.) That is, we have a linear order on each set $r^{-1}(v)$, $v \in V \setminus V_0$. An order on $\mathcal{P}(V_k, V_n)$ is defined as follows:

$$(e_{k+1}, e_{k+2}, \dots, e_n) < (f_{k+1}, f_{k+2}, \dots, f_n) \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{For some } i \text{ with } k+1 \leq i \leq n: \\ \text{(i)} \quad e_j = f_j \text{ for } i < j \leq n, \\ \text{(ii)} \quad e_i < f_i. \end{cases} \tag{2.1}$$

Given an ordered Bratteli diagram $\mathcal{B} = (V, E, \geq)$, let $X_{\mathcal{B}}$ denote the associated infinite path space, i.e. $X_{\mathcal{B}} = \{(e_1, e_2, e_3, \dots) \mid r(e_k) = s(e_{k+1}),\ k \geq 1\}$. We topologize $X_{\mathcal{B}}$ by postulating the following family of sets as a basis of open sets, namely the family of cylinder sets $U(e_1, e_2, \dots, e_k) = \{(f_1, f_2, \dots) \mid f_i = e_i,\ 1 \leq i \leq k\}$. Each $U(e_1, e_2, \dots, e_k)$ is also closed, as is easily seen, and so we note that $X_{\mathcal{B}}$ becomes a compact Hausdorff space with a countable basis of clopen sets. We call $X_{\mathcal{B}}$ with this topology the *Bratteli compactum* associated with $\mathcal{B}$. If $\mathcal{B}$ is simple and the cardinality of $X_{\mathcal{B}}$ is infinity, then it turns out that $X_{\mathcal{B}}$ is a Cantor set.

Let $E_{\max}$, $E_{\min}$ denote the maximal and minimal elements of the partially ordered set $E$, respectively. $(V, E, \geq)$ is *proper* if $(V, E)$ is a simple Bratteli diagram and there is a unique path $p_{\max} = (g_1, g_2, \dots) \in X_{\mathcal{B}}$ such that each $g_k \in E_{\max}$, $k \geq 1$ and similarly there is a unique minimal path $p_{\min} = (h_1, h_2, \dots) \in X_{\mathcal{B}}$ such that each $h_k \in E_{\min}$, $k \geq 1$.

Given an ordered Bratteli diagram $\mathcal{B} = (V, E, \geq)$ and a sequence $c_0 = 0 < c_1 < c_2 < c_3 \cdots$ in $\mathbb{Z}^+$, an ordered Bratteli diagram $(V', E', \geq')$ is a *telescoping* (or *contraction*) of $\mathcal{B}$ to $\{c_k\}_{k=0}^{\infty}$ if $(V', E')$ is a contraction of $(V, E)$ to $\{c_k\}_{k=0}^{\infty}$ and the order $\geq'$ is induced by $\geq$ as in (2.1).

2.3. *Bratteli–Vershik system.* Using a properly ordered Bratteli diagram, we can create a minimal dynamical system on the Bratteli compactum associated with it. Let $X_{\mathcal{B}}$ be the Bratteli compactum associated with $\mathcal{B} = (V, E, \geq)$. Let $p_{\max}$, $p_{\min}$ be the unique maximal and minimal path in $X_{\mathcal{B}}$ respectively. We define a homeomorphism $\lambda_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$ as follows.

Define $\lambda_{\mathcal{B}}(p_{\max}) = p_{\min}$. If $p = (e_1, e_2, \dots) \neq p_{\max}$, let $k = \min\{i \in \mathbb{N} \mid e_i \notin E_{\max}\}$ and $f_k$ denote the successor of $e_k$ in $E$. Let $(f_1, f_2, \dots, f_{k-1})$ denote the unique least

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path in $\mathcal{P}(V_{k-1})$ with $r(f_{k-1}) = s(f_k)$. Then $\lambda_{\mathcal{B}}(p) = (f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots)$. We call $\lambda_{\mathcal{B}}$ a *lexicographic map* (*Vershik map*). We can show that $\lambda_{\mathcal{B}}$ is a minimal homeomorphism. We call the pair $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ a *Bratteli–Vershik system*.

In [**HPS**], Herman, Putnam and Skau showed that the family of Cantor minimal systems coincides with the family of Bratteli–Vershik systems up to conjugacy.

2.4. *Dimension group associated with a Cantor minimal system.* $(G, G^+)$ is an *unperforated ordered group* if $G$ is a countable Abelian group and $G^+$ is a subset of $G$ so that the following statements are satisfied:

(1) $G^+ + G^+ \subseteq G^+$;
(2) $G^+ - G^+ = G$;
(3) $G^+ \cap (-G^+) = \{0\}$;
(4) (unperforatedness) if $g \in G$ satisfies $ng \in G^+$ for some $n \in \mathbb{N}$, then $g \in G^+$.

We will write $g \leq h$ if $h - g \in G^+$. $u \in G$ is an *order unit* if, for every $g \in G^+$, $g \leq nu$ for some $n \in \mathbb{Z}^+$. We say $(G, G^+, u)$ is a *unital ordered group*. Let $G$ and $H$ be ordered groups with ordered units $u$ and $v$ respectively. A *unital order homomorphism* between $G$ and $H$ is a group homomorphism $\varphi : G \to H$ such that $\varphi(G^+) \subseteq H^+$ and $\varphi(u) = v$. If, in particular, $\varphi$ is bijective and $\varphi(G^+) = H^+$, then we call $\varphi$ a *unital order isomorphism* and $(G, G^+, u)$, $(H, H^+, v)$ are said to be *order isomorphic*.

Suppose $\mathcal{M} = \{M^{(k)}\}_{k=1}^{\infty}$ is a sequence of matrices associated to a properly ordered Bratteli diagram $(V, E, \geq)$. For $k \geq 0$, define the order $\leq$ on $\mathbb{Z}^{|V_k|}$ as follows:

$$(z_1, z_2, \ldots, z_{|V_k|}) \leq (y_1, y_2, \ldots, y_{|V_k|}) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad z_i \leq y_i \text{ for } 1 \leq i \leq |V_k|.$$

Define $\mathbb{Z}^{|V_k|+} = \{(z_1, z_2, \ldots, z_{|V_k|}) \in \mathbb{Z}^{|V_k|} \mid z_i \geq 0 \text{ for } 1 \leq i \leq |V_k|\}$. Then it follows easily that $(\mathbb{Z}^{|V_k|}, \mathbb{Z}^{|V_k|+})$ is an unperforated ordered group.

Suppose $\{(G_k, G_k^+, u_k)\}_{k=0}^{\infty}$ is a sequence of unital ordered groups and $\Phi_{k+1} : G_k \to G_{k+1}$ is a unital order homomorphism for each $k \geq 0$. Let $\sqcup_{k=0}^{\infty} G_k = \{(g, k) \mid g \in G_k,\ k \geq 0\}$ be the disjoint union of $\{G_k\}$. For $n \geq m$, define $\Phi_{m,n} = \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_{m+1} \circ \Phi_m$. Then define an equivalence relation $\backsim$ on $\sqcup_{k=0}^{\infty} G_k$ by setting $(g, n) \backsim (h, m)$ if there exists $k > n, m$ such that $\Phi_{n+1,k}(g) = \Phi_{m+1,k}(h)$. Let $G = \sqcup_{k=0}^{\infty} G_k / \backsim$. Define the equivalence class of $(g, n)$ as $[g, n]$. A group structure can be defined on $G$ by $[g, n] + [h, m] = [\Phi_{n+1,k}(g) + \Phi_{m+1,k}(h), k]$ where $k$ is any integer $> n, m$. Define $G^+ = \{[g, n] \mid g \in G_n^+,\ n \geq 0\}$, inducing an ordering on $G$. Since each $\Phi_k$ is unital, $u = [u_n, n]$ is independent of $n$ and gives an order unit of $G$. The unital ordered group $(G, G^+, u)$ is called the *direct limit* of $\{(G_k, \Phi_{k+1})\}_{k=0}^{\infty}$ and denoted by $\varinjlim(G_k, \Phi_{k+1})$.

A countable unperforated ordered group is a *dimension group* if it has the Riesz interpolation property. That is, given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$. We say that $G$ is *simple* if it does not contain non-trivial order ideals. For details of dimension groups, see [**EHS**].

For $k \geq 1$, we define $\Phi_k : \mathbb{Z}^{|V_{k-1}|} \to \mathbb{Z}^{|V_k|}$ as $\Phi_k(\boldsymbol{x}) = M^{(k)}\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{Z}^{|V_{k-1}|}$. Define the direct limit $(D_{\mathcal{M}}, D_{\mathcal{M}}^+, u)$ of unital ordered group as follows:

$$(D_{\mathcal{M}}, D_{\mathcal{M}}^+, u) = \varinjlim(\mathbb{Z}^{|V_{k-1}|}, \Phi_k) = \mathbb{Z}^{|V_0|} \xrightarrow{\Phi_1} \mathbb{Z}^{|V_1|} \xrightarrow{\Phi_2} \mathbb{Z}^{|V_2|} \xrightarrow{\Phi_3} \cdots, u = [1, 0].$$

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We call $u = [1, 0]$ *a distinguished order unit*. It is easily shown that $(D_{\mathcal{M}}, D^+_{\mathcal{M}}, u)$ is a dimension group. So we write $K_0(V, E) = (D_{\mathcal{M}}, D^+_{\mathcal{M}}, u)$.

In the following, we will define a dimension group $K^0(X, T)$ of Cantor minimal system $(X, T)$. Let $C(X, \mathbb{Z})$ denote the continuous functions on $X$ with value in $\mathbb{Z}$. Let $B_T$ denote the (coboundary) subgroup $\mathrm{Im}(\mathrm{id} - T_*) = \{f - f \circ T^{-1} \mid f \in C(X, \mathbb{Z})\}$. Let $K^0(X, T)$ be the quotient group $C(X, \mathbb{Z})/B_T$ and define $K^0(X, T)^+ = \{[f] \mid f \geq 0, f \in C(X, \mathbb{Z})\}$, where $[\cdot]$ denotes the quotient map. In [**Pu**], Putnam showed that $K^0(X, T)$ with positive cone $K^0(X, T)^+$ is a simple, acyclic (i.e. $K^0(X, T) \ncong \mathbb{Z}$) dimension group with (canonical) distinguished order unit [1], where $1 = 1_X$ is the constant function 1. In addition, Herman, Putnam and Skau have shown in [**HPS**] that $K^0(X, T) \cong K_0(V, E)$ as ordered groups with distinguished order units, where $(V, E)$ is a Bratteli–Vershik representation of $(X, T)$, and that all (acyclic) simple dimension groups can be obtained in this (dynamical) way.

2.5. *Equivalence relation on Bratteli diagram.* We will define an equivalence relation $\sim$ on Bratteli diagrams and $\approx$ on ordered Bratteli diagrams as follows: $\mathcal{B} = (V, E) \sim (V', E') = \mathcal{B}'$ if there exists a Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E})$ such that $\tilde{\mathcal{B}}_{\mathrm{odd}}$ yields a telescoping of either $\mathcal{B}$ or $\mathcal{B}'$, and $\tilde{\mathcal{B}}_{\mathrm{even}}$ yields a telescoping of the other. Similarly $\mathcal{B} = (V, E, \geq) \approx (V', E', \geq') = \mathcal{B}'$ is an equivalence relation $\approx$ on ordered Bratteli diagrams. It is easy to see that $(V, E) \sim (V', E')$ if and only if $K_0(V, E) \cong K_0(V', E)$ as dimension groups via an isomorphism preserving the distinguished order units. Using these equivalence relations, Giordano, Herman, Putnam and Skau characterized strong orbit equivalence and conjugacy for Bratteli–Vershik systems (Cantor minimal systems) by the following theorem.

THEOREM 2.1. [**GPS, HPS**] *Suppose* $(X_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ *and* $(X_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ *are Bratteli–Vershik systems. Then we have:*

(1) $(X_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ *and* $(X_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ *are strongly orbit equivalent if and only if* $\mathcal{B}_1 \sim \mathcal{B}_2$*;*

(2) $(X_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ *and* $(X_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ *are (pointedly) conjugate if and only if* $\mathcal{B}_1 \approx \mathcal{B}_2$.

## 3. *Modification of Bratteli diagrams*

In this section we will show modification lemmas of Bratteli diagrams preserving the equal path number property.

*Notation.* For a Bratteli diagram $(V, E)$, let $\{M^{(n)} = [m^{(n)}_{i,j}]\}_{n \geq 1}$ denote the sequence of incidence matrices of $(V, E)$. For $k, n \in \mathbb{N}$ with $k < n$,

$$M^{(n,k)} = [m^{(n,k)}_{i,l}] = M^{(n)} M^{(n-1)} \cdots M^{(k+1)} M^{(k)},$$

$$\bar{m}^{(n)}_i = \sum_{j=1}^{|V_{n-1}|} m^{(n)}_{i,j}, \quad \bar{m}^{(n,k)}_i = \sum_{l=1}^{|V_{k-1}|} m^{(n,k)}_{i,l}.$$

($\bar{m}^{(n)}_i = |r^{-1}(v^{(n)}_i)|$ holds.) We recall that the incidence vector of $v^{(n)}_i$ is $M^{(n)}_i$ of the following form:

$$M^{(n)}_i = [m^{(n)}_{i,1}, m^{(n)}_{i,2}, \ldots, m^{(n)}_{i,|V_{n-1}|}].$$

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*Definition 3.1.* (The equal path number property) Let $(V, E)$ be a simple Bratteli diagram. We say that $(V, E)$ has the *equal path number property* if for any $n \in \mathbb{N}$ and any $u, v \in V_n$, $|r^{-1}(v)| = |r^{-1}(u)|$.

We note that the equal path number property is preserved under telescoping.

LEMMA 3.2. *Suppose $\mathcal{B} = (V, E)$ is a simple Bratteli diagram with the equal path number property and has at least one edge between every pair of vertices from consecutive levels. Then there exists a simple Bratteli diagram $\mathcal{B}' = (V', E')$ such that the following statements hold*:

(1) *$\mathcal{B}'$ has the equal path number property, so the number $m'_n = |r^{-1}(v_i'^{(n)})|$ is independent of the choice of $v_i'^{(n)} \in V'_n$;*
(2) *for all $n \in \mathbb{N}$, $|V'_n| < 2^{m'_n}$; and*
(3) *$\mathcal{B} \sim \mathcal{B}'$.*

*Proof.* Let $M^{(n)} = [m_{i,j}^{(n)}]$ be the $n$th level of incidence matrix of $\mathcal{B}$. For $n \in \mathbb{N}$, first we define an equivalence relation $\sim$ on $V_n$ by declaring $v_i^{(n)} \sim v_j^{(n)}$ if $M_i^{(n)} = M_j^{(n)}$. We define $V'_n = V_n/\sim$ and write $V'_n = \{v_i'^{(n)} \mid 1 \le i \le |V_n/\sim|\}$. Moreover define $M'^{(n)} = [m_{i,j}'^{(n)}]$ as

$$m_{i,j}'^{(n)} = \sum_{v_t^{(n-1)} \in v_j'^{(n-1)}} m_{s,t}^{(n)}, \quad \text{where } v_s^{(n)} \in v_i'^{(n)}.$$

We note that the definition of $m_{i,j}'^{(n)}$ is independent of the choice of $v_s^{(n)} \in v_i'^{(n)}$. We will show these statements.

(1) Let $m_n = |r^{-1}(v)|$ for any $v \in V_n$. Since $\mathcal{B}$ has the equal path number property, for any $1 \le i \le |V'_n|$, we have

$$\bar{m}_i'^{(n)} = \sum_{j=1}^{|V'_{n-1}|} \sum_{v_t^{(n-1)} \in v_j'^{(n-1)}} m_{s,t}^{(n)} = \sum_{t=1}^{|V_{n-1}|} m_{s,t}^{(n)} = m_n,$$

where $v_s^{(n)} \in v_i'^{(n)}$. So $\mathcal{B}'$ has the equal path number property and $m_n = m'_n$.

(2) Since $\{M_i^{(n)} \mid 1 \le i \le |V_n|\} \subset \{(n_1, \dots, n_{|V_{n-1}|}) \in \mathbb{N}^{|V_{n-1}|} \mid \sum_{i=1}^{|V_{n-1}|} n_i = m_n\}$ holds, we have

$$|V'_n| \le \sharp\left\{(n_1, \dots, n_{|V_{n-1}|}) \in \mathbb{N}^{|V_{n-1}|} \;\middle|\; \sum_{i=1}^{|V_{n-1}|} n_i = m_n\right\} = \binom{m_n - 1}{|V_{n-1}| - 1} < 2^{m_n} = 2^{m'_n}.$$

(3) We will construct a Bratteli diagram $\hat{\mathcal{B}} = (\hat{V}, \hat{E})$ so that $\hat{\mathcal{B}}_{\text{even}}$ corresponds to $\mathcal{B}$ and $\hat{\mathcal{B}}_{\text{odd}}$ corresponds to $\mathcal{B}'$. For $k \in \mathbb{N}$, we put $\hat{V}_{2k-1} = V'_k$ and $\hat{V}_{2k} = V_k$. We define the incidence matrix $\hat{M}^{(n)} = [\hat{m}_{i,j}^{(n)}] \in \mathbb{N}^{|\hat{V}_n| \times |\hat{V}_{n-1}|}$ as

$$\hat{m}_{i,j}^{(n)} = \begin{cases} m_{l,j}^{(k)}, & \text{if } n = 2k-1 \text{ and } v_l^{(k)} \in v_i'^{(k)}, \\ \delta_{i,j}^{(k)}, & \text{if } n = 2k, \end{cases}$$

where $\delta_{i,j}^{(k)}$ is defined by $\delta_{i,j}^{(k)} = 1$ if $v_i^{(k)} \in v_j'^{(k)}$, and $\delta_{i,j}^{(k)} = 0$ if $v_i^{(k)} \notin v_j'^{(k)}$. We will check that $\hat{M}^{(2k-1,2k-2)} = M'^{(k)}$ and $\hat{M}^{(2k,2k-1)} = M^{(k)}$ for all $k \in \mathbb{N}$, where

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$\hat{M}^{(0)} = [1] \in \mathbb{N}^{1\times 1}$. For $n \geq 1$, let $\phi_n : V_n \to V'_n$ be the quotient map defined by $\phi_n(i) = j$ if $v_i^{(n)} \in v_j'^{(n)}$. If $n = 0$, $\phi_0 : V_0 \to V'_0$ is defined by $\phi_0(1) = 1$, where $V_0 = \{v_1^{(0)}\}$ and $V'_0 = \{v_1'^{(0)}\}$. For $k \geq 1$, we have

$$\begin{aligned}
\hat{m}_{i,j}^{(2k,2k-1)} &= \sum_{l=1}^{|\hat{V}_{2k-1}|} \hat{m}_{i,l}^{(2k)} \hat{m}_{l,j}^{(2k-1)} = \sum_{l=1}^{|V'_k|} \delta_{i,l}^{(k)} m_{s,j}^{(k)} \quad (\text{where } v_s^{(k)} \in v_l'^{(k)}) \\
&= m_{s,j}^{(k)} \quad (s \in \phi_k^{-1} \circ \phi_k(i)) \\
&= m_{i,j}^{(k)} \quad (\text{because } \phi_k(i) = \phi_k(s)) \\
\hat{m}_{i,j}^{(2k-1,2k-2)} &= \sum_{l=1}^{|\hat{V}_{2k-2}|} \hat{m}_{i,l}^{(2k-1)} \hat{m}_{l,j}^{(2k-2)} = \sum_{l=1}^{|V_{k-1}|} m_{s,l}^{(k)} \delta_{l,j}^{(k-1)} \quad (\text{where } v_s^{(k)} \in v_i'^{(k)}) \\
&= \sum_{l \in \phi_{k-1}^{-1}(j)} m_{s,l}^{(k)} \quad (\text{because } \delta_{l,j}^{(k-1)} = 1 \Leftrightarrow v_l^{(k-1)} \in v_j'^{(k-1)}) \\
&= m_{i,j}'^{(k)}.
\end{aligned}$$

So we finish the proof. □

*Remarks.* (1) We give an example of $\mathcal{B}$, $\mathcal{B}'$ and $\hat{\mathcal{B}}$. (In this case, $m_1 = 10, m_2 = 30$ and $m_3 = 50$.)

$$M^{(1)} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} 10 & 10 & 10 \\ 15 & 5 & 10 \\ 15 & 5 & 10 \\ 5 & 10 & 15 \\ 10 & 10 & 10 \end{bmatrix}, \quad M^{(3)} = \begin{bmatrix} 15 & 5 & 20 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 5 & 5 & 30 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 15 & 10 & 5 & 10 \\ 15 & 5 & 20 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \end{bmatrix},$$

$$M'^{(1)} = \begin{bmatrix} 10 \end{bmatrix}, \quad M'^{(2)} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}, \quad M'^{(3)} = \begin{bmatrix} 20 & 25 & 5 \\ 20 & 20 & 10 \\ 10 & 35 & 5 \\ 20 & 25 & 5 \end{bmatrix},$$

$$\hat{M}^{(1)} = \begin{bmatrix} 10 \end{bmatrix}, \quad \hat{M}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{M}^{(3)} = \begin{bmatrix} 10 & 10 & 10 \\ 15 & 5 & 10 \\ 5 & 10 & 15 \end{bmatrix},$$

$$\hat{M}^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{M}^{(5)} = \begin{bmatrix} 15 & 5 & 20 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 5 & 5 & 30 & 5 & 5 \\ 10 & 15 & 10 & 5 & 10 \end{bmatrix}, \quad \hat{M}^{(6)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In the case above, $\phi_1(1, 2, 3) = (1, 1, 1)$, $\phi_2(1, 2, 3, 4, 5) = (1, 2, 2, 3, 1)$ and $\phi_3(1, 2, 3, 4, 5, 6, 7) = (1, 2, 3, 2, 4, 1, 2)$.

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(2) Since $\mathcal{B}$ has the equal path number property and by the construction of $\mathcal{B}'$, $M'^{(1)}$ is always a $1 \times 1$ matrix.

LEMMA 3.3. *Suppose $\mathcal{B} = (V, E)$ is a simple Bratteli diagram having the equal path number property. Let $N \in \mathbb{N}$ be given. Then there exists a simple Bratteli diagram $\mathcal{B}' = (V', E')$ such that the following statements hold:*

(1) *$\mathcal{B}'$ has the equal path number property;*
(2) *$\mathcal{B}'$ has at least $N$ edges between every pair of vertices from consecutive levels;*
(3) *let $m'_n = |r^{-1}(v)|$, where $v \in V'_n$. (By the condition (1), $m'_n$ is well defined.) For all $n \geq 2$, $|V'_n| < 2^{m'_n}$ and $(3/2)^{m'_n}(m'_n + 2)^{-8(|V'_{n-1}|+1)} > 5$;*
(4) *$\mathcal{B} \sim \mathcal{B}'$.*

*Proof.* We will first telescope $\mathcal{B}$ appropriately to $\mathcal{B}_0$. Then we will apply Lemma 3.2 to $\mathcal{B}_0$ to obtain $\mathcal{B}'$. In fact, we will define an increasing sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$, which are telescoped levels of $\mathcal{B}$, by induction. Define $n_1 = 1$. For $k \geq 2$, we assume $n_{k-1}$ have been defined. For $t > n_{k-1}$, let $m(t) = \bar{m}_i^{(t,n_{k-1}+1)}$ for any $v_i^{(t)} \in V_t$. We note that $m(t)$ is independent of the choice of $v_i^{(t)} \in V_t$ because the equal path number property is preserved under telescoping. Since $\mathcal{B}$ is simple, $m(t)$ is monotone increasing with respect to $t$. Therefore there is a $t > n_{k-1}$ such that all entries of incidence matrix $M^{(t,n_{k-1}+1)}$ are bigger than $N$ and $(3/2)^{m(t)}(m(t)+2)^{-8(|V_{n_{k-1}}|+1)} > 5$. We fix such a $t$ and define $n_k = t$. Let $\mathcal{B}_0$ be the telescoped diagram of $\mathcal{B}$ to $0, n_1, n_2, \ldots$. Applying Lemma 3.2 to $\mathcal{B}_0$ we get the Bratteli diagram $\mathcal{B}'$ satisfying the following:

- $\mathcal{B}'$ has the equal path number property. So the number $m'_n = |r^{-1}(v_i'^{(n)})|$ is independent of the choice of $v_i'^{(n)} \in V'_n$; since $m'_k = m(n_k)$ for all $k \in \mathbb{N}$, we see that
- for all $k \geq 2$, $|V'_k| < 2^{m'_k}$ and $(3/2)^{m'_k}(m'_k + 2)^{-8(|V_{n_{k-1}}|+1)} > 5$;
- $(\mathcal{B} \sim)\mathcal{B}_0 \sim \mathcal{B}'$.

Since the $k$th level of vertex set of $\mathcal{B}_0$ is $V_{n_k}$ and by the construction of $\mathcal{B}'$ in Lemma 3.2, we have

- $|V_{n_k}| \geq |V'_k|$ for all $k \in \mathbb{N}$;
- $\mathcal{B}'$ has at least $N$ edges between every pair of vertices from consecutive levels.

So $2^{m'_n}(m'_n + 2)^{-8(|V'_{n-1}|-1)} > 5$ holds for all $n \geq 2$ and we are done. □

*Definition 3.4.* (Ordered vector and distinct order) Let $\mathcal{B} = (V, E, \geq)$ be an ordered Bratteli diagram.

(1) For $k \geq 1$ and $v \in V_k$, let $\tau(v, j) \in V_{k-1}$ be the source vertex of $j$th element of $r^{-1}(v)$. We say that the following vector, consisting of vertices, is an *ordered vector* of $v$

$$\big(\tau(v, 1), \tau(v, 2), \ldots, \tau(v, |r^{-1}(v)| - 1), \tau(v, |r^{-1}(v)|)\big) \in (V_{k-1})^{|r^{-1}(v)|},$$

and denote it by $O.vec(v)$.

(2) We say edges to vertices of $V_k$ have *distinct orderings* (to put it simply that $V_k$ has distinct orderings) if for $v, v' \in V_k$, $O.vec(v) = O.vec(v')$ implies $v = v'$. If for any $n \geq 2$, $V_n$ has distinct orderings, we say that $\mathcal{B}$ has *distinct orderings for all levels*.

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LEMMA 3.5. *Suppose $\mathcal{B} = (V, E)$ is a simple Bratteli diagram having the equal path number property. Then there exists a properly ordered Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \geq)$ such that the following statements hold:*

(1) *$\tilde{\mathcal{B}}$ has the equal path number property and has at least one edge between every pair of vertices from consecutive levels;*
(2) *$\tilde{\mathcal{B}}$ has distinct ordering for all levels;*
(3) *for $n \geq 1$, $\tilde{v}_{\min}^{(n)} \in \tilde{V}_n$ ($\tilde{v}_{\max}^{(n)} \in \tilde{V}_n$) is the vertex which the unique infinite minimal (maximal) path goes through. Then $\sharp\{e \in \tilde{E}_{n+1} \mid s(e) = \tilde{v}_{\min}^{(n)},\ r(e) = v\} = 1$ and $\sharp\{e \in \tilde{E}_{n+1} \mid s(e) = \tilde{v}_{\max}^{(n)},\ r(e) = v\} = 1$ for any $v \in \tilde{V}_{n+1}$;*
(4) *let $\tilde{m}_n = |r^{-1}(v)|$, where $v \in \tilde{V}_n$. (By condition (1), $\tilde{m}_n$ is well defined.) For any $n \geq 2$ and $v \in \tilde{V}_n$, $\tau(v, 1) = \tilde{v}_{\min}^{(n-1)}$ and $\tau(v, \tilde{m}_n) = \tilde{v}_{\max}^{(n-1)}$;*
(5) *$\mathcal{B} \sim \tilde{\mathcal{B}}$.*

*Proof.* Applying Lemma 3.3, we may assume that $\mathcal{B}$ satisfies the following conditions:

(i) $\mathcal{B}$ satisfies the equal path number property. So the number $m_n = |r^{-1}(v)|$, where $v \in V_n$, is well defined;
(ii) $\mathcal{B}$ has at least 60 edges between every pair of vertices from consecutive levels;
(iii) for all $n \geq 2$, $|V_n| < 2^{m_n}$ and $(3/2)^{m_n}(m_n + 2)^{-8(|V_{n-1}|+1)} > 5$.

We will construct the incidence matrices $\{\tilde{M}^{(n)}\}_{n\geq 1}$ of $\tilde{\mathcal{B}}$ using $M^{(n)} = [m_{i,j}^{(n)}]$ which is the $n$th level of the incidence matrix of $\mathcal{B}$. For $n \geq 2$, we define $\theta_{i,j}^{(n)} \in \mathbb{N}$ and $\lambda_{i,j}^{(n)} \in \mathbb{Z}$ $(1 \leq i \leq |V_n|, 1 \leq j \leq |V_{n-1}|)$ as

$$m_{i,j}^{(n)} - 2\delta_1(j) = 4\theta_{i,j}^{(n)} + \lambda_{i,j}^{(n)}, \quad \lambda_{i,j}^{(n)} \in [0, 4), \tag{3.1}$$

where $\delta_l : \mathbb{N} \to \{0, 1\}$ is defined by $\delta_l(i) = 0$ if $i \neq l$, and $\delta_l(i) = 1$ if $i = l$. By (ii) and (3.1),

$$\theta_{i,j}^{(n)} > \frac{m_{i,j}^{(n)} - 6}{4} > 13 \quad \text{for any } i \text{ and } j. \tag{3.2}$$

Let $|\tilde{V}_n| = 4|V_n| + 2$ for all $n \in \mathbb{N}$ and $|\tilde{V}_0| = 1$. First we define $\tilde{M}^{(1)} \in \mathbb{N}^{6\times 1}$ as

$$\tilde{M}^{(1)} = [m^{(1)}, m^{(1)}, m^{(1)}, m^{(1)}, m^{(1)}, m^{(1)}]^{\mathrm{tr}},$$

where $M^{(1)} = [m^{(1)}] \in \mathbb{N}^{1\times 1} (= \mathbb{N})$. (We note that applying $\mathcal{B}$ in Lemma 3.3, $M^{(1)}$ is always $1 \times 1$ matrix.) For $n \geq 2$, we define $\tilde{M}^{(n)} = [\tilde{M}_{i,j}^{(n)}, 1 \leq i \leq |V_n|, 1 \leq j \leq |V_{n-1}|] \in \mathbb{N}^{|\tilde{V}_n|\times|\tilde{V}_{n-1}|}$, where $\tilde{M}_{i,j}^{(n)} \in \mathbb{N}^{(4+2\delta_1(i))\times(4+2\delta_1(j))}$, as follows:

$$\tilde{M}_{i,j}^{(n)} = (l_{\zeta,\xi}), \quad l_{\zeta,\xi} = \begin{cases} 1 & \text{if } \xi = \delta_1(j), 2\delta_1(j) \\ \theta_{i,j}^{(n)} + 1 & \text{if } 2\delta_1(j) < \xi \leq 2\delta_1(j) + \lambda_{i,j}^{(n)} \\ \theta_{i,j}^{(n)} & \text{if } 2\delta_1(j) + \lambda_{i,j}^{(n)} < \xi \leq 4 + 2\delta_1(j). \end{cases}$$

(By the previous definition, $\xi = 0$ implies that $l_{\zeta,0}$ is an empty entry.) We write $\tilde{M}^{(n)} = [\tilde{m}_{i',j'}^{(n)}]$, where $1 \leq i' \leq |\tilde{V}_n|$ and $1 \leq j' \leq |\tilde{V}_{n-1}|$. By the previous definition of $\tilde{M}^{(n)}$, we calculate

$$\sum_{t=-1-2\delta_1(j)}^{2} \tilde{m}_{4i+s,4j+t}^{(n)} = m_{i,j}^{(n)}, \quad \text{where } s \in [-1 - 2\delta_1(i), 2]. \tag{3.3}$$

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In the following, we will check that $\bar{\tilde{m}}_i^{(n)} = m_n$ for any $1 \le i \le |\tilde{V}_n|$. This implies that statement (1) holds. For $i$, there is $l \in \mathbb{N}$ such that $4l - 1 - 2\delta_1(l) \le i \le 4l + 2$. By (3.3), we have

$$\bar{\tilde{m}}_i^{(n)} = \sum_{j=1}^{|\tilde{V}_{n-1}|} \tilde{m}_{i,j}^{(n)} = \sum_{k=1}^{|V_{n-1}|} \sum_{t=-1-2\delta_1(k)}^{2} \tilde{m}_{i,4k+t}^{(n)} = \sum_{k=1}^{|V_{n-1}|} m_{l,k}^{(n)} = \bar{m}_l^{(n)} = m_n.$$

So we let $\tilde{m}_n = m_n$. We write $\tilde{V}_n = \{\tilde{v}_i^{(n)} \mid 1 \le i \le |\tilde{V}_n|\}$ and we let $\tilde{v}_{\min}^{(n)} = \tilde{v}_1^{(n)}$ and $\tilde{v}_{\max}^{(n)} = \tilde{v}_2^{(n)}$. For $\tilde{v}_i^{(n)}$, define $\mathrm{Dist}(\tilde{v}_i^{(n)})$ as

$$\mathrm{Dist}(\tilde{v}_i^{(n)}) = \sharp\{(v_1, v_2, \dots, v_{\tilde{m}_n}) \subset (\tilde{V}_{n-1})^{\tilde{m}_n} \mid v_1 = \tilde{v}_{\min}^{(n-1)}, v_{\tilde{m}_n} = \tilde{v}_{\max}^{(n-1)}, \\ \sharp\{k \mid v_k = \tilde{v}_j^{(n-1)}\} = \tilde{m}_{i,j}^{(n)} \text{ for any } 1 \le j \le |\tilde{V}_{n-1}|\}.$$

$\mathrm{Dist}(\tilde{v}_i^{(n)})$ denotes the maximal possible number for arranging edges in $r^{-1}(\tilde{v}_i^{(n)})$ with $\tau(\tilde{v}_i^{(n)}, 1) = \tilde{v}_{\min}^{(n-1)}$ and $\tau(\tilde{v}_i^{(n)}, \tilde{m}_n) = \tilde{v}_{\max}^{(n-1)}$. Then we see that

$$\mathrm{Dist}(\tilde{v}_i^{(n)}) = \frac{(\tilde{m}_n - 2)!}{\prod_{j=3}^{|\tilde{V}_{n-1}|} \tilde{m}_{i,j}^{(n)}!}.$$

Now we will show that for $\tilde{v}_i^{(n)}$, $|\tilde{V}_n| \le \mathrm{Dist}(\tilde{v}_i^{(n)})$. We claim the following inequality for $n \in \mathbb{N}$:

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+2}{e}\right)^{n+2}. \tag{3.4}$$

If $n = 1$, the inequality holds trivially. If $n \ge 2$, then $e^n = \sum_{k=0}^{\infty} n^k/k! > n^n/n!$. Therefore the first part of the inequality holds. Next, we can calculate

$$\log(n+1)! = \sum_{k=1}^{n+1} \log k < \sum_{k=1}^{n+1} \int_k^{k+1} \log x \, dx = \int_1^{n+2} \log x \, dx \\ = (n+2)\log(n+2) - (n+1).$$

Since $\log(n+1) > 1$ for $n \ge 2$, we get

$$\log n! < (n+2)\log(n+2) - (n+1) - \log(n+1) \\ < (n+2)\{\log(n+2) - 1\} = \log\left((n+2)/e\right)^{n+2}.$$

So the second part of the inequality also holds.

For $\tilde{v}_i^{(n)}$, there is $l \in \mathbb{N}$ so that $4l - 1 - 2\delta_1(l) \le i \le 4l + 2$. Using (3.4), we calculate

$$\mathrm{Dist}(\tilde{v}_i^{(n)}) > \frac{\{(\tilde{m}_n - 2)/e\}^{(\tilde{m}_n - 2)}}{\prod_{j=3}^{|\tilde{V}_{n-1}|} \{(\tilde{m}_{i,j}^{(n)} + 2)/e\}^{(\tilde{m}_{i,j}^{(n)} + 2)}} \\ = \prod_{j=3}^{|\tilde{V}_{n-1}|} \left(\frac{\tilde{m}_n - 2}{\tilde{m}_{i,j}^{(n)} + 2}\right)^{\tilde{m}_{i,j}^{(n)}} \times \left(\frac{\tilde{m}_n - 2}{e} \times \prod_{j=3}^{|\tilde{V}_{n-1}|} \frac{\tilde{m}_{i,j}^{(n)} + 2}{e}\right)^{-2} \\ > \prod_{k=1}^{|V_{n-1}|} \left(\frac{m_n - 2}{\theta_{l,k}^{(n)} + 3}\right)^{m_{l,k}^{(n)} - 2\delta_1(k)} \times \left\{(\tilde{m}_n - 2) \times \prod_{j=3}^{|\tilde{V}_{n-1}|} (\tilde{m}_n + 2)\right\}^{-2}$$

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$$
\begin{aligned}
&> \prod_{k=1}^{|V_{n-1}|} \left( \frac{4\theta_{l,k}^{(n)} - 2}{\theta_{l,k}^{(n)} + 3} \right)^{m_{l,k}^{(n)} - 2\delta_1(k)} \times (\tilde{m}_n + 2)^{-2|\tilde{V}_{n-1}|} \\
&> \frac{3^{m_n}}{4^2} \times (m_n + 2)^{-8|V_{n-1}|-4} \quad \left( \text{because } 3 < \frac{4\theta_{l,k}^{(n)} - 2}{\theta_{l,k}^{(n)} + 3} < 4 \text{ by (3.2)} \right) \\
&> 2^{m_n} \times (3/2)^{m_n} (m_n + 2)^{-8(|V_{n-1}|+1)} \\
&> 5 \times 2^{m_n} > 4|V_n| + 2 = |\tilde{V}_n| \quad \text{(by (iii))}.
\end{aligned}
$$

It is now obvious that we can assign a partial order $\geq$ on $\tilde{E}_n$ for each $n \geq 2$, so that statements (2), (3) and (4) hold.

Finally we will check the statement (5). To show this, we may construct a Bratteli diagram $\hat{\mathcal{B}} = (\hat{V}, \hat{E})$ so that the $\hat{\mathcal{B}}_{\text{odd}}$ corresponds to $\mathcal{B}$ and $\hat{\mathcal{B}}_{\text{even}}$ corresponds to $\tilde{\mathcal{B}}$. For $k \in \mathbb{N}$, we put $\hat{V}_{2k-1} = V_k$ and $\hat{V}_{2k} = \tilde{V}_k$. We define the incidence matrix $\hat{M}^{(n)} = [\hat{m}_{i,j}^{(n)}] \in \mathbb{N}^{|\hat{V}_n| \times |\hat{V}_{n-1}|}$ as

$$
\hat{m}_{i,j}^{(n)} = \begin{cases} \tilde{m}_{4i-1,j}^{(k)}, & \text{if } n = 2k-1, \\ \displaystyle\sum_{l=-1-2\delta_1(j)}^{2} \delta_i(4j + l), & \text{if } n = 2k. \end{cases}
$$

We will check that $\hat{M}^{(2k,2k-1)} = \tilde{M}^{(k)}$ and $\hat{M}^{(2k-1,2k-2)} = M^{(k)}$ and for all $k \in \mathbb{N}$.

$$
\begin{aligned}
\hat{m}_{i,j}^{(2k,2k-1)} &= \sum_{l=1}^{|\hat{V}_{2k-1}|} \hat{m}_{i,l}^{(2k)} \hat{m}_{l,j}^{(2k-1)} = \sum_{l=1}^{|V_k|} \left\{ \left( \sum_{t=-1-2\delta_1(l)}^{2} \delta_i(4l + t) \right) \tilde{m}_{4l-1,j}^{(k)} \right\} \\
&= \tilde{m}_{4l_0-1,j}^{(k)} \quad \begin{array}{l} \text{(there exists a unique } l_0 \text{ such that} \\ 4l_0 - 1 - 2\delta_1(l_0) \leq i \leq 4l_0 + 2) \end{array} \\
&= \tilde{m}_{i,j}^{(k)}, \\
\hat{m}_{i,j}^{(2k-1,2k-2)} &= \sum_{l=1}^{|\hat{V}_{2k-2}|} \hat{m}_{i,l}^{(2k-1)} \hat{m}_{l,j}^{(2k-2)} = \sum_{l=1}^{|\tilde{V}_{k-1}|} \left\{ \tilde{m}_{4i-1,l}^{(k)} \left( \sum_{t=-1-2\delta_1(j)}^{2} \delta_l(4j + t) \right) \right\} \\
&= \sum_{t=-1-2\delta_1(j)}^{2} \tilde{m}_{4i-1,4j+t}^{(k)} = m_{i,j}^{(k)} \quad \text{(by (3.3))}.
\end{aligned}
$$

Therefore $\mathcal{B} \sim \tilde{\mathcal{B}}$ holds. So we finish the proof. □

*Remark.* We give an example of $\mathcal{B}$, $\mathcal{B}'$ and $\tilde{\mathcal{B}}$. If $M^{(1)} = [60]$, $M^{(2)} = \left[\begin{smallmatrix} 75 \\ 75 \end{smallmatrix}\right]$ and $M^{(3)} = \left[\begin{smallmatrix} 100 & 90 \\ 60 & 130 \\ 75 & 115 \end{smallmatrix}\right]$, then $\tilde{M}^{(n)}$ and $\hat{M}^{(n)}$ $(n = 1, 2, 3)$ are the following:

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$$\tilde{M}^{(1)} = \begin{bmatrix} 60 \\ 60 \\ 60 \\ 60 \\ 60 \\ 60 \end{bmatrix}, \quad \tilde{M}^{(2)} = \begin{bmatrix} 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \end{bmatrix},$$

$$\tilde{M}^{(3)} = \begin{bmatrix} 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 15 & 15 & 14 & 14 & 33 & 33 & 32 & 32 \\ 1 & 1 & 15 & 15 & 14 & 14 & 33 & 33 & 32 & 32 \\ 1 & 1 & 15 & 15 & 14 & 14 & 33 & 33 & 32 & 32 \\ 1 & 1 & 15 & 15 & 14 & 14 & 33 & 33 & 32 & 32 \\ 1 & 1 & 19 & 18 & 18 & 18 & 29 & 29 & 29 & 28 \\ 1 & 1 & 19 & 18 & 18 & 18 & 29 & 29 & 29 & 28 \\ 1 & 1 & 19 & 18 & 18 & 18 & 29 & 29 & 29 & 28 \\ 1 & 1 & 19 & 18 & 18 & 18 & 29 & 29 & 29 & 28 \end{bmatrix},$$

$$\hat{M}^{(1)} = \begin{bmatrix} 60 \end{bmatrix}, \quad \hat{M}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{M}^{(3)} = \begin{bmatrix} 1 & 1 & 19 & 18 & 18 & 18 \\ 1 & 1 & 19 & 18 & 18 & 18 \end{bmatrix},$$

$$\hat{M}^{(4)} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \hat{M}^{(5)} = \begin{bmatrix} 1 & 1 & 25 & 25 & 24 & 24 & 23 & 23 & 22 & 22 \\ 1 & 1 & 15 & 15 & 14 & 14 & 33 & 33 & 32 & 32 \\ 1 & 1 & 19 & 18 & 18 & 18 & 29 & 29 & 29 & 28 \end{bmatrix},$$

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$$\hat{M}^{(6)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

## 4. *Proof of the main theorem*

In this section, we will show the main theorem (Theorem 1.2). We recall that $(X, T)$ is *expansive* if there exists $\delta > 0$ such that for any $x, y \in X$ with $x \neq y$, $\sup_{n\in\mathbb{Z}} d(T^n x, T^n y) > \delta$, where $d$ is a metric yielding the topology of $X$. One can show that the definition of expansiveness is independent of the choice of metric $d$. (However, the *expansive constant* $\delta$ depends on $d$.)

The next theorem is a sufficient condition in order that a Bratteli–Vershik system is expansive.

THEOREM 4.1. *Suppose $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is a Bratteli–Vershik system, where $\tilde{\mathcal{B}}$ satisfies conditions (1)–(4) in Lemma 3.5. Then $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is expansive.*

*Proof.* Take any $x, y \in X_{\tilde{\mathcal{B}}}$ with $x \neq y$ and fix them. (If $x = (x_1, x_2, \dots) \in X_{\tilde{\mathcal{B}}}$, we write $(x)_k = x_k \in E_k$ and $\pi_k$ is a truncation map from $X_{\tilde{\mathcal{B}}}$ to $\mathcal{P}(\tilde{V}_k)$ defined by $\pi_k(x) = (x_1, x_2, \dots, x_k)$.) We write $S = \lambda_{\tilde{\mathcal{B}}}$. It suffices to show that there is $m \in \mathbb{Z}$ so that $\pi_1(S^m x) \neq \pi_1(S^m y)$ cf. [**GJ1**]. If $(x)_1 \neq (y)_1$, the claim would have been proven. Therefore assume that there is $k \in \mathbb{N}$ for which $\pi_k(x) = \pi_k(y)$ and $\pi_{k+1}(x) \neq \pi_{k+1}(y)$. Let $n \leq 0$ so that $-n$ is the smallest number such that the edges of $\pi_{k+1}(S^n x)$ lie in $\tilde{E}_{\min}$. Then we have the following two cases:

(i) $\pi_k(S^n x) = \pi_k(S^n y)$;
(ii) $\pi_k(S^n x) \neq \pi_k(S^n y)$.

We consider case (i). We note that if $\pi_k(S^n x) = \pi_k(S^n y)$, then the edges of $\pi_{k+1}(S^n y)$ always lie in $\tilde{E}_{\min}$ (i.e. $(S^n y)_{k+1} \in \tilde{E}_{\min}$). Since $s((S^n x)_{k+1}) = \tilde{v}^{(k)}_{\min} = s((S^n y)_{k+1})$ and by condition (3), $\{e \in \tilde{E}_{k+1} \mid s(e) = \tilde{v}^{(k)}_{\min},\ r(e) = r((S^n y)_{k+1})\} = \{(S^n y)_{k+1}\}$. Therefore by condition (4), $(S^n y)_{k+1} \in \tilde{E}_{\min}$. Let $v = r((S^n x)_{k+1})$ and $u = r((S^n y)_{k+1})$. Then by the assumption, $v \neq u$. Since $\tilde{V}_{k+1}$ has distinct orderings, there exists $1 < n' < \tilde{m}_{k+1}$ such that $\tau(v, n') \neq \tau(u, n')$. Let $\tilde{n} = n + (n' - 1)|\mathcal{P}(\tilde{v})|$, where $\tilde{v} \in \tilde{V}_k$. (We recall that $\mathcal{P}(\tilde{v})$ denotes the set of all paths from $\tilde{V}_0$ to $\tilde{v}$.) Then we see that $s((S^{\tilde{n}} x)_{k+1}) = \tau(v, n')$ and $s((S^{\tilde{n}} y)_{k+1}) = \tau(u, n')$. This implies that $\pi_k(S^{\tilde{n}} x) \neq \pi_k(S^{\tilde{n}} y)$.

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Both cases (i) and (ii) imply that there exists $N \in \mathbb{Z}$ such that $\pi_k(S^N x) \neq \pi_k(S^N y)$ holds. By repeating this procedure, we get $\pi_1(S^m x) \neq \pi_1(S^m y)$ for some $m \in \mathbb{Z}$. So we finish the proof. □

*Proof of Theorem 1.2.* Let $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be a Bratteli–Vershik system so that properly ordered Bratteli diagram $\mathcal{B}$ has the equal path number property. By telescoping, we may assume that $\mathcal{B}$ has at least one edge between every pair of vertices from consecutive levels. We apply Lemma 3.5 and get the properly ordered Bratteli diagram $\tilde{\mathcal{B}}$ which satisfies the conditions in Lemma 3.5. By Theorem 4.1, the Bratteli–Vershik system $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is expansive. Now $\mathcal{B} \sim \tilde{\mathcal{B}}$, and so by Theorem 2.1, $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ are strongly orbit equivalent. Moreover, since $\tilde{\mathcal{B}}$ has the equal path number property and by Theorem 1.1, $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is a Toeplitz flow. So we finish the proof. □

*Acknowledgements.* The author especially thanks Yuji Ito and Toshihiro Hamachi for their many helpful comments and valuable advice. He would also wish to thank the referees for the ideas which improved this paper.

The author was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

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